J. Math. Anal. Appl. 380 (2011) 768–781

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Norm inequalities for vector functions

B.A. Bhayo [a], V. Božin [b], D. Kalaj [c], M. Vuorinen [a,*]

[a] Department of Mathematics, University of Turku, 20014 Turku, Finland
[b] Faculty of Mathematics, University of Belgrade, Studentski trg 16, Belgrade, Serbia
[c] University of Montenegro, Faculty of Mathematics, Dzordza Vaöingtona b.b., Podgorica, Montenegro

ARTICLE INFO

Article history:
Received 27 August 2010
Available online 12 February 2011
Submitted by Steven G. Krantz

Keywords:
Quasiconformal map
Normed linear space

ABSTRACT

We study vector functions of $\mathbb{R}^n$ into itself, which are of the form $x \mapsto g(|x|)x$, where $g : (0,\infty) \to (0,\infty)$ is a continuous function and call these radial functions. In the case when $g(t) = t^c$ for some $c \in \mathbb{R}$, we find upper bounds for the distance of image points under such a radial function. Some of our results refine recent results of L. Maligranda and S.S. Dragomir. In particular, we study quasiconformal mappings of this simple type and obtain norm inequalities for such mappings.

1. Introduction

In 2006 L. Maligranda [10] studied the following function

$$\alpha_p(x,y) = \big||x|^{p-1}x - |y|^{p-1}y\big|, \quad p \in \mathbb{R}, \tag{1.1}$$

for $x, y \in \mathbb{R}^n \setminus \{0\}$, termed the p-angular distance between x and y. It is clear that α_p satisfies the triangle inequality and thus it defines a metric. Note that $\alpha_0(x,y)$ equals $2\sin(\omega/2)$ where $\omega \in [0,\pi]$ is the angle between the segments $[0,x]$ and $[0,y]$. He proved in [10, Theorem 2] the following theorem in the context of normed spaces.

1.2. Theorem.

$$\alpha_p(x,y) \leqslant \begin{cases} (2-p)\frac{|x-y|\max\{|x|^p,|y|^p\}}{(\max\{|x|,|y|\})} & \text{if } p \in (-\infty,0) \text{ and } x,y \neq 0; \\ (2-p)\frac{|x-y|}{(\max\{|x|,|y|\})^{1-p}} & \text{if } p \in [0,1] \text{ and } x,y \neq 0; \\ p(\max\{|x|,|y|\})^{p-1}|x-y| & \text{if } p \in (1,\infty). \end{cases}$$

Soon thereafter, in 2009, S.S. Dragomir [6, Theorem 1] refined this result and gave the following upper bound for the p-angular distance for nonzero vectors x, y.

1.3. Theorem.

$$\alpha_p(x,y) \leqslant \begin{cases} |x-y|(\max\{|x|,|y|\})^{p-1} + \big||x|^{p-1} - |y|^{p-1}\big|\min\{|x|,|y|\} & \text{if } p \in (1,\infty); \\ \frac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \big||x|^{1-p} - |y|^{1-p}\big|\min\{\frac{|x|^p}{|y|^{1-p}}, \frac{|y|^p}{|x|^{1-p}}\} & \text{if } p \in [0,1]; \\ \frac{|x-y|}{(\min\{|x|,|y|\})^{1-p}} + \frac{||x|^{1-p}-|y|^{1-p}|}{(\max\{|x|^{-p}|y|^{1-p},|y|^{-p}|x|^{1-p}\})} & \text{if } p \in (-\infty,0). \end{cases}$$

* Corresponding author.
E-mail addresses: barbha@utu.fi (B.A. Bhayo), bozinv@turing.mi.sanu.ac.rs (V. Božin), davidk@ac.me (D. Kalaj), vuorinen@utu.fi (M. Vuorinen).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.02.029

Generalizations for operators were discussed very recently in [5]. For general information about norm inequalities see [11, Chapter XVIII].

Studying sharp constants connected to the p-Laplace operator J. Byström [4, Lemma 3.3] proved in 2005 the following result.

1.4. Theorem. *For $p \in (0,1)$ and $x, y \in \mathbb{R}^n$, we have*

$$\alpha_p(x,y) \leqslant 2^{1-p}|x-y|^p$$

with equality for $x = -y$.

In this paper we study a two exponent variant of the function $x \mapsto |x|^{p-1}x$ defined for $a, b > 0$, $x \in \mathbb{R}^n$,

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & \text{if } |x| < 1, \\ |x|^{b-1}x & \text{if } |x| \geqslant 1. \end{cases} \tag{1.5}$$

This function, like its one exponent version (the special case $a = b$), defines a quasiconformal mapping and it has been used in many examples to illuminate various properties of these maps [12, p. 49]. For instance, if $a \in (0,1)$ the function $\mathcal{A}_{a,b}$ is Hölder-continuous at the origin.

We prove that the change of distance under this function is maximal in the radial direction, up to a constant, in the sense of the next theorem (observe that the points x and z are on the same ray). Note that the result is sharp for $a \to 1$. This result is natural to expect, but the proof is somewhat involved. For brevity we write $\mathcal{A} = \mathcal{A}_{a,b}$ if $0 < a \leqslant 1 \leqslant b$.

1.6. Theorem. *Let $0 < a \leqslant 1 \leqslant b$ and*

$$C(a,b) = \sup_{|x| \leqslant |y|} Q(x,y),$$

where

$$Q(x,y) = \frac{|\mathcal{A}(x) - \mathcal{A}(y)|}{|\mathcal{A}(x) - \mathcal{A}(z)|}, \quad x, y \in \mathbb{R}^n \setminus \{0\} \text{ with } x \neq y,$$

and

$$z = \frac{x}{|x|}\big(|x| + |x-y|\big).$$

Then

$$C(a,b) = \frac{2}{3^a - 1} \quad \text{and} \quad \lim_{a \to 1} C(a,b) = 1.$$

Because $\mathcal{A}_{a,b}$ agrees with $x \mapsto |x|^{a-1}x$ in $\mathbb{B}^n$, we can compare Theorem 1.6 to Theorems 1.2, 1.3, and 1.4. We also have the following upper bound for α_p:

1.7. Theorem. *For all $x, y \in \mathbb{R}^n$ and $p \in (0,1)$*

$$\alpha_p(x,y) \leqslant \big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\big|, \tag{1.8}$$

and furthermore, if $|x| \leqslant |y|$, we have also

$$\alpha_p(x,y) \leqslant \big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\big| \leqslant \frac{2}{3^p - 1}\big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(z)\big| \tag{1.9}$$

where z is as in Theorem 1.6.

For a systematic comparison of the above results, see Section 5 where it is shown that sometimes the bound in Theorem 1.7 is better than the other bounds in Theorems 1.2, 1.3, 1.4.

We also discuss some properties of the distortion function $\varphi_K(r)$ associated with the quasiconformal Schwarz lemma, see [9].

2. Preliminary results

We prove here some inequalities for elementary functions that will be applied in later sections. These inequalities deal with the logarithm and some of them may be new results. Note also in the paper [7] some elementary Bernoulli type inequalities were proved and used as a key tool. We use the notation sh, ch, th, arsh, arch and arth to denote the hyperbolic sine, cosine, tangent and their inverse functions, respectively.

As well known, conformal invariants of geometric function theory are on one hand closely linked with function theoretic extremal problems and on the other hand with special functions such as complete elliptic integrals, elliptic functions and hypergeometric functions. The connection between conformal invariants and special functions is provided by conformal maps which can be applied to express maps of quadrilaterals and ring domains onto canonical ring domains such as a rectangle and an annulus.

For example, the quasiconformal version of the Schwarz lemma says that for a K-quasiconformal map of the unit disk $\mathbb{B}^2$ onto itself keeping 0 fixed, we have for all $z \in \mathbb{B}^2$ the sharp bound [9, p. 64]

$$\bigl|f(z)\bigr| \leqslant \varphi_K\bigl(|z|\bigr), \qquad \varphi_K(r) = \mu^{-1}\bigl(\mu(r)/K\bigr) \tag{2.1}$$

where $\mu : (0,1) \to (0,\infty)$ is a decreasing homeomorphism defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \qquad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \tag{2.2}$$

and where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$. The function $\varphi_K(r)$ has numerous applications to quasiconformal mapping theory, see [9,8,2], which motivates the study of its properties. One of the challenges is to find bounds, in the range $(0,1)$, and yet asymptotically sharp when $K \to 1$. For instance, the change of hyperbolic distances under K-quasiconformal mappings of the unit disk onto itself can be estimated in terms of the function φ_K, see [2,9].

2.3. Lemma. *The following functions are monotone increasing from* $(0,\infty)$ *onto* $(1,\infty)$;

(1) $$f(x) = \frac{(1+x)\log(1+x)}{x}.$$

(2) $$g(x) = \frac{x}{\log(1+x)}.$$

(3) *For a fixed* $t \in (0,1)$, *the function* $h(K) = K(1-t^{2/K})$ *is monotone increasing on* $(1,\infty)$.

Proof. For the proof of (1) see [7, p. 7]. For (2), we get

$$g'(x) = \frac{1}{\log(1+x)} - \frac{x}{(1+x)(\log(1+x))^2} = \frac{(1+x)\log(1+x) - x}{(1+x)(\log(1+x))^2},$$

and $g'(x) > 0$ by (1). Moreover, g tends to 1 and ∞ when x tends 0 and ∞. Proof of (3) follows easily because $x \mapsto (1-a^x)/x$ is decreasing on $(0,1)$ for each $a \in (0,1)$ [2, 1.58(3)]. □

2.4. Corollary. *For a fixed* $x \in (0,1)$, *the following functions,* (1) $f(a) = (1+ax)^{1/a}$, (2) $g(a) = (\log(1+x^a))^{1/a}$ *are decreasing and increasing on* $(1,\infty)$, *respectively.* (3) *The following inequality holds for* $x \geqslant 0$ *and* $a \in [0,1]$,

$$\log\bigl(1+x^a\bigr) \leqslant \max\bigl\{\log(1+x), \log^a(1+x)\bigr\}.$$

2.5. Lemma. *For* $K > 1$, $r \in (0,1)$, $u = \operatorname{arch}(1/r)/K$, *the following functions*

(1) $f(K) = r \operatorname{arth}(1/\operatorname{ch}(u))\operatorname{sh}(u)$,
(2) $g(K) = rK \operatorname{arth}(1/\operatorname{ch}(u))\operatorname{sh}(u)$

are strictly decreasing and increasing, respectively. Moreover, both functions tend to $\sqrt{1-r^2}\operatorname{arth}(r)$ *when* K *tends to* 1.

Proof. Differentiating f with respect to K we get

$$f'(K) = -\frac{r\operatorname{arth}(1/r)}{K^2}\left(\operatorname{arth}\left(\frac{1}{\operatorname{ch}(u)}\right)\operatorname{ch}(u) - 1\right) \leqslant 0,$$

$$g'(K) = r\left(\operatorname{ch}\left(\frac{1}{r}\right)\left(1 - \operatorname{arth}\left(\frac{1}{\operatorname{ch}(u)}\right) + K\operatorname{arth}\left(\frac{1}{\operatorname{ch}(u)}\right)\operatorname{sh}(u)\right)\right) \geqslant 0,$$

respectively. We obtain

$$f(1)=g(1)=r\,\mathrm{arth}(r)\sqrt{\big(\mathrm{ch}\big(\mathrm{arch}(1/r)\big)\big)^2-1}=\sqrt{1-r^2}\,\mathrm{arth}(r). \qquad \square$$

2.6. Lemma.

(1) *For a fixed $t>0$, the following function is monotone increasing in $K>1$. Moreover, for $t=t_0=(e-1)/(e+1)$, the function is increasing from $(1,\infty)$ onto $(m_1,1)$,*

$$f(K)=\frac{K-\log(1/t)}{t^{2/K}(K+\log(1/t))},\qquad m_1=\frac{1+\log t_0}{t_0^2(1-\log t_0)}\approx 0.6027\ldots.$$

(2) *The following function is monotone increasing from $(1,\infty)$ onto $(m_2,1)$,*

$$g(K)=\frac{t_0^{1/K}\log(1/t_0^2)}{K(1-t_0^{2/K})},\qquad m_2=\frac{2t_0\log t_0}{t_0^2-1}\approx 0.9072\ldots.$$

Proof. Differentiating f with respect to K we get

$$\begin{aligned}
f'(K)&=\frac{t^{-2/K}}{K+\log(1/t)}-\frac{t^{-2/K}(K-\log(1/t))}{(K+\log(1/t))^2}+\frac{2t^{-2/K}(K-\log(1/t))\log t}{K^2(K+\log(1/t))^2}\\
&=\frac{2t^{-2/K}(K^2\log(1/t)+K^2\log t-(\log(1/t))^2\log t)}{K^2(K+\log(1/t))^2}\\
&=\frac{2t^{-2/K}(\log(1/t))^3}{K^2(K+\log(1/t))^2}>0.
\end{aligned}$$

For $t=t_0$, f tends to m_1 and 1 when K tends to 1 and ∞, respectively.

For the proof of (2), we differentiate g with respect to K and get,

$$\begin{aligned}
g'(K)&=-\frac{t_0^{1/K}\log(1/t_0^2)}{K^2(1-t_0^{2/K})}-\frac{2t_0^{3/K}\log(1/t_0^2)\log t_0}{K^3(1-t_0^{2/K})^2}-\frac{t_0^{1/K}\log(1/t_0^2)\log t_0}{K^3(1-t_0^{2/K})}\\
&=t_0^{1/K}\log\big(1/t_0^2\big)\big(-K\big(1-t_0^{2/K}\big)-\big(1+t_0^{2/K}\big)\log t_0\big)/\big(K^3\big(1-t_0^{2/K}\big)^2\big)\\
&=t_0^{1/K}\log\big(1/t_0^2\big)\big(t_0^{2/K}\big(1+\log(1/t_0)\big)-\big(K-\log(1/t_0)\big)\big)/\big(K^3\big(1-t_0^{2/K}\big)^2\big)\\
&=\frac{t_0^{3/K}\log(1/t_0^2)(1+\log(1/t_0))}{K^3(1-t_0^{2/K})^2}\left(1-\frac{K-\log(1/t_0)}{t_0^{2/K}(1+\log(1/t_0))}\right)>0
\end{aligned}$$

by (1). We can see that g tends to m_2 and 1 when K tends to 1 and ∞, respectively. This completes the proof. $\square$

2.7. Lemma. *The following inequality holds for $K\geqslant 1$ and $t\in[t_0,1)$, $t_0=(e-1)/(e+1)$*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right)\leqslant K\log\left(\frac{1+t}{1-t}\right). \tag{2.8}$$

Proof. Write $h(t)=K\,\mathrm{arth}(t)-\mathrm{arth}(t^{1/K})$. Differentiating h with respect to t we get

$$\begin{aligned}
h'(t)&=\frac{K}{1-t^2}-\frac{t^{1/K-1}}{K(1-t^{2/K})}=\frac{K^2t(1-t^{2/K})-t^{1/K}(1-t^2)}{tK(1-t^2)(1-t^{2/K})}\\
&\geqslant\frac{Kt(1-t^2)-t^{1/K}(1-t^2)}{tK(1-t^2)(1-t^{2/K})}=\frac{Kt-t^{1/K}}{Kt(1-t^{2/K})}\geqslant 0
\end{aligned}$$

the first inequality holds by Lemma 2.3(3) and the second one holds when $Kt\geqslant t^{1/K}\Leftrightarrow t\geqslant(1/K)^{K/(K-1)}=c_1(K)$. We see that $c_1(K)\to 1/e\approx 0.3679\ldots$ and 0 when $K\to 1$ and ∞ respectively, hence $h(t)$ is increasing in $t\geqslant 1/e$.

We can see that $h(t_0)=K(1-2\,\mathrm{arth}(t_0^{1/K})/K)/2$. Now it is enough to prove that $f(K)=2\,\mathrm{arth}(t_0^{1/K})/K<1$. Differentiating f with respect to K we get

$$\begin{aligned}
f'(K) &= \frac{-2\,\mathrm{arth}(t_0^{1/K})}{K^2} - \frac{2t_0^{1/K}\log(t_0)}{K^3(1-t_0^{2/K})} \\
&= 2\big(-K\big(1-t_0^{2/K}\big)\,\mathrm{arth}\big(t_0^{1/K}\big) - t_0^{1/K}\log(t_0)\big)/\big(K^3\big(1-t_0^{2/K}\big)\big) \\
&\leqslant 2\big(-K\big(1-t_0^{2/K}\big)\,\mathrm{arth}(t_0) + t_0^{1/K}\log(1/t_0)\big)/\big(K^3\big(1-t_0^{2/K}\big)\big) \\
&= 2\left(-\big(K\big(1-t_0^{2/K}\big)/2\big)\log\left(\frac{1+t_0}{1-t_0}\right) + t_0^{1/K}\log(1/t_0)\right)/\big(K^3\big(1-t_0^{2/K}\big)\big) \\
&= \big(t_0^{1/K}\log\big(1/t_0^2\big) - K\big(1-t_0^{2/K}\big)\big)/\big(K^3\big(1-t_0^{2/K}\big)\big) \\
&= \frac{1}{K^2}\left(\frac{t_0^{1/K}\log(1/t_0^2)}{K(1-t_0^{2/K})} - 1\right) < 0
\end{aligned}$$

by Lemma 2.6(2), hence f is a monotone decreasing function from $(1, K)$ onto $(0, 1/2)$. This implies the proof. □

2.9. Lemma. *The following inequality holds for $K \geqslant 1$ and $t \in (0, t_0]$, $t_0 = (e-1)/(e+1)$*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right) \leqslant K\left(\log\left(\frac{1+t}{1-t}\right)\right)^{1/K}. \tag{2.10}$$

Proof. Write

$$F(t) = K - \frac{\log((1+t^{1/K})/(1-t^{1/K}))}{(\log(1+t)/(1-t))^{1/K}}.$$

For the proof of (2.10) we show that $F(t)$ is decreasing in t and $F(t_0) \geqslant 0$. Differentiating F with respect to t we get,

$$F'(t) = \frac{\log(\frac{1+t}{1-t})^{(K-1)/K}(2t^{1/K}(t^2-1)\log(\frac{1+t}{1-t}) - 2t(t^{2/K}-1)\log(\frac{1+t^{1/K}}{1-t^{1/K}}))}{Kt(t^2-1)(t^{2/K}-1)}.$$

Now we show that

$$t\big(t^{2/K}-1\big)\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right) \geqslant t^{1/K}\big(t^2-1\big)\log\left(\frac{1+t}{1-t}\right). \tag{$*$}$$

For the proof of $(*)$, it is enough to prove that $t(t^{2/K}-1) \geqslant t^{1/K}(t^2-1)$. We get

$$\begin{aligned}
t\big(t^{2/K}-1\big) - t^{1/K}\big(t^2-1\big) &= \big(t^{1/K+1}+t\big)\big(t^{1/K}-1\big) - \big(t^{1/K+1}+t^{1/K}\big)(t-1) \\
&= t^{1/K+1/K+1} + t^{1/K} - t - t^{1/K+1+1} \\
&= t^{1/K}\big(t^{1/K+1}+1\big) - t\big(t^{1/K+1}+1\big) \\
&= \big(t^{1/K+1}+1\big)\big(t^{1/K}-t\big) \geqslant 0,
\end{aligned}$$

this implies that $F(t)$ is decreasing in t. Now we prove that $F(t_0)$ is positive as a function of K. We write

$$f(K) = K - \log\left(\frac{1+t_0^{1/K}}{1-t_0^{1/K}}\right) = K - 2\,\mathrm{arth}\big(t_0^{1/K}\big) = F(t_0).$$

Differentiating f with respect to K we get

$$f'(K) = 1 + \frac{2t_0^{1/K}\log(t_0)}{K^2(1-t_0^{2/K})} \geqslant 1 - \frac{t_0^{1/K}\log(1/t_0^2)}{K^2(1-t_0^{2/K})} > 0$$

by Lemma 2.6(2), hence f is increasing in K. This implies the proof. □

2.11. Corollary. *The following inequality holds for $K \geqslant 1$ and $t \in [0, 1)$*

$$\log\left(\frac{1+t^{1/K}}{1-t^{1/K}}\right) \leqslant K\max\left\{\left(\log\left(\frac{1+t}{1-t}\right)\right)^{1/K}, \log\left(\frac{1+t}{1-t}\right)\right\}. \tag{2.12}$$

Proof. The proof follows easily from inequalities (2.8) and (2.10). □

The next function tells us how the hyperbolic distances from the origin are changed under the radial selfmapping of the unit disk, $z \mapsto |z|^{1/K-1}z$, $K > 1$, which is the restriction of $\mathcal{A}_{1/K,1/K}(z)$ to the unit disk. See also [3].

2.13. Theorem. *The following inequality holds for* $K \geqslant 1$, $|z| < 1$

$$\rho\big(0, \mathcal{A}_{1/K,K}(z)\big) \leqslant K \max\big\{\rho\big(0,|z|\big), \rho^{1/K}\big(0,|z|\big)\big\} \tag{2.14}$$

where ρ *is the hyperbolic metric* [13, *p.* 19].

Proof. Proof follows easily from inequality (2.12) and the formula $\rho(0,r) = \log((1+r)/(1-r))$. □

2.15. Remark. The constant K cannot be replaced by $K^{9/10}$ in (2.14), because for $t = t_0$, the inequality (2.14) is equivalent to $1 - 2\,\mathrm{arth}(t_0^{1/K})/K^{9/10} \geqslant 0$. Write $f(K) = 1 - 2\,\mathrm{arth}(t_0^{1/K})/K^{9/10}$, and we get

$$f'(K) = \frac{9\,\mathrm{arth}(t_0^{1/K})}{5K^{19/10}} + \frac{2t^{1/K}\log(t_0)}{K^{29/10}(1-t_0^{2/K})},$$

we see that $f'(1.005) = -0.004 < 0$, $f(K)$ is not increasing in K.

2.16. Lemma. *For* $K > 1$ *the function*

$$F(r) = \frac{2\,\mathrm{arth}(1/\mathrm{ch}(\mathrm{arch}(1/r)/K))}{\max\{2\,\mathrm{arth}(r), (2\,\mathrm{arth}(r))^{1/K}\}}$$

is monotone increasing in $(0, t_0)$ *and decreasing in* $(t_0, 1)$.

Proof. (1) Let $u = \mathrm{arch}(1/r)/K$ and

$$f(r) = \frac{\mathrm{arth}(1/\mathrm{ch}(u))}{\mathrm{arth}(r)}.$$

Differentiating f with respect to r we get

$$\begin{aligned} f'(r) &= -\frac{\mathrm{arth}(1/\mathrm{ch}(u))}{(1-r^2)(\mathrm{arth}(r))^2} + \frac{(1/\mathrm{ch}(u))\,\mathrm{th}(u)}{K\sqrt{1/r-1}\sqrt{1+1/r}r^2\,\mathrm{arth}(r)(1-(1/\mathrm{ch}(u))^2)} \\ &= -\frac{Kr\,\mathrm{arth}(1/\mathrm{ch}(u))\,\mathrm{sh}(u) - \sqrt{1-r^2}\,\mathrm{arth}(r)}{Kr(1-r^2)(\mathrm{arth}(r))^2\,\mathrm{sh}(u)} \leqslant 0, \end{aligned}$$

by Lemma 2.5(2), hence f is decreasing in $r \in (0,1)$.

(2) Let

$$g(r) = \frac{2^{1-1/K}\mathrm{arth}(1/\mathrm{ch}(u))}{(\mathrm{arth}(r))^{1/K}}.$$

Differentiating g with respect to K we get

$$g'(r) = \xi\big(\big(1-r^2\big)\mathrm{arth}(r) - r\sqrt{1-r^2}\mathrm{arth}\big(1/\mathrm{ch}(u)\big)\mathrm{sh}(u)\big) \geqslant 0$$

by Lemma 2.5(1), here

$$\xi = \frac{2^{1-1/K}(\mathrm{arth}(r))^{-(1+K)/K}}{Kr(1-r^2)^{3/2}\,\mathrm{sh}(u)}.$$

Hence g is increasing in $r \in (0,1)$. We see that $f(t_0) = g(t_0)$. Thus $F(r)$ increases in $r \in (0,t_0)$ and decreases in $t \in (t_0, 1)$. □

For instance it is well known that for all $K > 1$, $r \in (0,1)$

$$\log\left(\frac{1+\varphi_K(r)}{1-\varphi_K(r)}\right) > K\log\left(\frac{1+r}{1-r}\right) \tag{2.17}$$

[1, (4.5)]. In the next theorem we study a function $p(r)$ which by [2, Theorem 10.14] is a minorant of $\varphi_K(r)$.

2.18. Theorem. *The following inequality holds for* $K \geqslant 1$, $r \in (0,1)$, $t_0 = (e-1)/(e+1)$,

$$\log\left(\frac{1+p(r)}{1-p(r)}\right) \leqslant c_3(K)\max\left\{\log\left(\frac{1+r}{1-r}\right), \left(\log\left(\frac{1+r}{1-r}\right)\right)^{1/K}\right\}$$

here $p(r) = 1/\operatorname{ch}(\operatorname{arch}(1/r)/K)$ *and* $c_3(K) = 2\operatorname{arth}(p(t_0))$. *Moreover,* $c_3(K) \to 1$ *when* $K \to 1$.

Proof. The inequality follows easily from Lemma 2.16, because the maximum value of the function given in Lemma 2.16 is $c_3(K) = 1/\operatorname{ch}(\operatorname{arch}(1/t_0)/K)$. $\square$

We remark in passing that an inequality similar to (2.18) but with $p(r)$ replaced with $\varphi_K(r)$ and $c_3(K)$ replaced with a constant $c(K)$ was proved in [3, Lemma 4.8].

3. Quasiinvariance of the distance ratio metric

Our goal in this section is to study how the distances in the j-metric are transformed under the function (1.5) following closely the paper [7]. The main result here is Corollary 3.3.

3.1. Lemma. *The following inequality holds for* $K \geqslant 1$:

$$\log\left(1 + \frac{|\mathcal{A}_{1/K,K}(x) - \mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|, |\mathcal{A}_{1/K,K}(y)|\}}\right) \leqslant 2^{1-1/K}\max\left\{\log^{1/K}(t), \log(t)\right\} \tag{3.2}$$

here $t = 1 + \frac{|x-y|}{\min\{|x|,|y|\}}$, *for all* $x, y \in \mathbb{B}^n$.

Proof. By Theorem 1.4 and Corollary 2.4(1) we get

$$\begin{aligned} 1 + \frac{|\mathcal{A}_{1/K,K}(x) - \mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|, |\mathcal{A}_{1/K,K}(y)|\}} &\leqslant 1 + 2^{1-1/K}\frac{|x-y|^{1/K}}{\min\{|x|^{1/K}, |y|^{1/K}\}} \\ &\leqslant \left(1 + \left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)^{1/K}\right)^{2^{1-1/K}}. \end{aligned} \tag{$*$}$$

Taking log both sides to $(*)$ and by Corollary 2.4(3) we get

$$\begin{aligned} &\log\left(1 + \frac{|\mathcal{A}_{1/K,K}(x) - \mathcal{A}_{1/K,K}(y)|}{\min\{|\mathcal{A}_{1/K,K}(x)|, |\mathcal{A}_{1/K,K}(y)|\}}\right) \\ &\quad\leqslant \log\left(\left(1 + \left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)^{1/K}\right)^{2^{1-1/K}}\right) \\ &\quad\leqslant 2^{1-1/K}\max\left\{\log\left(1+\frac{|x-y|}{\min\{|x|,|y|\}}\right), \left(\log\left(1+\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right)^{1/K}\right\}. \qquad \square \end{aligned}$$

We denote by ∂G the boundary of a domain G and define

$$d(z) = \min\left\{|z-m|\colon m \in \partial G\right\}.$$

For a domain $G \subset \mathbb{R}^n$, $G \neq \mathbb{R}^n$, the following formula

$$j(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right), \quad x, y \in G$$

defines j as a metric in G (see [13, p. 28]).

3.3. Corollary. *Let* $D = \mathbb{R}^n \setminus \{0\}$, *then we have*

$$j_D\big(\mathcal{A}_{1/K,K}(x), \mathcal{A}_{1/K,K}(y)\big) \leqslant 2^{1-1/K}\max\left\{j_D(x,y), j_D^{1/K}(x,y)\right\}$$

for all $K \geqslant 1$, $x, y \in \mathbb{B}^n \cap D$.

Proof. Follows from inequality (3.2). $\square$

4. Radial functions

4.1. Definition. Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a homeomorphism. We say that f is a *radial function* if there exists a homeomorphism $g : (0,\infty) \to (0,\infty)$ such that $f(x) = g(|x|)x$, $x \in \mathbb{R}^n \setminus \{0\}$.

The following functions are examples of the radial functions:

(1) $h(x) = \frac{x}{|x|^2}$, $x \in \mathbb{R}^n \setminus \{0\}$, $h(0) = \infty$, $h(\infty) = 0$.
(2) For $a, b > 0$,

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & \text{if } |x| \leqslant 1, \\ |x|^{b-1}x & \text{if } |x| > 1. \end{cases}$$

4.2. Remark. Properties of $\mathcal{A}$:

(1) For $|x| < 1$ and $a, b, c, d > 0$

$$\begin{aligned} \mathcal{A}_{a,b}\big(\mathcal{A}_{c,d}(x)\big) &= \mathcal{A}_{a,b}\big(|x|^{c-1}x\big) = \big||x|^{c-1}x\big|^{a-1}|x|^{c-1}x \\ &= |x|^{ac-c}|x|^{c-1}x = |x|^{ac-1}x. \end{aligned}$$

(2) For $|x| > 1$

$$\begin{aligned} \mathcal{A}_{a,b}\big(\mathcal{A}_{c,d}(x)\big) &= \mathcal{A}_{a,b}\big(|x|^{d-1}x\big) = \big||x|^{d-1}x\big|^{b-1}|x|^{d-1}x \\ &= |x|^{bd-d}|x|^{d-1}x = |x|^{bd-1}x. \end{aligned}$$

(1) and (2) imply that $\mathcal{A}_{a,b}(\mathcal{A}_{c,d}(x)) = \mathcal{A}_{ac,bd}(x)$.

(3) $\mathcal{A}^{-1}_{a,b}(x) = \mathcal{A}_{1/a,1/b}(x)$.

4.3. Lemma. *(See [13, (1.5)].) An inversion in $S^{n-1}(a,r)$ is defined as,*

$$h(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \qquad h(a) = \infty, \qquad h(\infty) = a.$$

Moreover,

$$\big|h(x) - h(y)\big| = \frac{r^2|x-y|}{|x-a||y-a|}. \tag{4.4}$$

One of the goals of this section is to find a partial counterpart of the distance formula (4.4) for $\mathcal{A}$ and to prove Theorem 1.6.

4.5. Lemma. *Let $h(w) = r^2 w/|w|^2$, $r > 0$, $w \in \mathbb{R}^n \setminus \{0\}$ and let $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \leqslant |y|$. Then with $\lambda = (|x| + |x-y|)/|x|$ and $z = \lambda x$ we have*

$$\big|h(x) - h(z)\big| \leqslant \big|h(x) - h(y)\big| \leqslant 3\big|h(x) - h(z)\big|.$$

Equality holds in the upper bound for $x = -y$.

Proof. For the proof of first inequality we observe that

$$\begin{aligned} \big|h(x) - h(z)\big| &= \left|h(x) - \frac{\lambda}{|\lambda|^2}h(x)\right| = \frac{|\lambda - 1|}{\lambda}\frac{r^2}{|x|} \\ &= \frac{r^2|x-y|}{|x|(|x| + |x-y|)} \\ &\leqslant \frac{r^2|x-y|}{|x||y|} = \big|h(x) - h(y)\big| \end{aligned}$$

by triangle inequality.

For the second inequality, we have

$$\frac{|h(x)-h(y)|}{|h(x)-h(z)|} = \frac{|x-y|}{|x||y|}\frac{|x|(|x|+|x-y|)}{|x-y|} = \frac{|x|}{|y|}+\frac{|x-y|}{|y|} \leqslant 1+\frac{|x|+|y|}{|y|} \leqslant 3.$$

Note that here equality holds for $x=-y$. □

4.6. Lemma. *The following inequality holds for $K \geqslant 1$:*

$$\big||x|^{K-1}x-|y|^{K-1}y\big| \leqslant e^{\pi(K-1/K)}|x|^{K-1/K}\max\{|x-y|^{1/K},|x-y|^{K}\}$$

for all $x, y \in \mathbb{C}\setminus\overline{\mathbb{B}}^2$.

Proof. By [2, Theorem 14.18, (14.4)] we get because $f: x \mapsto |x|^{K-1}x$ is K-quasiconformal [12, 16.2]

$$\big||x|^{K-1}x, f(0), |y|^{K-1}y, f(\infty)\big| \leqslant \eta^*_{K,2}\big(|x,0,y,\infty|\big) = \eta_{K,2}\left(\frac{|x-y|}{|x|}\right).$$

Finally by [2, Theorem 10.24] and [13, Remark 10.31] we have

$$\begin{aligned}\big||x|^{K-1}x-|y|^{K-1}y\big| &\leqslant |x|^K\eta_{K,2}\left(\frac{|x-y|}{|x|}\right)\\ &\leqslant \lambda(K)|x|^K\max\left\{\left(\frac{|x-y|}{x}\right)^{1/K},\left(\frac{|x-y|}{x}\right)^{K}\right\}\\ &\leqslant e^{\pi(K-1/K)}|x|^{K-1/K}\max\{|x-y|^{1/K},|x-y|^K\}.\end{aligned}$$ □

4.7. Lemma. *The following inequality holds for $K \geqslant 1$ and for all $x, y \in \mathbb{R}^n\setminus\overline{\mathbb{B}}^n$:*

$$\big||x|^{\beta-1}x-|y|^{\beta-1}y\big| \leqslant c(K)|x|^{\beta-\alpha}\max\{|x-y|^{\alpha},|x-y|^{\beta}\}$$

here $c(K)=2^{K-1}K^K\exp(4K(K+1)\sqrt{K-1})$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Proof. By [2, Theorem 14.18] we get because $f: x \mapsto |x|^{\beta-1}x$ is K-quasiconformal [12, 16.2]

$$\big||x|^{\beta-1}x, f(0), |y|^{\beta-1}y, f(\infty)\big| \leqslant \eta^*_{K,n}\big(|x,0,y,\infty|\big),$$

and this is equivalent to

$$\big||x|^{\beta-1}x-|y|^{\beta-1}y\big| \leqslant |x|^{\beta}\eta^*_{K,n}\left(\frac{|x-y|}{|x|}\right).$$

By [2, Theorem 14.6] we get

$$\begin{aligned}\big||x|^{\beta-1}x-|y|^{\beta-1}y\big| &\leqslant c(K)|x|^{\beta}\max\left\{\left(\frac{|x-y|}{x}\right)^{\alpha},\left(\frac{|x-y|}{x}\right)^{\beta}\right\}\\ &\leqslant c(K)|x|^{\beta-\alpha}\max\{|x-y|^{\alpha},|x-y|^{\beta}\}.\end{aligned}$$ □

4.8. Corollary. *The following inequalities hold for $K \geqslant 1$:*

$$\left|\frac{x}{|x|^{1+1/K}}-\frac{y}{|y|^{1+1/K}}\right| \leqslant 2^{1-1/K}\frac{|x-y|^{1/K}}{(|x||y|)^{1/K}} \tag{4.9}$$

for all $x, y \in \mathbb{R}^n\setminus\mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+\beta}}-\frac{y}{|y|^{1+\beta}}\right| \leqslant \frac{c(K)}{|x|^{\beta-\alpha}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{\alpha},\left(\frac{|x-y|}{|x||y|}\right)^{\beta}\right\} \tag{4.10}$$

for all $x, y \in \mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+K}}-\frac{y}{|y|^{1+K}}\right| \leqslant \frac{e^{\pi(K-1/K)}}{|x|^{K-1/K}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{1/K},\left(\frac{|x-y|}{|x||y|}\right)^{K}\right\} \tag{4.11}$$

for all $x, y \in \mathbb{B}^2$.

Proof. For the proof of (4.9) we define

$$g(z) = \mathcal{A}_{1/K,K}\big(h(z)\big) = \frac{z}{|z|^{1+1/K}}, \qquad h(z) = \frac{z}{|z|^2}, \quad z \in \mathbb{R}^n \setminus \mathbb{B}^n.$$

By Theorem 1.4 and (4.4) we get,

$$\begin{aligned}\big|g(x)-g(y)\big| &= \left|\frac{x}{|x|^{1+1/K}} - \frac{y}{|y|^{1+1/K}}\right| \\ &\leqslant 2^{1-1/K}\big|h(x)-h(y)\big|^{1/K} \leqslant 2^{1-1/K}\frac{|x-y|^{1/K}}{(|x||y|)^{1/K}}.\end{aligned}$$

Again for the proof of (4.10) we define

$$g(z) = \mathcal{A}_{\alpha,\beta}\big(h(z)\big) = \frac{z}{|z|^{1+\beta}}, \qquad h(z) = \frac{z}{|z|^2}, \quad z \in \mathbb{B}^n.$$

By Lemma 4.7 and (4.4) we get,

$$\begin{aligned}\big|g(x)-g(y)\big| &\leqslant c(K)\big|h(x)\big|^{\beta-\alpha}\max\big\{\big|h(x)-h(y)\big|^{\alpha}, \big|h(x)-h(y)\big|^{\beta}\big\} \\ &= \frac{c(K)}{|x|^{\beta-\alpha}}\max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{\alpha}, \left(\frac{|x-y|}{|x||y|}\right)^{\beta}\right\}.\end{aligned}$$

Similarly, inequality (4.11) follows from Lemma 4.6 and (4.4). □

4.12. Lemma. *For $0 < a \leqslant 1 \leqslant p < \infty$ and $0 \leqslant s \leqslant 2\pi$ we have*

$$\frac{(1+p^{2a}-2p^a\cos s)^{1/2}}{(-1+(1+X)^a)} \leqslant \frac{1+p^a}{(-1+(2+p)^a)}, \quad X = \sqrt{1+p^2-2p\cos s}.$$

Proof. Let

$$f_{p,a}(s) = \frac{(1+p^{2a}-2p^a\cos s)}{(-1+(1+X)^a)^2}.$$

Then

$$f'_{p,a}(s) = 2\frac{(-a(p^{1-a}+p^{a+1}-2p\cos s)/X + (1+X-(1+X)^{1-a}))\sin s}{p^{-a}(1+X)^{1-a}(-1+(1+X)^a)^3}.$$

As

$$p^{1-a} + p^{a+1} \leqslant 1 + p^2$$

because

$$p^{1+a}\big(1-p^{1-a}\big) \leqslant 1 - p^{1-a}$$

it follows that

$$f'_{p,a}(s)/\sin s \geqslant 2\frac{(-aX + (1+X-(1+X)^{1-a}))}{p^{-a}(1+X)^{1-a}(-1+(1+X)^a)^3}.$$

As

$$(1+X)^{1-a} < 1 + (1-a)X,$$

it follows that

$$f'_{p,a}(s) = 0 \quad \text{if and only} \quad s = 0 \quad \text{or} \quad s = \pi.$$

For $s = 0$, the function $f_{p,a}$ achieves its minimum

$$f_{p,a}(0) = \left(\frac{-1+p^a}{-1+p^a}\right)^2 = 1$$

and for $s = \pi$ its maximum

$$f_{p,a}(\pi) = \left(\frac{1+p^a}{(-1+(2+p)^a)}\right)^2. \qquad \square$$

4.13. Lemma. *For $p \geqslant 1$, and $0 < d \leqslant 1$ there holds*

$$\frac{1+p^d}{(2+p)^d-1} \leqslant \frac{2}{3^d-1}. \tag{4.14}$$

Proof. Let

$$h(p) = \big(3^d-1\big)\big(1+p^d\big) - 2\big((2+p)^d-1\big).$$

We need to show that $h(p) \leqslant 0$. First of all

$$h'(p) = d\big(\big(3^d-1\big)p^{d-1} - 2(2+p)^{d-1}\big).$$

Then

$$h'(p) \leqslant 0 \quad \Leftrightarrow \quad \left(\frac{2}{p}+1\right)^{1-d} \leqslant \frac{2}{3^d-1}.$$

Since

$$\left(\frac{2}{p}+1\right)^{1-d} \leqslant 3^{1-d}$$

we need to show that

$$(3)^{1-d} \leqslant \frac{2}{3^d-1},$$

but this is equivalent to

$$3^d \leqslant 3$$

which is obviously true. Thus $h'(p) \leqslant 0$, and consequently $h(p) \leqslant h(1) = 0$ and this inequality coincides with (4.14). □

Proof of Theorem 1.6. *The case* $1 \leqslant |x| \leqslant |y|$. Let us show that $Q(x, y) \leqslant 1$. Without loss of generality, we can assume that $x = r$ and z are positive real numbers, and $y = Re^{it}$. Then $z = r + |r - Re^{it}|$. Let

$$p = \frac{R}{r}.$$

Then $p \geqslant 1$. Next we have

$$\begin{aligned}
\frac{|\mathcal{A}(x) - \mathcal{A}(y)|}{|\mathcal{A}(x) - \mathcal{A}(z)|} &= \frac{|1 - p^b e^{it}|}{(1 + |1 - pe^{it}|)^b - 1} \\
&\leqslant \frac{|1 - p^b e^{it}|}{(1 + |1 - p|)^{b-1}(1 + |1 - pe^{it}|) - 1} \\
&= \frac{|1 - p^b e^{it}|}{p^{b-1}(1 + |1 - pe^{it}|) - 1} \\
&= \frac{|1 - p^b e^{it}|}{p^{b-1} - 1 + |p^{b-1} - p^b e^{it}|} \\
&= \frac{|1 - p^{b-1} + p^{b-1} - p^b e^{it}|}{p^{b-1} - 1 + |p^{b-1} - p^b e^{it}|} \leqslant 1.
\end{aligned}$$

If $|x| \leqslant |y| \leqslant 1$ and $|z| \leqslant 1$, then by Lemmas 4.12 and 4.13 we get

$$\begin{aligned}
\frac{|\mathcal{A}(x) - \mathcal{A}(y)|}{|\mathcal{A}(x) - \mathcal{A}(z)|} &= \frac{|r^a - R^a e^{it}|}{(r + |r - Re^{it}|)^a - r^a} \\
&\leqslant \frac{1 + p^a}{(2 + p)^a - 1} \leqslant \frac{2}{3^a - 1}.
\end{aligned}$$

If $|x| \leqslant |y| \leqslant 1$ and $|z| \geqslant 1$, then it follows from Lemmas 4.12 and 4.13 and $|z|^b \geqslant |z|^a$ that

$$\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|} \leqslant \frac{|r^a - R^a e^{it}|}{(r+|r-Re^{it}|)^a - r^a} \leqslant \frac{1+p^a}{(2+p)^a - 1} \leqslant \frac{2}{3^a-1}.$$

The case $|x| \leqslant 1 \leqslant |y|$ and $r^{a-1} > R^{b-1}$. Then there holds

$$Q(x,y) \leqslant \frac{2}{3^a-1}.$$

First of all

$$\frac{|\mathcal{A}(x)-\mathcal{A}(y)|}{|\mathcal{A}(x)-\mathcal{A}(z)|} = \frac{|r^a - R^b e^{it}|}{(r+|r-Re^{it}|)^b - r^a} = \frac{|\alpha - e^{it}|}{(\beta+|\beta-e^{it}|)^b - \alpha}$$

where $\alpha = \frac{r^a}{R^b}$ and $\beta = \frac{r}{R}$. Take the continuous function $k(q) = \beta^q$, $a \leqslant q \leqslant 1$. Since

$$\beta = k(1) = \frac{r}{R} \leqslant \alpha = \frac{r^a}{R^b} \leqslant k(a) = \frac{r^a}{R^a}$$

it follows that there exists a constant c with $a \leqslant c \leqslant 1$ such that $k(c) = \beta^c = \alpha$. Then

$$\frac{|\alpha - Re^{it}|}{(\beta+|\beta-e^{it}|)^b - \alpha} = \frac{|\beta^c - e^{it}|}{(\beta+|\beta-e^{it}|)^b - \beta^c} \leqslant \frac{|\beta^c - e^{it}|}{(\beta+|\beta-e^{it}|)^c - \beta^c} \leqslant \frac{1+\beta^c}{(2+\beta)^c - \beta^c} \leqslant \frac{2}{3^c-1} \leqslant \frac{2}{3^a-1},$$

the second inequality follows from Lemma 4.12 and the third inequality follows from Lemma 4.13 by taking $p = 1/\beta$ and $c = d$.

Finally, let us show that $Q(a,b) \geqslant 2/(3^a-1)$. Suppose that $x \in \mathbb{R}^n \setminus \{0\}$ is such that $3|x| < 1$, i.e. $0 < |x| < 1/3$ and $y = -x$. Then $z = x(|x| + |x-y|)/|x| = 3x$ and

$$Q(x,-x) = \frac{2|x|^a}{(3|x|)^a - |x|^a} = \frac{2}{3^a-1},$$

and hence $C(a,b) \geqslant 2/(3^a-1)$. □

5. Conclusions

5.1. Proof of Theorem 1.7

For $|x|, |y| < 1$ we have

$$\alpha_p(x,y) = \big||x|^{p-1}x - |y|^{p-1}y\big| = \big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\big|.$$

Consider the case $|x| < 1 < |y|$. It is obvious that

$$\cos\theta \leqslant 1 < \frac{|x|^{-p}(|y|^{1/p} + |y|^p)}{2},$$

this is equivalent to

Table 1
Sample points with $K < \min\{B, D, M\}$.

x	y	B	D	M	K
$-2.00-2.65i$	$2.65-2.65i$	3.0496	143.4290	3.6030	2.6591
$2.25-0.75i$	$2.65+1.30i$	2.0438	38.9860	1.8236	1.5158
$1.35+0.50i$	$1.95-0.65i$	1.6107	14.8000	1.3571	1.2768
$1.10+2.30i$	$-2.40+2.10i$	2.6479	82.4142	2.9447	2.3646

Table 2
Sample points with $D < \min\{B, K, M\}$.

x	y	B	K	M	D
$0.80-0.50i$	$-1.80+1.45i$	3.6968	45.3884	3.2066	2.5495
$2.25-0.75i$	$0.00-0.05i$	15.5147	32.3855	2.7931	2.6174
$2.55+1.50i$	$-1.10+1.70i$	2.8148	76.9511	3.1879	2.7039
$-2.70+3.00i$	$1.50+0.60i$	4.2727	106.6320	3.6118	3.1104

Table 3
Sample points with $B < \min\{D, K, M\}$.

x	y	D	K	M	B
$-2.45-2.205i$	$-1.2+0.55i$	2.92	43.55	2.42	2.40
$-1.65+1.45i$	$2.15+2.75i$	3.01	92.27	3.22	2.83
$-0.2-3i$	$-0.4+0.2i$	5.21	34.64	2.77	2.53
$0.9-2.9i$	$-1.4+1.35i$	3.74	115.15	4.16	3.11

$$\begin{aligned}
&\cos\theta \leqslant 1 < \frac{(|y|^{1/p}-|y|^p)(|y|^{1/p}+|y|^p)}{2|x|^p(|y|^{1/p}-|y|^p)} \\
\Leftrightarrow\quad & 2|x|^p|y|^{1/p}\cos\theta - 2|x|^p|y|^p\cos\theta < |y|^{2/p}-|y|^{2p} \\
\Leftrightarrow\quad & |y|^{2p} - 2|x|^{p-1}|y|^{p-1}|x||y|\cos\theta < |y|^{2/p} - 2|x|^{p-1}|y|^{1/p-1}|x||y|\cos\theta \\
\Leftrightarrow\quad & \big||x|^{p-1}x\big|^2 + \big||y|^{p-1}y\big|^2 - 2|x|^{p-1}|y|^{p-1}xy < \big||x|^{p-1}x\big|^2 + \big||y|^{1/p-1}y\big|^2 - 2|x|^{p-1}|y|^{1/p-1}xy \\
\Leftrightarrow\quad & \big||x|^{p-1}x - |y|^{p-1}y\big|^2 < \big||x|^{p-1}x - |y|^{1/p-1}y\big|^2 = \big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\big|^2.
\end{aligned}$$

Consider now the case $1 < |x| < |y|$. Starting with the observation that the function $t \mapsto t^{1/p} - t^p$ is increasing for $t > 1$ when $p \in (0,1)$, we see that

$$\begin{aligned}
&\frac{|x|^{1/p}}{|x|^p}\left(\left(\frac{|y|}{|x|}\right)^{1/p} - 1\right) > \left(\left(\frac{|y|}{|x|}\right)^{p} - 1\right) \quad\Leftrightarrow\quad \left(|y|^{1/p}-|x|^{1/p}\right)^2 > \left(|y|^p - |x|^p\right)^2 \\
\Leftrightarrow\quad & |x|^{2/p} - |y|^{2p} + |y|^{2/p} - |x|^{2p} > 2|x|^{1/p}|y|^{1/p} - 2|x|^p|y|^p.
\end{aligned}$$

Now it is clear that

$$\begin{aligned}
&\cos\theta \leqslant 1 < \frac{|x|^{2/p} - |y|^{2p} + |y|^{2/p} - |x|^{2p}}{2|x|^{1/p}|y|^{1/p} - 2|x|^p|y|^p} \\
\Leftrightarrow\quad & |x|^{2p} + |y|^{2p} - 2|x|^p|y|^p\cos\theta < |x|^{2/p} + |y|^{2/p} - 2|x|^{1/p}|y|^{1/p}\cos\theta \\
\Leftrightarrow\quad & \big||x|^{p-1}x\big|^2 + \big||y|^{p-1}y\big|^2 - 2|x|^{p-1}|y|^{p-1}xy < \big||x|^{1/p-1}x\big|^2 + \big||y|^{1/p-1}y\big|^2 - 2|x|^{1/p-1}|y|^{1/p-1}xy \\
\Leftrightarrow\quad & \big||x|^{p-1}x\big|^2 + \big||y|^{p-1}y\big|^2 - 2|x|^{p-1}|y|^{p-1}xy < \big||x|^{p-1}x\big|^2 + \big||y|^{1/p-1}y\big|^2 - 2|x|^{p-1}|y|^{1/p-1}xy \\
\Leftrightarrow\quad & \big||x|^{p-1}x - |y|^{p-1}y\big|^2 < \big||x|^{1/p-1}x - |y|^{1/p-1}y\big|^2 = \big|\mathcal{A}_{p,1/p}(x) - \mathcal{A}_{p,1/p}(y)\big|^2.
\end{aligned}$$

5.2. Comparison of the bounds

In what follows, we use the symbols M, D, B, K for the bounds given by Theorems 1.2, 1.3, 1.4, 1.6, respectively. In the case of the complex plane, we will show by numerical examples that each of these four bounds can occur as minimal. To this end, for each of the symbols M, D, B, K, we give a table of four x, y pairs and the corresponding upper bound values associated with the four symbols M, D, B, K, such that the bound associated with the symbol in question is the least one. For the computation of the K bound it should be observed that in Theorem 1.7 we have the constraint $|x| \leqslant |y|$. If this is not the situation to begin with, we have swapped the points for computation. In Tables 1–4 the parameter $p = 0.5$. □

Table 4
Sample points with $M < \min\{B, D, K\}$.

x	y	B	D	K	M
$0.30 + 0.50i$	$-0.15 + 2.95i$	2.23	3.69	23.73	2.17
$0.95 + 1.85i$	$0.55 + 1.55i$	1.00	0.53	5.18	0.52
$1.60 - 0.25i$	$1.10 - 0.35i$	1.01	0.64	3.93	0.60
$-0.60 + 0.30i$	$-3.00 + 1.95i$	2.41	4.02	32.84	2.31

Table 5
Sample points with $M < \min\{(4.9), D\}$.

x	y	(4.9)	D	M
$2.25 + 2.45i$	$-0.01 + 2.95i$	0.27	0.27	0.24
$-2.60 + 0.40i$	$-0.70 - 0.60i$	1.23	3.30	1.19
$0.75 - 0.75i$	$-2.90 - 2.50i$	1.32	4.53	1.23
$2.90 + 1.90i$	$1.20 + 0.85i$	0.75	1.67	0.71

Table 6
Sample points with $(4.9) < \min\{D, M\}$.

x	y	D	M	(4.9)
$-2.60 - 1.05i$	$-1.35 - 1.40i$	0.70	0.65	0.56
$-0.45 - 1.05i$	$2.35 + 1.80i$	3.95	1.83	1.46
$-1.15 + 2.30i$	$2.70 + 0.65i$	0.99	2.12	0.96
$-0.10 + 1.25i$	$2.90 + 2.45i$	0.71	0.94	0.60

Table 7
Sample points with $D < \min\{(4.9), M\}$.

x	y	(4.9)	M	D
$1.35 + 2.95i$	$-1.35 + 2.90i$	0.59	1.07	0.43
$-0.80 + 2.75i$	$-1.85 + 2.40i$	0.38	0.49	0.25
$2.65 + 2.20i$	$-2.45 + 2.40i$	0.49	0.64	0.49
$1.20 - 0.70i$	$1.30 + 0.70i$	1.05	1.96	0.91

In conclusion, Tables 1–4 demonstrate that each of the above four bounds is sometimes smaller than the minimum of the other three bounds. Some further results, in addition to Theorems 1.2, 1.3, 1.4, 1.6 can be found in the papers [10,6]. The tables were compiled with the help of the Mathematica software package.

In Tables 5–7 we compare (4.9), M and D, for $x, y \in \mathbb{R}^n \setminus \mathbb{B}^n$, $p = -0.6$.

Acknowledgments

The first author is indebted to the Graduate School of Mathematical Analysis and Its Applications for support. He also wishes to acknowledge the expert help of Dr. H. Ruskeepää in the use of the Mathematical® software.

The fourth author was, in part, supported by the Academy of Finland, Project 2600066611.

References

[1] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Dimension-free quasiconformal distortion in n-space, Trans. Amer. Math. Soc. 297 (1986) 687–706.
[2] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, Conformal Invariants, Inequalities and Quasiconformal Maps, J. Wiley, 1997, 505 pp.
[3] B. A. Bhayo, M. Vuorinen, On Mori's theorem for quasiconformal maps in the n-space, Trans. Amer. Math. Soc., in press, arXiv:0906.2853 [math.CV].
[4] J. Byström, Sharp constants for some inequalities connected to the p-Laplace operator, JIPAM. J. Inequal. Pure Appl. Math. 6 (2) (2005), Article 56, 8 pp. (electronic).
[5] F. Dadipour, M. Fujii, M.S. Moslehian, Dunkl–Williams inequality for operators associated with p-angular distance, arXiv:1006.1941 [math.OA].
[6] S.S. Dragomir, Inequalities for the p-angular distance in normed linear spaces (English summary), Math. Inequal. Appl. 12 (2) (2009) 391–401.
[7] R. Klén, V. Manojlović, M. Vuorinen, Distortion of normalized quasiconformal mappings, arXiv:0808.1219 [math.CV].
[8] R. Kühnau, Handbook of Complex Analysis: Geometric Function Theory, vols. 1–2, North-Holland, Amsterdam, 2002/2005.
[9] O. Lehto, K.I. Virtanen, Quasiconformal Mappings in the Plane, translated from German by K.W. Lucas, second ed., Grundlehren Math. Wiss., Band 126, Springer-Verlag, New York/Heidelberg, 1973, viii+258 pp.
[10] L. Maligranda, Simple norm inequalities, Amer. Math. Monthly 113 (3) (2006) 256–260.
[11] D.S. Mitrinović, J.E. Pecarić, A.M. Fink, Classical and New Inequalities in Analysis, Math. Appl. (East European Series), vol. 61, Kluwer Academic Publishers Group, Dordrecht, 1993, xviii+740 pp.
[12] J. Väisälä, Lectures on n-Dimensional Quasiconformal Mappings, Lecture Notes in Math., vol. 229, Springer-Verlag, Berlin, 1971.
[13] M. Vuorinen, Conformal Geometry and Quasiregular Mappings, Lecture Notes in Math., vol. 1319, Springer, Berlin, 1988.