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When is the Isbell topology a group topology? ☆

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ABSTRACT

Conditions on a topological space X under which the space $C(X,\mathbb{R})$ of continuous real-valued maps with the Isbell topology κ is a topological group (topological vector space) are investigated. It is proved that the addition is jointly continuous at the zero function in $C_\kappa(X,\mathbb{R})$ if and only if X is infraconsonant. This property is (formally) weaker than consonance, which implies that the Isbell and the compact-open topologies coincide. It is shown the translations are continuous in $C_\kappa(X,\mathbb{R})$ if and only if the Isbell topology coincides with the fine Isbell topology. It is proved that these topologies coincide if X is prime (that is, with at most one non-isolated point), but do not even for some sums of two consonant prime spaces.

1. Introduction

In [14] and [15] Isbell introduced and studied a topology on the space $C(X,Z)$ of continuous functions from a topological space X to a topological space Z, defined in terms of (what is now called) *compact families* of open subsets of X and open subsets of Z. The *Isbell topology* is finer than the *compact-open* topology and coarser than the *natural topology* (that is, the topological reflection of the *natural convergence*, most often called *continuous convergence*). Recently Jordan introduced in [17] several intermediate topologies, finer than the Isbell and coarser than the natural topology, that turn out to be instrumental in understanding function spaces. One of them is the so-called *fine Isbell topology*.

The Isbell topology and the natural topology coincide on $C(X,\$)$ (that can be identified with the set of closed subsets of X) and on the homeomorphic space $C(X,\$^*)$ (of open subsets of X) where it is homeomorphic to the *Scott topology*.[1] The open sets for the Scott topology on $C(X,\$)$ are precisely the compact families of open subsets of X. A topological space X is called *consonant* [5] if these topologies on $C(X,\$^*)$ coincide with the compact-open topology.[2]

It is known that if X is consonant, then the Isbell topology on $C(X,\mathbb{R})$ coincides with the compact-open topology. We prove that the converse is true for completely regular spaces, partially answering [21, Problem 62]. Answering [21, Problem 61] positively, we also show that the Isbell topology on $C(X,Z)$ is completely regular whenever Z is.

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[1] $\$:= \{\emptyset, \{1\}, \{0,1\}\}$ and $\$^* := \{\emptyset, \{0\}, \{0,1\}\}$ are two homeomorphic representations of the *Sierpiński topology* on $\{0,1\}$.

[2] In other words, if each compact family on X is compactly generated.

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There are consonant examples (e.g., [10, Example 5.12], [11]) of spaces, for which the Isbell topology is strictly coarser than the natural topology, but to our knowledge there is so far no characterization of X for which the Isbell topology and the natural topology coincide on $C(X,\mathbb{R})$.

The natural convergence is always a group convergence, in particular, it is invariant under translations,[3] hence the natural topology is also invariant under translations as the topological reflection of the natural convergence (see [7]), but need not be a group topology, e.g. [16]. In [20], B. Papadopoulos proposes a sufficient condition on a topological space X for the Isbell topological space $C_\kappa(X,\mathbb{R})$ to be a vector space topology. However, it seems that no example has been known so far of a space X, for which $C_\kappa(X,\mathbb{R})$ is *not* a vector space topology.

In this note, we investigate under what conditions the Isbell topology is a group topology, equivalently a vector space topology, because we prove that multiplication by scalars is jointly continuous for the Isbell topology.

In general, a topology on an abelian group is a group topology if and only if the translations and the inversion are continuous, and if the group operation is (jointly) continuous at the neutral group element. As the inversion is a homeomorphism for the Isbell topology on $C(X,\mathbb{R})$, we are confronted with two quests about the Isbell topology on $C(X,\mathbb{R})$:

(1) invariance by translations, and
(2) continuity of the addition at the zero function $\bar{0}$, that is, the property

$$\mathcal{N}_\kappa(\bar{0})+\mathcal{N}_\kappa(\bar{0})\geqslant\mathcal{N}_\kappa(\bar{0}). \tag{1.1}$$

More specifically, we show that the space $C_\kappa(X,\mathbb{R})$ of real-valued continuous functions on X endowed with the Isbell topology is invariant under translations if and only if the Isbell and fine Isbell topologies coincide. In [17], Jordan provides an example of a topological space X, for which the Isbell and fine Isbell topologies on $C(X,\mathbb{R})$ do *not* coincide. This shows that there exists X for which $C_\kappa(X,\mathbb{R})$ is not invariant by translations.

We call a space *infraconsonant* if every compact family $\mathcal{A}$ contains another compact family $\mathcal{B}$ such that every pairwise intersection of elements of $\mathcal{B}$ belongs to $\mathcal{A}$, and we show that (1.1) holds if and only if X is infraconsonant. Of course, every consonant space is infraconsonant. There are infraconsonant and non-consonant spaces, but we do not know yet of a completely regular one.

Problem 1.1. Does there exist a completely regular infraconsonant space that is not consonant?

We call a topological space *prime* if it has at most one non-isolated point. We show that $C_\kappa(X,\mathbb{R})$ is invariant under translations if X is a prime space and that there are prime spaces that are not infraconsonant. In other words, $C_\kappa(X,\mathbb{R})$ may be translation-invariant without satisfying (1.1). We also show that $C_\kappa(X,\mathbb{R})$ may fail to have either of these properties. However, we do not know if it can satisfy (1.1) without being invariant under translations. In other words:

Problem 1.2. Does there exist a completely regular infraconsonant space X such that $C_\kappa(X,\mathbb{R})$ has discontinuous translations?

A positive solution to this problem would also provide a positive answer to Problem 1.1, because $C_\kappa(X,\mathbb{R})$ is a topological group if X is consonant.[4] We do not know if the converse is true:

Problem 1.3. Does there exist a non-consonant completely regular space X such that $C_\kappa(X,\mathbb{R})$ is a topological group?

In view of our result, a prime positive solution to Problem 1.1 would also provide a positive answer to Problem 1.3.

2. Generalities

If $\mathcal{A}$ is a family of subsets of a topological space X then $\mathcal{O}_X(\mathcal{A})$ denotes the family of open subsets of X containing an element of $\mathcal{A}$. In particular, if $A\subset X$ then $\mathcal{O}_X(A)$ denotes the family of open subsets of X containing A. A family $\mathcal{A}=\mathcal{O}_X(\mathcal{A})$ is *compact* if whenever $\mathcal{P}\subset\mathcal{O}_X$ and $\bigcup\mathcal{P}\in\mathcal{A}$ then there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup\mathcal{P}_0\in\mathcal{A}$. Of course, for each compact subset K of X, the family $\mathcal{O}_X(K)$ is compact. The following proposition extends the fact that continuous functions are bounded on compact sets.

Proposition 2.1. *If $\mathcal{A}$ is a compact family on X and $f\in C(X,\mathbb{R})$, then there is $A\in\mathcal{A}$ such that $f(A)$ is bounded.*

Proof. As $\bigcup_{n<\omega}f^{-}(\{r\colon\ |r|<n\})=X\in\mathcal{A}$ and f is continuous, there exists $n<\omega$ such that $f^{-}(\{r\colon\ |r|<n\})\in\mathcal{A}$ by the compactness of $\mathcal{A}$. $\square$

[3] Actually, the natural convergence is a convergence vector space.
[4] Problem 1.2 has been recently solved in the negative in [6].

We denote by $\kappa(X)$ the collection of compact families on X. Seen as a family of subsets of $\mathcal{O}_X$ (the set of open subsets of X), $\kappa(X)$ is the family of open subsets of the *Scott topology*; hence every union of compact families is compact, in particular $\bigcup_{K\in\mathcal{K}}\mathcal{O}_X(K)$ is compact if $\mathcal{K}$ is a family of compact subsets of X. A topological space is called *consonant* if every compact family $\mathcal{A}$ is *compactly generated*, that is, there is a family $\mathcal{K}$ of compact sets such that $\mathcal{A}=\bigcup_{K\in\mathcal{K}}\mathcal{O}_X(K)$. Similarly, $k(X):=\{\mathcal{O}(K)\colon\ K\subseteq X \text{ compact}\}$ is a basis for a topology on $\mathcal{O}_X$, and a space X is consonant if and only if this topology coincides with the Scott topology. Bouziad calls *weakly consonant* [1] a space X in which for every compact family $\mathcal{A}$ there is a compact subset K of X such that $\mathcal{O}_X(K)\subseteq\mathcal{A}$.

Lemma 2.2. *([17, Lemma 36]) Consonance and weak consonance are equivalent among regular topological spaces.*

The *Isbell topology* on $C(X,Z)$ can be defined by the following subbase of open sets

$$[\mathcal{A},U]:=\big\{f\in C(X,Z)\colon\ \exists A\in\mathcal{A},\ f(A)\subseteq U\big\},$$

where $\mathcal{A}$ ranges over $\kappa(X)$ and U ranges over open subsets of Z. We write $C_\kappa(X,Z)$ for the set $C(X,Z)$ endowed with the Isbell topology, and $C_k(X,Z)$ if it is endowed with the compact-open topology. A space X is called Z*-consonant* if $C_\kappa(X,Z)=C_k(X,Z)$. Note that if \$ denotes the Sierpiński space, then \$-consonant means consonant. Moreover, the following is immediate.

Proposition 2.3. *X is consonant if and only if it is Z-consonant for every Z. In particular, if X is consonant, then $C_\kappa(X,\mathbb{R})$ is a topological vector space.*

[21, Problem 62] asks for what spaces Z (other than \$) does Z-consonance imply consonance. We have the following partial answer, which refines [4, Theorem 4.4] which was announced without proof and proved in [19, Theorem 4.17].[5]

The *grill of* a family $\mathcal{A}$ of subsets of X is the family $\mathcal{A}^\#:=\{B\subseteq X\colon\ \forall A\in\mathcal{A},\ A\cap B\neq\emptyset\}$. Note that if $\mathcal{A}=\mathcal{O}(\mathcal{A})$, then

$$A\in\mathcal{A}\quad\Longleftrightarrow\quad A^c\notin\mathcal{A}^\#.$$

Proposition 2.4. *If X is completely regular and $\mathbb{R}$-consonant, then it is consonant.*

Proof. If X is $\mathbb{R}$-consonant then in particular $\mathcal{N}_k(\bar{0})\geqslant\mathcal{N}_\kappa(\bar{0})$ where $\bar{0}$ denotes the zero function. Hence for every $\mathcal{A}\in\kappa(X)$ there exist a compact subset K of X and $r>0$ such that $[K,B_r]\subseteq[\mathcal{A},B_{\frac{1}{2}}]$ where $B_r:=(-r,r)$. In view of Lemma 2.2, it is sufficient to show that $\mathcal{O}(K)\subseteq\mathcal{A}$. Assume on the contrary that there is an open set U such that $K\subseteq U$ and $U\notin\mathcal{A}$. Then the closed set $F:=X\setminus U$ is disjoint from K and $F\in\mathcal{A}^\#$. As X is completely regular, there is $h\in C(X,\mathbb{R})$ such that $h(K)=\{0\}$ and $h(F)=\{1\}$. Then $h\in[K,B_r]$ but $h\notin[\mathcal{A},B_{\frac{1}{2}}]$ because $1\in h(A)$ for every $A\in\mathcal{A}$; a contradiction. □

In the proof above, we used the well-known fact that *if A is a compact subset of a completely regular space X and F is a closed subset of X such that $A\cap F=\emptyset$, then there exists $h\in C(X,[0,1])$ such that $h(A)=\{0\}$ and $h(F)=\{1\}$*. We extend this fact to a closed set and a compact family.

Lemma 2.5. *If $\mathcal{A}=\mathcal{O}(\mathcal{A})$ is a compact family of subsets of a completely regular topological space X, and F is a closed subset of X with $F^c\in\mathcal{A}$, then there are $A\in\mathcal{A}$ and $h\in C(X,[0,1])$ such that $h(A)=\{0\}$ and $h(F)=\{1\}$.*

Proof. By complete regularity, for every $x\notin F$, there are an open neighborhood O_x of x and $h_x\in C(X,[0,1])$ such that $h_x(O_x)=\{0\}$ and $h_x(F)=\{1\}$. Therefore $F^c=\bigcup_{x\notin F}O_x\in\mathcal{A}$, so that by the compactness of $\mathcal{A}$ there are $n<\omega$ and $x_1,\dots,x_n\notin F$ such that $A=\bigcup_{1\leqslant i\leqslant n}O_{x_i}\in\mathcal{A}$. The continuous function $\min_{1\leqslant i\leqslant n}h_{x_i}$ is 0 on A and 1 on F. □

Consequently, for each $O\in\mathcal{A}$ there exists a continuous function h valued in $[0,1]$ such that $O\supset\{x\colon\ h(x)<\frac{1}{2}\}\supset\{x\colon\ h(x)=0\}\supset\operatorname{int}\{x\colon\ h(x)=0\}\in\mathcal{A}$, that is,

Corollary 2.6. *An* (*openly isotone*) *compact family of subsets of a completely regular topological space has a base of co-zero sets and a base of the interiors of zero sets.*

Papadopoulos says that a space X has *property* (A^*) if whenever $\mathcal{A}\in\kappa(X)$ and A_1 and A_2 are open subsets of X such that $A_1\cup A_2\in\mathcal{A}$, there exist filters $\mathcal{F}_i$ such that $A_i\in\mathcal{F}_i$, $i=1,2$ such that $\mathcal{O}_X(\mathcal{F}_i)\in\kappa(X)$ and $\mathcal{O}(\mathcal{F}_1)\cap\mathcal{O}(\mathcal{F}_2)\subseteq\mathcal{A}$. The main result of [20] is that property (A^*) is sufficient for the Isbell topology on $C(X,\mathbb{R})$ to be a vector space topology. If X is regular, this result follows immediately from Proposition 2.3 because of the following:

[5] Note that the notion of $\mathbb{R}$-consonance introduced in [19] (coincidence of the natural and compact-open topologies on $C(X,\mathbb{R})$) is stronger than our notion and should not be confused.

Proposition 2.7. *Let X be a regular topological space. Then X is consonant if and only if X has property (A^*).*

Proof. Assume that X is consonant and that $A_1 \cup A_2 \in \mathcal{A}$ where $\mathcal{A} \in \kappa(X)$. Because X is consonant, there is a compact set $K \subseteq A_1 \cup A_2$ such that $\mathcal{O}(K) \subseteq \mathcal{A}$. By regularity and compactness, there are finitely many closed sets C_i such that each C_i is a subset of either A_1 or A_2 and $K \subseteq \bigcup_{i=1}^{i=n} C_i$. Therefore, there exist compact subsets K_1 of A_1 and K_2 of A_2 such that $K = K_1 \cup K_2$, so that $\mathcal{O}(K_1)$ and $\mathcal{O}(K_2)$ are the sought compact filters. Conversely, if X satisfies (A^*) then for every $A \in \mathcal{A}$ there is a compact filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{O}(\mathcal{F}) \subseteq \mathcal{A}$. Because $\mathcal{F}$ is a compact filter in a regular space, $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\operatorname{adh}\mathcal{F})$ and $\operatorname{adh}\mathcal{F}$ is compact (e.g. [5, Proposition 2.2]). Therefore, $\mathcal{A}$ is compactly generated and X is consonant. □

Lemma 2.8. *([3]) If $\mathcal{A} \in \kappa(X)$ and C is a closed subset of X such that $C \in \mathcal{A}^\#$ then*

$$\mathcal{A} \vee C := \mathcal{O}\big(\{A \cap C\colon\ A \in \mathcal{A}\}\big)$$

is a compact family on X.

Lemma 2.9. *If $\mathcal{A} \in \kappa(X)$ and $A_0 \in \mathcal{A}$ then*

$$\mathcal{A} \downarrow A_0 := \mathcal{O}\big(\{A \in \mathcal{A}\colon\ A \subseteq A_0\}\big)$$

is a compact family on X.

Proof. If $\bigcup_{i\in I} O_i \in \mathcal{A} \downarrow A_0$ then there is $A \in \mathcal{A}$ such that $A \subseteq A_0$ and $A \subseteq \bigcup_{i\in I} O_i$ so that $A \subseteq \bigcup_{i\in I}(O_i \cap A_0)$. By compactness of $\mathcal{A}$ there is a finite subset F of I such that $\bigcup_{i\in F}(O_i \cap A_0) \in \mathcal{A}$. But $\bigcup_{i\in F}(O_i \cap A_0) \subseteq A_0$ so that $\bigcup_{i\in F}(O_i \cap A_0) \in \mathcal{A} \downarrow A_0$ and $\bigcup_{i\in F} O_i \in \mathcal{A} \downarrow A_0$. □

The following theorem answers [21, Problem 61].

Theorem 2.10. *If Z is completely regular, then $C_\kappa(X, Z)$ is completely regular.*

Proof. Let $f \in [\mathcal{A}, O]$ where $\mathcal{A} \in \kappa(X)$ and O is Z-open. As $\mathcal{A}$ is compact and f continuous, $\mathcal{O}_Z(f(\mathcal{A}))$ is compact, and since Z is completely regular, by Lemma 2.5, there are $A \in \mathcal{A}$ and $h \in C(Z, [0, 1])$ such that $h(f(A)) = \{0\}$ and $h(Z\backslash O) = \{1\}$. Define

$$F(g) := \inf_{A\in\mathcal{A}} \sup_{x\in A} h\big(g(x)\big) = \sup_{H\in\mathcal{A}^\#} \inf_{x\in H} h\big(g(x)\big)$$

for each $g \in C(X, Z)$. Then $F(f) = 0$ and $F(g) = 1$ for each $g \notin [\mathcal{A}, O]$. Moreover, $F : C_\kappa(X, Z) \to [0, 1]$ is continuous. To see that $F^-([0, r))$ is open for each $r \in [0, 1]$, notice that $F(g) < r$ if and only if there is $A_r \in \mathcal{A}$ such that $g(A_r) \subset [0, r)$, that is, if and only if $g \in [\mathcal{A}, h^-([0, r))]$. On the other hand, if $0 \leqslant s < 1$ and $s < F(g)$, then, by the second equality, there exist $s < t < F(g)$ and a closed set $H \in \mathcal{A}^\#$ such that $t \leqslant h(g(x))$ for each $x \in H$, thus $g(H) \subset h^-(s, 1]$. By Lemma 2.8, $\mathcal{A} \vee H$ is compact, and if an open set includes H then it belongs to $\mathcal{A} \vee H$, in particular $g^-h^-(s, 1] \in \mathcal{A} \vee H$, that is, g belongs to the open set $[\mathcal{A} \vee H, h^-(s, 1]]$. If now $b \in [\mathcal{A} \vee H, h^-(s, 1]]$, then there is $A \in \mathcal{A}$ such that $h(b(A \cap K)) \subset (s, 1]$, hence

$$s < \sup_{A\in\mathcal{A}} \inf_{x\in A\cap H} h\big(b(x)\big) \leqslant \inf_{A\in\mathcal{A}} \sup_{x\in A\cap H} h\big(b(x)\big) \leqslant \inf_{A\in\mathcal{A}} \sup_{x\in A} h\big(b(x)\big) = F(b). \qquad \square$$

As $[\mathcal{A}, -U] = -[\mathcal{A}, U]$ for every $U \subset \mathbb{R}$ and each compact family $\mathcal{A}$, the inversion is a homeomorphism in $C_\kappa(X, \mathbb{R})$. More generally,

Proposition 2.11. *Multiplication by scalars is jointly continuous for the Isbell topology.*

Proof. Let $f \in C(X, \mathbb{R})$ and $r \in \mathbb{R}$ be such that $rf \in [\mathcal{A}, O]$, where $\mathcal{A}$ is a compact family on X and O is an open subset of $\mathbb{R}$. If $r = 0$ then it is enough to consider $O = B(0, \varepsilon)$ with $\varepsilon > 0$. By Proposition 2.1 there exist $A_0 \in \mathcal{A}$ and $R > 0$ such that $f(A_0) \subset B(0, R)$ and thus $f(A) \subset B(0, R)$ for each basic element A of $\mathcal{A}_0 := \mathcal{A} \downarrow A_0$. Therefore $f \in [\mathcal{A}_0, B(0, R)]$ and $B(0, \frac{\varepsilon}{R})[\mathcal{A}_0, B(0, R)] \subset [\mathcal{A}, O]$. Let $|r| > 0$. Since $\mathcal{O}((rf)(\mathcal{A}))$ is a compact family of the consonant space $\mathbb{R}$, there exists a compact subset K of O such that $\mathcal{O}_\mathbb{R}(K) \subset \mathcal{O}((rf)(\mathcal{A}))$, hence there exist $A_0 \in \mathcal{A}$ and $\varepsilon > 0$ such that $(rf)(A_0) \subset B(K, \varepsilon) \subset B(K, 2\varepsilon) \subset O$. If $\mathcal{A}_0 := \mathcal{A} \downarrow A_0$, then $f(A) \subset \frac{1}{r} B(K, \varepsilon)$ for a base of elements A of $\mathcal{A}_0$, hence $f \in [\mathcal{A}_0, \frac{1}{r} B(K, \varepsilon)]$. On the other hand, there is $\delta > 0$ such that $B(1, \frac{\delta}{|r|})B(K, \varepsilon) \subset B(K, 2\varepsilon)$ and thus $B(r, \delta)[\mathcal{A}_0, \frac{1}{r} B(K, \varepsilon)] \subset O$. □

Corollary 2.12. *If $C_\kappa(X, \mathbb{R})$ is a topological group then it is a topological vector space.*

3. Structure of $C_\kappa(X,\mathbb{R})$ at the zero function

As usual, if A and B are subsets of a group, $A+B := \{a+b\colon\ a\in A, b\in B\}$ and if $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets, $\mathcal{A}+\mathcal{B} := \{A+B\colon\ A\in\mathcal{A},\ B\in\mathcal{B}\}$.

As we have mentioned, a topology on an abelian group is a group topology if and only if translations are continuous and $\mathcal{N}(0)+\mathcal{N}(0)\geqslant\mathcal{N}(0)$. In this subsection, we investigate the latter property, that is,

$$\mathcal{N}_\kappa(\bar{0})+\mathcal{N}_\kappa(\bar{0})\geqslant\mathcal{N}_\kappa(\bar{0}), \tag{3.1}$$

for the space $C_\kappa(X,\mathbb{R})$. If (p_n) is a decreasing sequence of positive numbers that tends to zero, then

$$\left[\bigcap_{i=1}^{n}\mathcal{A}_i, \left(-\max_{i=1}^{n} p_i, \max_{i=1}^{n} p_i\right)\right]\subseteq\bigcap_{i=1}^{n}\left[\mathcal{A}_i, (-p_i, p_i)\right],$$

and thus $\mathcal{N}_\kappa(\bar{0})$ has a filter base of the form

$$\left\{\left[\mathcal{A}, (-p_n, p_n)\right]\colon\ \mathcal{A}\in\kappa(X),\ n\in\mathbb{N}\right\}, \tag{3.2}$$

because a finite intersection of compact families is compact.

We call a topological space X *infraconsonant* if for every compact family $\mathcal{A}$ on X there is a compact family $\mathcal{B}$ such that $\mathcal{B}\vee\mathcal{B} := \{B\cap C\colon\ B\in\mathcal{B},\ C\in\mathcal{B}\}$ is a (not necessarily compact) subfamily of $\mathcal{A}$. Note that if X is consonant then every compact family includes a compact filter of the form $\mathcal{O}(K)$ for a compact set K. Taking $\mathcal{B}=\mathcal{O}(K)$ gives infraconsonance, so that every consonant space is infraconsonant.

Theorem 3.1. *Let $(G,+)$ be an abelian topological group. If X is infraconsonant, then the addition is continuous at $\bar{0}$ in $C_\kappa(X,G)$. Moreover if X is completely regular, then the addition is continuous at $\bar{0}$ in $C_\kappa(X,\mathbb{R})$ if and only if X is infraconsonant.*

Proof. Assume that X is infraconsonant. Let $\mathcal{A}\in\kappa(X)$ and $V\in\mathcal{N}_G(0)$. By infraconsonance, there exists a compact subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}\vee\mathcal{B}\subseteq\mathcal{A}$. If $W\in\mathcal{N}_G(0)$ such that $W+W\subseteq V$, then $[\mathcal{B},W]+[\mathcal{B},W]\subseteq[\mathcal{A},V]$, which proves (3.1).

Conversely, assume that X is not infraconsonant. Let $\mathcal{A}$ be a compact family witnessing the definition of non-infraconsonance. Note that $\mathcal{B}\vee\mathcal{C}\nsubseteq\mathcal{A}$ for every pair of compact families $\mathcal{B}$ and $\mathcal{C}$ for otherwise $\mathcal{D}=\mathcal{B}\cap\mathcal{C}$ would be a compact subfamily of $\mathcal{A}$ such that $\mathcal{D}\vee\mathcal{D}\subseteq\mathcal{A}$. Let $V=(-\frac{1}{2},\frac{1}{2})$. We claim that for any pair $(\mathcal{B},\mathcal{C})$ of compact families and any pair (U,W) of $\mathbb{R}$-neighborhood of 0, $[\mathcal{B},U]+[\mathcal{C},W]\nsubseteq[\mathcal{A},V]$. Indeed, there exist $B\in\mathcal{B}$ and $C\in\mathcal{C}$ such that $B\cap C\notin\mathcal{A}$. Then $B^c\cup C^c\in\mathcal{A}^{\#}$. Moreover, $B^c\notin\mathcal{B}^{\#}$ so that by Lemma 2.5, there exist $B_1\in\mathcal{B}$ and $f\in C(X,\mathbb{R})$ such that $f(B_1)=\{0\}$ and $f(B^c)=\{1\}$. Similarly, $C^c\notin\mathcal{C}$ so that there exist $C_1\in\mathcal{C}$ and $g\in C(X,\mathbb{R})$ such that $g(C_1)=\{0\}$ and $g(C^c)=\{1\}$. Then $f+g\in[\mathcal{B},U]+[\mathcal{C},W]$ but $1\in(f+g)(A)$ for all $A\in\mathcal{A}$ so that $f+g\notin[\mathcal{A},V]$. □

Complete regularity cannot be relaxed (to regularity) in Theorem 3.1.

Example 3.2. There exist regular non-infraconsonant spaces X, for which the addition is jointly continuous at $\bar{0}$ in $C_\kappa(X,\mathbb{R})$. In [12] Herrlich builds a regular space, on which each continuous function is constant. To this purpose, for each regular space Y he constructs a regular space $H(Y)$ such that Y is closed in $H(Y)$, and each $f\in C(H(Y),\mathbb{R})$ is constant on Y. Define $H^0(Y):=Y$, $H^{n+1}(Y):=H(H^n(Y))$ and $X:=\bigcup_{n<\omega}H^n(Y)$ with the finest topology for which all the injections are continuous. Then each continuous (real-valued) function on X is constant. Moreover it can be shown that X is regular. This fact is stated in [12] in case where Y is a singleton, but is true for an arbitrary regular space Y. Let Y be a regular non-infraconsonant space, for instance the space from Example 3.7. As all continuous functions on X are constant, the continuity of the (joint) addition on $C_\kappa(X,\mathbb{R})$ follows from the continuity of the addition on $\mathbb{R}$. If X were infraconsonant, then its closed subset Y would be infraconsonant, in contradiction with the assumption.

As we have mentioned, $C_\kappa(X,\$^*)$ is the lattice of open subsets of X endowed with the Scott topology, in which open sets are exactly the compact families of open subsets of X. Dually, $C_\kappa(X,\$)$ is the set of closed subsets of X endowed with the *upper Kuratowski topology*, in which $\mathcal{F}$ is open if the family $\mathcal{F}_c=\{X\setminus F\colon\ F\in\mathcal{F}\}$ is compact. The following was prompted by a conversation with Ahmed Bouziad (University of Rouen) in June 2008 (in Erice), who asked us if infraconsonance was related to the joint continuity of the union operation on $C_\kappa(X,\$)$.

Lemma 3.3. *If X is regular and infraconsonant, then for every $\mathcal{A}\in\kappa(X)$ and every $A\in\mathcal{A}$, there is $\mathcal{C}\in\kappa(X)$ such that $A\in\mathcal{C}$ and $\mathcal{C}\vee\mathcal{C}\subseteq\mathcal{A}$.*

Proof. Assume that X is infraconsonant and regular. If $\mathcal{A}$ is a compact family on X, then for each element A of $\mathcal{A}$ there is $A_0\in\mathcal{A}$ such that $\operatorname{cl}A_0\subset A$. The family $\mathcal{A}_1=\mathcal{A}\downarrow A_0$ is compact by Lemma 2.9 so that there is a compact family $\mathcal{B}$ such that $\mathcal{B}\vee\mathcal{B}\subseteq\mathcal{A}_1$. For each B_1 and B_2 in $\mathcal{B}$, there is $B\in\mathcal{A}$, $B\subseteq A_0$ such that $B\subseteq B_1\cap B_2$. Therefore the family $\mathcal{B}_1=\mathcal{B}\vee\operatorname{cl}A_0$ contains A, satisfies $\mathcal{B}_1\vee\mathcal{B}_1\subseteq\mathcal{A}_1\subseteq\mathcal{A}$, and is compact by Lemma 2.8. □

We shall consider binary maps: the intersection $\cap$ and the union $\cup$, defined by $\bigcap(A,B) := A\cap B$ and $\bigcup(A,B) := A\cup B$.

Proposition 3.4. *Let X be a regular topological space. The following are equivalent*:

(1) *X is infraconsonant*;
(2) *The intersection $\cap : C_\kappa(X,\$^*)\times C_\kappa(X,\$^*)\to C_\kappa(X,\$^*)$ is (jointly) continuous for the Scott topology*;
(3) *The union $\cup : C_\kappa(X,\$)\times C_\kappa(X,\$)\to C_\kappa(X,\$)$ is (jointly) continuous for the upper Kuratowski topology.*

Proof. The equivalence between (2) and (3) is immediate, because these topologies are isomorphic by complementation. Assume X is infraconsonant and let U and V be two open subsets of X. Let $\mathcal{A}$ be a Scott open neighborhood of $\bigcap(U,V)$, i.e., a compact family containing $U\cap V$. By Lemma 3.3, there is a compact family $\mathcal{C}$ containing $U\cap V$ such that

$$\bigcap(\mathcal{C},\mathcal{C}) = \mathcal{C}\vee\mathcal{C}\subseteq\mathcal{A}.$$

Note that $\mathcal{C}$ is a common Scott neighborhood of U and V so that $\cap$ is continuous.

Conversely, assume that $\cap$ is continuous and consider a compact family $\mathcal{A}$. Since $\bigcap^{-1}(\mathcal{A})$ has non-empty interior there are compact families $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{B}\times\mathcal{C}\subseteq\bigcap^{-1}(\mathcal{A})$. The compact family $\mathcal{D}=\mathcal{B}\cap\mathcal{C}$ then satisfies $\mathcal{D}\vee\mathcal{D}\subseteq\mathcal{A}$ so that X is infraconsonant. □

Note that the implication (2) $\Longrightarrow$ (1) does not use regularity. It is well known (e.g. [22]) that the natural convergence on $C(X,\$)$ is topological if and only if X is core-compact. A topological space X is *core-compact* if for every $x\in X$ and every $U\in\mathcal{O}(x)$ there is $V\in\mathcal{O}(x)$ that is relatively compact in U.

Therefore, if X is core-compact then the Isbell topology and the natural convergence coincide on $C(X,\$)$, which is easily seen to make the map $\cap$ jointly continuous. In view of Proposition 3.4, X is then infraconsonant. Moreover X is locally compact if and only if the natural convergence coincides with the compact-open topology on $C(X,\$)$ (e.g. [22, Proposition 2.19]), that is, every compact family is compactly generated. Therefore, if X is core-compact but not locally compact, then X is infraconsonant but not consonant. Such a space is constructed in [13, Section 7].

Corollary 3.5. *There exists a (non-Hausdorff) infraconsonant space that is not consonant.*

We will now exhibit a class of prime (hence completely regular) non-infraconsonant spaces. Recall that if $(\mathcal{F}_n)_{n<\omega}$ is a sequence of filters, then the *contour* $\mathcal{F}$ is defined by $\mathcal{F} := \bigcup_{p<\omega}\bigcap_{n\geqslant p}\mathcal{F}_n$ [8]. A prime space X (with only non-isolated point ∞) is a *contour space* if there exists a family $\{X_n:\ n\in\omega\}$ of disjoint infinite subsets of $X\setminus\{\infty\}$ such that $X=\bigcup_{n<\omega}X_n\cup\{\infty\}$ and

$$\mathcal{N}(\infty)=\{\infty\}\wedge\bigcup_{p<\omega}\bigcap_{n\geqslant p}\mathcal{F}_n,$$

where each $\mathcal{F}_n$ is a free filter on X_n. Notice that the sets $\{\infty\}\cup\bigcup_{n\geqslant p}F_n$, where $F_n\in\mathcal{F}_n$ and $p<\omega$ form a filter base of $\mathcal{N}(\infty)$. Therefore

$$\forall_{n<\omega}\quad X_n\notin\mathcal{N}(\infty)^{\#}, \tag{3.3}$$

$$\forall_{V\in\mathcal{N}(\infty)}\quad \bigl|\{n\in\omega:\ X_n\cap V=\emptyset\}\bigr|<\omega. \tag{3.4}$$

Compact sets in a contour space are finite. In fact, if K is compact, then $K_n := K\cap X_n$ is finite, because $\mathcal{F}_n$ is finer than the cofinite filter of X_n for each $n<\omega$; as $\mathcal{N}(\infty)\geqslant\bigcap_{n\in\omega}\mathcal{F}_n$ the set $\bigcup_{n<\omega}K_n$ is closed and does not contain ∞, so that it consists of isolated points, and thus is finite.

In particular, ∞ is a *compact-repellent* point, that is, $\infty\notin \mathrm{cl}(K\setminus\{\infty\})$ for each compact set. On the other hand, X_n is closed for each n and the upper and the lower limit of $(X_n)_{n<\omega}$ coincide and are equal to $\{\infty\}$[6]; Thus, by [5, Theorem 6.1], contour spaces are not consonant. Actually,

Theorem 3.6. *Contour prime spaces are not infraconsonant.*

Proof. The family

$$\mathcal{D} := \bigl\{D\in\mathcal{O}(\infty):\ \forall_{n\in\omega},\ D\cap X_n\neq\emptyset\bigr\}$$

is non-empty and compact. Indeed, if $\{O_\alpha:\ \alpha\in I\}$ is an open cover of $D\in\mathcal{D}$, there is $\alpha_0\in I$ such that $\infty\in O_{\alpha_0}$ and by (3.4), the set $J := \{n\in\omega:\ X_n\cap O_{\alpha_0}=\emptyset\}$ is finite. For each $n\in J$, there is $x_n\in D\cap X_n$, and there is $\alpha_n\in I$ such that $x_n\in O_{\alpha_n}$. Then $\bigcup_{n\in J\cup\{0\}}O_{\alpha_n}\in\mathcal{D}$.

[6] That is, $\{\infty\}=\bigcap_{p<\omega}\mathrm{cl}(\bigcup_{n\geqslant p}X_n)=\bigcap_{N\in[\omega]^\omega}\mathrm{cl}(\bigcup_{n\in N}X_n)$.

On the other hand, if $\mathcal{C} \subseteq \mathcal{O}(\infty)$ is a compact family, there is n_0 such that $\mathcal{C}$ is free on X_{n_0}, that is, the restriction of $\mathcal{C}$ to X_{n_0} is finer than the cofinite filter of X_{n_0}. Otherwise, for each $n \in \omega$ there would be a finite subset $F(n)$ of X_n such that $F(n) \# \mathcal{C}$.

On the other hand, $V := X \setminus \bigcup_{n\in\omega} F(n) \in \mathcal{O}(\infty)$, hence, by the compactness of $\mathcal{C}$, there exists a finite set T such that $V \cup T \in \mathcal{C}$ and $(V \cup T) \cap \bigcup_{n\in\omega} F(n) \subset T$. This is a contradiction with $F(n) \# \mathcal{C}$ for all n.

By the compactness of $\mathcal{C}$ there exists $C_0 \in \mathcal{C}$ such that $C_0 \cap X_{n_0}$ is finite. As $\mathcal{C}$ is free on X_{n_0}, there is $C \in \mathcal{C}$ disjoint from $C_0 \cap X_{n_0}$, so that $\mathcal{C} \vee \mathcal{C} \nsubseteq \mathcal{D}$. Hence X is not infraconsonant. □

Example 3.7 *(The Arens space is not infraconsonant).* Recall that the Arens space has underlying set $\{\infty\} \cup \{x_{n,k}\colon n \in \omega,\ k \in \omega\}$ and carries the topology in which every point but ∞ is isolated and a base of neighborhoods of ∞ is given by sets of the form

$$\{\infty\} \cup \bigcup_{n \geqslant p} \{x_{n,k}\colon k \geqslant f(n)\},$$

where p ranges over ω and f ranges over ω^ω.

In [9] a notion of *sequential contour* of arbitrary order was introduced. A sequential contour of rank 1 is a free sequential filter (that is, the cofinite filter of a countable set); a sequential contour of rank $\alpha > 1$ is a contour of the sequence $(\mathcal{F}_n)_{n<\omega}$, where $\mathcal{F}_n$ is a sequential contour of rank α_n on a countable set X_n, $\{X_n\colon n < \omega\}$ are disjoint, and $\alpha = \sup_{n<\omega}(\alpha_n + 1)$. It follows that:

Corollary 3.8. *For every countable ordinal α the prime topology determined by a sequential contour of rank α is not infraconsonant.*

We do not know however if there are completely regular infraconsonant spaces that are not consonant.

A. Bouziad pointed out to us that, in view of Proposition 3.4, Theorem 3.6 also shows that the assumption of separation is essential in the result of J. Lawson stating that a compact Hausdorff semitopological lattice is topological [18]. Indeed, if X is regular but not infraconsonant (for instance the Arens space), then $C(X, \$^*)$ is a T_0 compact semitopological lattice (i.e., $\cap$ is separately continuous) which is not a topological lattice, because $\cap$ is not jointly continuous.

4. Continuity of translations

As we mentioned, Francis Jordan introduced in [17] the *fine Isbell topology* on the set $C(X, Y)$. We shall now prove that translations are always continuous for the fine Isbell topology, and that the neighborhood filters at the zero function $\bar{0}$ for the Isbell and for the fine Isbell topologies coincide. Therefore translations are continuous for the Isbell topology if and only if it coincides with the fine Isbell topology.

If N and M are two subsets of $X \times Y$, the set N is *buried in* M, in symbols $N \ll M$, if for every $x \in X$ there exist $V \in \mathcal{O}_X(x)$ and $W \in \mathcal{O}_Y(N(x))$ such that $V \times W \subseteq M$. If $f \in C(X, Y)$ and $A \subseteq X$, we denote by $f_{|A}$ the graph of the restriction of f to A. A subbase for the fine Isbell topologies is given by sets of the form:

$$\langle \mathcal{A}, M \rangle := \{f \in C(X, Y)\colon \exists A \in \mathcal{A},\ f_{|A} \ll M\},$$

where $\mathcal{A}$ ranges over compact families of X and M ranges over open subsets of $X \times Y$. We denote by $C_{\bar{\kappa}}(X, Y)$ the set $C(X, Y)$ endowed with the fine Isbell topology. If $(G, +)$ is a topological group, we denote by 0 its neutral element and by $\bar{0}$ the constant function zero of $C(X, G)$.

Theorem 4.1. *Let $(G, +)$ be a topological group. The fine Isbell topological space $C_{\bar{\kappa}}(X, G)$ is invariant by translations.*

The neighborhood filters at $\bar{0}$ for the fine Isbell and the Isbell topologies coincide.

Proof. (1) $\mathcal{N}_\kappa(\bar{0}) \leqslant \mathcal{N}_{\bar{\kappa}}(\bar{0})$: Is clear. Consider now $\langle \mathcal{A}, M \rangle$ such that $\bar{0} \in \langle \mathcal{A}, M \rangle$, $\mathcal{A} \in \kappa(X)$ and M is open in $X \times G$. There is $A \in \mathcal{A}$ such that for every $x \in A$, there are $V_x \in \mathcal{O}(x)$ and $W_x \in \mathcal{O}_G(0)$ such that $V_x \times W_x \subseteq M$. Since $\mathcal{A}$ is compact and $A = \bigcup_{x\in A} V_x$ there is a finite subset F of A such that $B = \bigcup_{x\in F} V_x \in \mathcal{A}$. But then $W = \bigcap_{x\in F} W_x \in \mathcal{O}_G(0)$ and $B \times W \subseteq M$ so that

$$\bar{0} \in [\mathcal{A} \downarrow B, W] \subseteq \langle \mathcal{A}, M \rangle.$$

(2) $\mathcal{N}_{\bar{\kappa}}(f) \geqslant f + \mathcal{N}_\kappa(\bar{0})$: Let $\mathcal{A} \in \kappa(X)$, $B \in \mathcal{O}_G(0)$. Consider $M := \bigcup_{x\in X}\{x\} \times (f(x) + B)$. Then $f \in \langle \mathcal{A}, M \rangle$ and $\langle \mathcal{A}, M \rangle \subseteq f + [\mathcal{A}, B]$. Indeed, if $h \in \langle \mathcal{A}, M \rangle$ then there is $A \in \mathcal{A}$ such that for all $x \in A$, there are an open neighborhood V_x of x and an open neighborhood W_x of $h(x)$ such that $V_x \times W_x \subseteq M$. In particular, $\{x\} \times W_x \subseteq M$ so that $W_x \subseteq f(x) + B$ and $(h - f)(x) \in B$. Therefore $(h - f)(A) \subseteq B$.

(3) $\mathcal{N}_{\bar{\kappa}}(f) \leqslant f + \mathcal{N}_\kappa(\bar{0})$: Let $\mathcal{A} \in \kappa(X)$ and let M be an open subset of $X \times G$ such that $f \in \langle \mathcal{A}, M \rangle$, that is, there is $A \in \mathcal{A}$ such that for all $x \in A$, there are an open neighborhood V_x of x and an open neighborhood $W_x = f(x) + B_x$ of $f(x)$, where $B_x \in \mathcal{O}_G(0)$ such that $V_x \times W_x \subseteq M$. By continuity of f we may assume that $f(V_x) \subseteq f(x) + B'_x \subseteq W_x$ for each x,

where $B'_x \in \mathcal{O}_G(0)$ and $B'_x + B'_x \subseteq B_x$. Since $\mathcal{A}$ is compact and $A = \bigcup_{x\in A} V_x$ there is a finite subset F of A such that $A_1 = \bigcup_{x\in F} V_x \in \mathcal{A}$. Let $W \in \mathcal{O}_G(0)$ be such that $W = -W$ and $W \subseteq \bigcap_{x\in F} B'_x \in \mathcal{O}_G(0)$. Then $f + [\mathcal{A} \downarrow A_1, W] \subseteq \langle \mathcal{A}, M\rangle$. Indeed, if $h \in [\mathcal{A} \downarrow A_1, W]$ then there is $A_2 \in \mathcal{A}$, $A_2 \subseteq A_1$ such that $h(A_2) \subseteq W$. For each $x \in A_2$, there is $t_x \in F$ such that $x \in V_{t_x}$. Note that $V_{t_x} \times W_{t_x} \subseteq M$ and that $f(x) \in f(V_{t_x})$ and

$$f(V_{t_x}) + W \subseteq f(x) + B'_x + B'_x \subseteq W_{t_x}$$

so that $V_{t_x} \times (f(x) + W) \subseteq M$ which completes the proof because $f(x) + W \in \mathcal{O}((f + h)(x))$ and $V_{t_x} \in \mathcal{O}(x)$. □

Corollary 4.2. *$C_\kappa(X, G)$ is invariant by translation if and only if $C_\kappa(X, G) = C_{\overline{\kappa}}(X, G)$.*

The result above also provides a more handy description of the fine Isbell topology on $C(X, G)$ when G is a topological group (for instance for $C_{\overline{\kappa}}(X, \mathbb{R})$):

$$\mathcal{N}_{\overline{\kappa}}(f) = f + \big\{[\mathcal{A}, B]\colon \mathcal{A} \in \kappa(X),\ B \in \mathcal{O}_G(0)\big\}.$$

Theorem 4.3. *The following are equivalent*:

(1) *$C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological vector space*;
(2) *$C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological group*;
(3) *X is infraconsonant.*

Proof. If $C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological group then $\mathcal{N}_{\overline{\kappa}}(\bar{0}) + \mathcal{N}_{\overline{\kappa}}(\bar{0}) = \mathcal{N}_{\overline{\kappa}}(\bar{0})$. But $\mathcal{N}_{\overline{\kappa}}(\bar{0}) = \mathcal{N}_\kappa(\bar{0})$ so that by Theorem 3.1, X is infraconsonant. Conversely, if X is infraconsonant, then $\mathcal{N}_{\overline{\kappa}}(\bar{0}) + \mathcal{N}_{\overline{\kappa}}(\bar{0}) = \mathcal{N}_{\overline{\kappa}}(\bar{0})$ and translations are continuous in $C_{\overline{\kappa}}(X, \mathbb{R})$ so that $C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological group. Remains to see that if $C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological group, it is also a topological vector space.

First, note that for each fixed $f \in C(X, \mathbb{R})$ the map $S_f : \mathbb{R} \to C_{\overline{\kappa}}(X, \mathbb{R})$ defined by $S_f(r) = rf$ is continuous. Indeed, for each $\mathcal{A} \in \kappa(X)$, there are $A \in \mathcal{A}$ and $R \in \mathbb{R}$ such that $f(A) \subseteq B(0, R)$ by Proposition 2.1. Therefore for each $O \in \mathcal{N}_{\mathbb{R}}(0)$ there is $\delta > 0$ such that $B(0, \delta) \cdot f(A) \subseteq O$. Thus, if $rf + [\mathcal{A}, O] \in \mathcal{N}_{\overline{\kappa}}(f)$ then $B(r, \delta) \cdot f = rf + B(0, \delta) \cdot f \subseteq rf + [\mathcal{A}, O]$.

Note also that $S : \mathbb{R} \times C_{\overline{\kappa}}(X, \mathbb{R}) \to C_{\overline{\kappa}}(X, \mathbb{R})$ defined $S(r, f) = rf$ is continuous at $(r, \bar{0})$ for each r by Proposition 2.11. Let now $r \in \mathbb{R}$ and $f \in C(X, \mathbb{R})$ and consider $rf + W \in \mathcal{N}_{\overline{\kappa}}(f)$, where $W \in \mathcal{N}_{\overline{\kappa}}(\bar{0})$. Since $C_{\overline{\kappa}}(X, \mathbb{R})$ is a topological group, there is $V \in \mathcal{N}_{\overline{\kappa}}(\bar{0})$ such that $V + V \subseteq W$. By continuity of S_f, there is $T \in \mathcal{N}_{\mathbb{R}}(r)$ such that $T \cdot f \subseteq rf + V$. Moreover, by continuity of S at $(r, \bar{0})$ there are $T' \in \mathcal{N}_{\mathbb{R}}(r)$, $T' \subseteq T$ and $U \in \mathcal{N}_{\overline{\kappa}}(\bar{0})$ such that $T' \cdot U \subseteq V$. Then

$$T' \cdot (f + U) = T' \cdot f + T' \cdot U \subseteq rf + V + V \subseteq rf + W,$$

which proves continuity of S at (r, f) because $f + U \in \mathcal{N}_{\overline{\kappa}}(f)$. □

In particular, in view of Example 3.7, the fine Isbell topology does not need to be a group topology. In [17, Example 1] Jordan shows that if X and Y are two zero-dimensional consonant spaces such that the topological sum $Z := X \oplus Y$ is not consonant then $C_\kappa(Z, \mathbb{R}) < C_{\overline{\kappa}}(Z, \mathbb{R})$. In view of Corollary 4.2 we obtain:

Example 4.4. There exists a space Z such that translations of $C_\kappa(Z, \mathbb{R})$ fail to be continuous.

Following [17], we say that a space X is *sequentially inaccessible* provided that for any sequence $(\mathcal{F}_n)_{n\in\omega}$ of countably based z-filters (filters based in zero sets) on X

$$(\forall_n\ \operatorname{adh} \mathcal{F}_n = \emptyset) \quad \Longrightarrow \quad \operatorname{adh}\Big(\bigcap_{n\in\omega} \mathcal{F}_n\Big) = \emptyset.$$

Jordan proved [17] that if X is completely regular, Lindelöf and sequentially inaccessible, then the fine Isbell topology and the natural topology coincide on $C(X, \mathbb{R})$. This result can be combined with Theorem 4.3 to obtain the following Banach–Dieudonné like result.[7]

Theorem 4.5. *If X is a completely regular, Lindelöf and sequentially inaccessible space, then the natural topology on $C(X, \mathbb{R})$ is a group (and vector) topology if and only if X is infraconsonant.*

As we have seen, translations in $C(X, \mathbb{R})$ are, in general, not continuous for the Isbell topology. They are however continuous if X is prime (that is, has at most one non-isolated point). More generally,

[7] The classical theorem of Banach–Dieudonné (and its many variants) can be seen as providing sufficient conditions on a topological vector space for the natural topology on its dual space to be a group topology. See [16,2] for details.

Proposition 4.6. *If X is prime and G is an abelian consonant topological group, then the Isbell topology on $C(X,G)$ is translation invariant.*

Proof. If a family $\mathcal{A}$ is compact on X, then for each $x \in X$ the family $\mathcal{A}_x := \mathcal{O}_X(x) \cap \mathcal{A}$ is compact included in $\mathcal{A}$. Therefore it is enough to consider basic neighborhoods for the Isbell topology of the form $[\mathcal{A}_x, U]$ where U is an open subset of G.

Consider $f \mapsto g + f$ for $g \in C(X,G)$. Let $x \in X$ and $\mathcal{A}$ be a compact family included in $\mathcal{O}_X(x)$ and U is an open subset of G. Let $f_0 + g \in [\mathcal{A}, U]$.

The family $\mathcal{D} := \mathcal{O}_G((f_0+g)(\mathcal{A}))$ is compact in a consonant space G, hence there is a compact set $K \subset U$ such that $\mathcal{O}_G(K) \subset \mathcal{D}$. Let $W = -W$ be a closed neighborhood of 0 in G such that $K + 3W \subset U$. Then there is $A \in \mathcal{A}$ such that $(f_0+g)(A) \subset K + W$. Furthermore there exists $A_0 \in \mathcal{A}$ such that $A_0 \subset A$ and $f_0(A_0)$ is bounded and $(f_0+g)(A_0) \subset K + W$.

Let V_0 be an element of $\mathcal{O}_X(x)$ included in A_0 such that $f_0(V_0) \subset f_0(x_\infty) + W$ and $g(V_0) \subset g(x_\infty) + W$.

Then there is a finite subset F of A_0 (disjoint from V_0) such that $A_1 := V_0 \cup F \in \mathcal{A}$. Then let $\mathcal{A}_1 := \mathcal{A} \downarrow A_1$. Of course, $f_0 + g \in [\mathcal{A}_1, K + W]$ and $\mathcal{A}_1 \in \kappa(X)$.

Then there are $n < \omega$ and finite sets $F_1, \ldots, F_n$ such that $f_0(F_k) - f_0(F_k) \subset W$ and $g(F_k) - g(F_k) \subset W$ for each $1 \leqslant k \leqslant n$, and moreover $F_1 \cup \cdots \cup F_n = F$. Finally, let $\mathcal{D}_0 := \mathcal{A}_1 \vee V_0$ and $\mathcal{D}_k := \mathcal{A}_1 \vee F_k$ for $1 \leqslant k \leqslant n$. Note that $\mathcal{A}_1 = \bigcap_{k=0}^n \mathcal{D}_k$. On the other hand, there exist $x_k \in F_k$ for $1 \leqslant k \leqslant n$, such that

$$f_0 \in \bigcap_{k=0}^{n} \big[\mathcal{D}_k, f_0(x_k) + W\big],$$

where $x_0 := x_\infty$. If now $f \in \bigcap_{k=0}^n [\mathcal{D}_k, f_0(x_k) + 2W]$ then

$$f + g \in \bigcap_{k=0}^{n} \big[\mathcal{D}_k, f_0(x_k) + g(x_k) + 3W\big] \subset [\mathcal{A}_1, U] \subset [\mathcal{A}, U]. \qquad \square$$

Corollary 4.7. *If X is a prime topological space, then $C_\kappa(X,\mathbb{R})$ is translation invariant.*

Since, by Corollary 4.2, $C_\kappa(X,\mathbb{R})$ is translation invariant if and only if it coincides with $C_{\overline{\kappa}}(X,\mathbb{R})$, Corollary 4.7 implies a result [17, Theorem 18] of Jordan that the Isbell and the fine Isbell topologies coincide provided that the underlying topology is prime.

Corollary 4.7 combined with Theorem 3.1 and Corollary 4.2 leads to the following results.

Theorem 4.8. *If X is prime, then $C_\kappa(X,\mathbb{R})$ is a topological group (or topological vector space) if and only if X is infraconsonant.*

Note that if X is as in Example 3.7, then $C_\kappa(X,\mathbb{R})$ is invariant by translation but not a topological group.

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