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Note

A note on univalent functions starlike with respect to a boundary point

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Dedicated to the memory of Professor Walter Hengartner[1]

Abstract

The object of this paper is to prove the sufficiency of a recently established necessary condition for a univalent function to be starlike with respect to a boundary point.

1. Introduction and statement of Main theorem

Let $\mathbb{C}$ be the complex plane and let $\mathbf{D}$ be the open unit disc $\{z\colon |z| < 1\}$. A complex region Ω with $0 \in \partial\Omega$ is called *starlike with respect to the origin* if for every point $w \in \Omega$ the line segment $(0, w] = \{tw\colon 0 < t \leqslant 1\}$ lies in Ω. Also, we call a univalent function f of $\mathbf{D}$ onto Ω *starlike with respect to the* (*boundary point at the*) *origin*. Denote by $\mathcal{S}_0^*$ the class of all such functions.

Let f be an analytic function of $\mathbf{D}$ and let $\zeta \in \partial\mathbf{D}$. We say that f has the *asymptotic value* $a \in \mathbb{C} \cup \{\infty\}$ at ζ if there exists a Jordan arc Γ that ends at ζ and lies in $\mathbf{D}$ except for ζ such that

$$f(z) \to a \quad \text{as } z \in \Gamma,\ z \to \zeta.$$

Also, we say that f has the *angular limit* $a \in \mathbb{C} \cup \{\infty\}$ at ζ if

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$$f(z) \to a \quad \text{as } z \in A,\ z \to \zeta,$$

for every Stolz angle A at ζ, $A = \{z \in \mathbf{D}\colon\ |\arg(1 - \bar{\zeta} z)| < \pi/2 - \delta\}$, where $0 < \delta < \pi/2$. For these notions see [4, p. 267].

Since the origin is an accessible point in Ω, there exist infinitely many functions $f \in \mathcal{S}_0^*$ whose angular limits at 1 is zero [5, Corollary 2.17].

Univalent functions starlike with respect to a boundary point were first introduced in 1981 by Robertson [6]. In his paper, the following two classes of univalent functions were introduced:

(i) The class $\mathcal{G}$ of univalent functions f of $\mathbf{D}$ that satisfy $f(0) = 1$ and

$$\Re\left\{2z\frac{f'(z)}{f(z)} + \frac{1+z}{1-z}\right\} > 0 \quad (z \in \mathbf{D});$$

(ii) The class $\mathcal{G}^*$ of univalent functions f of $\mathbf{D}$ that satisfy $f(0) = 1$, $\lim_{r\to 1^-} f(r) = 0$, $f(\mathbf{D})$ is starlike with respect to the origin, and $\Re\{e^{i\alpha} f(z)\} > 0$ for some real α and all $z \in \mathbf{D}$.

Further, Robertson proved that $\mathcal{G} \subset \mathcal{G}^*$ and conjectured that $\mathcal{G}^* \subset \mathcal{G}$. The conjecture was resolved positively by Lyzzaik in [3] where a short proof of the former set-inclusion was also given.

It is immediate that $f^2 \in \mathcal{S}_0^*$ if $f \in \mathcal{G}$; conversely if $f(0) = 1$, $\lim_{r\to 1^-} f(r) = 0$, and $f^2 \in \mathcal{S}_0^*$, then $f \in \mathcal{G}$. However, if $f^2 \in \mathcal{S}_0^*$, and $f(0) \neq 1$ or $\lim_{r\to 1^-} f(r) \neq 0$, then there exists a real β such that $\lim_{t\to 1^-} f(te^{i\beta}) = 0$ and, consequently, $f(e^{i\beta}z)/f(0) \in \mathcal{G}$. This gives at once a complete analytic definition of the functions $f \in \mathcal{S}_0^*$.

Let $\mathcal{B}$ be the class of all analytic functions from $\mathbf{D}$ to itself. For $\alpha > 0$, let $\mathcal{B}(\alpha)$ be the subclass of $\mathcal{B}$ consisting of all functions ω whose angular limits of $(1 - \omega(z))/(1 - z)$ at 1 is α.

In order to establish another analytic definition of the functions $f \in \mathcal{S}_0^*$, Lecko proved recently the following result [2, Theorem 3.2].

Theorem 1. *Let f be an analytic function of $\mathbf{D}$ with angular limit zero at 1. If $f \in \mathcal{S}_0^*$, then there exists $\omega \in \mathcal{B}(\alpha)$, $\alpha \in (0, 1]$, such that*

$$-(1-z)^2\frac{f'(z)}{f(z)} = 4\frac{1-\omega(z)}{1+\omega(z)}, \quad z \in \mathbf{D}. \tag{1}$$

In an attempt to prove the converse of this theorem, Lecko also proved the following result [2, Theorem 3.3].

Theorem 2. *Let f be an analytic function of $\mathbf{D}$ with angular limit zero at 1. If there exist $\omega \in \mathcal{B}$ and $\alpha \in (0, 1]$ such that $\lim_{z\to 1} \omega(z) = 1$, $\lim_{z\to 1}(1 - \omega(z))/(1 - z) = \alpha$, and (1) holds, then $f \in \mathcal{S}_0^*$.*

The object of this note is to prove the converse of Theorem 1 stated as follows.

Main theorem. *Let f be an analytic function of* **D** *with asymptotic value zero at* 1*. If there exists $\omega \in \mathcal{B}(\alpha)$, $\alpha \in (0,1]$, such that* (1) *holds, then $f \in S_0^*$.*

Observe that this is a stronger converse of Theorem 1 in view of the weaker conditions on both f and its associated function ω.

2. Proof of Main theorem

Let $g = f^{1/2\alpha}$. For $0 < x < 1$, $(1-\omega(x))/(1-x) \to \alpha$ as $x \to 1$. Then there exists a sequence (x_n), $0 < x_n < 1$, such that $x_n \to 1$, $\omega(x_n) \to 1$, and $(1-|\omega(x_n)|)/(1-x_n) \to A \leqslant \alpha$. Then, by the theorem of Carathéodory–Landau–Valiron [1, Theorem 1.5, p. 9],

$$\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2} \leqslant A\frac{|1-z|^2}{1-|z|^2}, \quad z \in \mathbf{D}.$$

Thus $A > 0$ and

$$\sup_{z\in\mathbf{D}} \frac{|1-\omega(z)|^2}{1-|\omega(z)|^2}\frac{1-|z|^2}{|1-z|^2} \leqslant \alpha. \tag{2}$$

Using (1) and (2), we obtain

$$\left(1-|z|^2\right)\frac{|g'(z)|}{1+|g(z)|^2} \leqslant \frac{2}{\alpha}\frac{|g(z)|}{1+|g(z)|^2}\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2}\frac{1-|z|^2}{|1-z|^2} \leqslant 1. \tag{3}$$

Thus g is a normal function. Since g, like f, assumes zero as an asymptotic value at 1, it has the angular limit zero at 1 [4, Theorem 9.3].

For $k > 0$, let $\gamma_k(\theta) = (1+ke^{i\theta})/(1+k)$, $\theta \in [0, 2\pi]$; this is the positively-oriented circle centered at $1/(1+k)$ and tangent to the unit circle at 1. By virtue of (1), we conclude

$$\begin{aligned}\frac{d}{d\theta}\arg g\circ\gamma_k(\theta) &= \frac{d}{d\theta}\Im\left[\log g\circ\gamma_k(\theta)\right]\\ &= \frac{1}{4\alpha}\Re\left[-\left(1-\gamma_k(\theta)\right)^2\frac{f'\circ\gamma_k(\theta)}{f\circ\gamma_k(\theta)}\right]\Big/\Re\left(1-\gamma_k(\theta)\right) > 0\end{aligned}$$

for $0 < \theta < 2\pi$ and $\arg g\circ\gamma_k(\theta)$ is strictly increasing in $(0, 2\pi)$; see [2].

Fix $z_0 \in \mathbf{D}$. Note that there exists a unique $k > 0$ such that $z_0 \in \gamma_k$. Denote by O_k the horocycle of γ_k; this is the finite open disc bounded by γ_k. Let $w_0 = g(z_0)$ and let $[0, w_0]$ be the line segment from 0 to w_0. Since g' is nonvanishing in $\mathbf{D}$ and $\arg g\circ\gamma_k(\theta)$ is strictly increasing, there exists a unique arc σ_0 from z_0 to $z_1 \in \gamma_k$ lying in O_k except for the endpoints such that g maps $\sigma_0 \setminus \{z_1\}$ homeomorphically onto an open–closed line segment $(w_1, w_0] \subset [0, w_0]$. We contend that σ_0 is a cross-cut of O_k. Because g' is nonvanishing, σ_0 admits no self intersections. Furthermore, if $z_0 = z_1$, then in this case one would obtain $w_0 = g(z_0) = g(z_1) = w_1$, which would yield a contradiction. This proves our contention.

Suppose $z_1 \neq 1$. Let δ_k, $1 \notin \delta_k$, be the subarc of γ_k ending in z_0 and z_1, and let G be the Jordan domain bounded by δ_k and σ_0. Direct σ_0 so that ∂G is the arc-product $\sigma_0\delta_k$. In this case the winding number $n(\partial G, z)$, $z \in G$, is one. This yields $n(g(\partial G), 0) > 0$ which, by the argument principle, implies that g vanishes at some points in $\mathbf{D}$. Thus we have a contradiction and $z_1 = 1$.

Since g has the angular limit zero at 1, again by [4, Theorem 9.3], $w_1 = 0$ and $g : \sigma_0 \to [0, w_0]$ is a homeomorphism. It follows that, since $f = g^{2\alpha}$, f belongs to $\mathcal{S}_0^*$ if it is shown to be univalent in $\mathbf{D}$.

There exists a single-valued analytic branch of $\log g$ in $\mathbf{D}$. We claim that $\log g$ is a univalent function in $\mathbf{D}$. Suppose that z_0 and z_1 are two points in $\mathbf{D}$ with $\log g(z_0) = \log g(z_1)$. Then $g(z_0) = g(z_1)$. There exists $k, r > 0$ such that $z_0 \in \gamma_k$ and $z_1 \in \gamma_r$. We may assume that $k \geqslant r > 0$; then $z_1 \in \bar{O}_k \setminus \{1\}$. If $z_1 \in \gamma_k$, then let λ, $1 \notin \lambda$, be the subarc of γ_k from z_0 to z_1. Since $g(z_0) = g(z_1)$ and $\arg g \circ \gamma_k(\theta)$ is strictly increasing, $n(g \circ \lambda, 0) = m$, where m is a nonzero integer. Thus $\log g(z_1) - \log g(z_0) = 2m\pi i$ and $\log g(z_0) \neq \log g(z_1)$. Hence $z_1 \in O_k$. In this case, as shown above, we can find a directed cross-cut σ_1 of O_r from 1 to z_1 such that $g : \sigma_1 \to [0, g(z_1)]$ is a homeomorphism. Since g' is nonvanishing in $\mathbf{D}$, σ_1 continues through z_1 to a Jordan arc σ in $\mathbf{D}$ that terminates at a point $\zeta \in \partial\mathbf{D}$ and maps under g homeomorphically to a line-segment $[0, w_1]$, with w_1 possibly infinity, containing $g(z_1)$ as an interior point. Since g is a normal function, $\zeta \neq 1$ or else $w_1 = 0$ as zero is the only asymptotic value of g at 1; once again by [4, Theorem 9.3]. Hence σ intersects γ_k at some point, ξ. In this case, let λ be the arc-product of the subarc of γ_k from z_0 to ξ that avoids 1 with the subarc of σ^{-1} from ξ to z_1. This implies $n(g \circ \lambda, 0) = m$, where m is a nonzero integer, $\log g(z_1) - \log g(z_0) = 2m\pi i$ and $\log g(z_0) \neq \log g(z_1)$. Hence the above claim holds.

Recall that for every $z \in \mathbf{D}$ there exists a Jordan arc σ from 1 to z with $\sigma \setminus \{1\} \subset \mathbf{D}$ such that $g : \sigma \to [0, g(z)]$ is a homeomorphism. This means that $\log g$ is convex is the direction of the real axis in the sense, referred to henceforth by the *restricted horizontal convexity*, that every horizontal line meets $\log g(\mathbf{D})$, if at all, in an interval $s + it_0$, $s < s_0$ for some s_0.

Observe that g satisfies the Visser–Ostrowski condition at 1 [5, p. 81]; namely,

$$(z-1)\frac{g'(z)}{g(z)} = \frac{1}{2\alpha}(z-1)\frac{f'(z)}{f(z)} = \frac{4}{2\alpha}\,\frac{1-\omega(z)}{1-z}\,\frac{1}{1+\omega(z)} \to 1 \tag{4}$$

as $z \to 1$ in every Stolz angle of 1. This gives

$$(r-1)\frac{\partial}{\partial r}\log\bigl|g(r)\bigr| \to 1$$

as $r \to 1^-$; hence $\log|g(r)|$ is strictly decreasing in some interval $[\rho, 1)$, $0 < \rho < 1$.

Through every point $\log g(r)$, $\rho \leqslant r < 1$, there exists a unique maximal vertical interval $\log|g(r)| + it$, $a_r \leqslant t \leqslant b_r$, which lies in $\log g(\mathbf{D})$ except for its endpoints; a_r or b_r could possibly be $-\infty$ or ∞, respectively. We claim that a_r and b_r are monotone decreasing and increasing functions of r in $\rho \leqslant r < 1$, respectively. For $\rho \leqslant r < 1$, consider the horizontal semi-strip

$$S_r = \bigl\{s + it\colon\ s < \log\bigl|g(r)\bigr|,\ a_r < t < b_r\bigr\}.$$

The restricted horizontal convexity of $\log g(\mathbf{D})$ yields $S_r \subset \log g(\mathbf{D})$. Fix r_1, $\rho \leqslant r_1 < 1$. There exists r_1', $r_1 < r_1' < 1$, such that for every $r_1 < r < r_1'$, $\log g(r) \in S_{r_1}$ and, consequently, $a_r \leqslant a_{r_1}$ and $b_r \geqslant b_{r_1}$. By appealing to the same argument for any r, $r_1 < r < r_1'$, instead of r_1, we infer that a_r and b_r are monotone decreasing and increasing functions of r in (r_1, r_1'), respectively. This together with the uniform continuity of $\log g(r)$ on any compact subinterval $[r_1, r_2]$ of $[\rho, 1)$ yield $a_{r_2} \leqslant a_{r_1}$ and $b_{r_1} \leqslant b_{r_2}$ which proves our claim.

For $0 < r < 1$, let

$$g_r(z) = g\left(\frac{z+r}{1+rz}\right) \Big/ g(r). \tag{5}$$

Then $\log g_r$, with $\log 1 = 0$, is a univalent function in $\mathbf{D}$. With $s = (z+r)/(1+rz)$, and once again by (4), we have,

$$\frac{d}{dz} \log\left\{\frac{1+z}{1-z} g_r(z)\right\} = \frac{2}{1-z^2} + \frac{1+r}{(1+rz)(1-z)}(1-s)\frac{g'(s)}{g(s)} \to 0$$

as $r \to 1^-$ locally uniformly in $\mathbf{D}$; hence, likewise is the convergence

$$\log g_r \to \log\left\{\frac{1-z}{1+z}\right\}.$$

Let $G_r = \log g_r(\mathbf{D})$, $\rho \leqslant r < 1$. By the Carathéodory kernel theorem, we conclude that G_r converges to the horizontal strip $S = \{s + it\colon |t| < \pi/2\}$ with respect to the origin in the sense of the Carathéodory kernel convergence [5, p. 14]. Fix $0 < \epsilon < 1$. It follows that there exist a sequence $\{r_n\}$, $r_n \to 1^-$ as $n \to \infty$, and points $\tau_n, \tau_n' \in \partial \log g_{r_n}(\mathbf{D})$ such that $|\tau_n - (-1 + i\pi/2)| < \epsilon/2$ and $|\tau_n' - (-1 - i\pi/2)| < \epsilon/2$. Observe that, by (5), each $\log g_{r_n}(\mathbf{D})$ contains the translate of S_{r_n} by $-\log g(r_n)$; namely, the horizontal semi-strip $\{u + iv\colon\ u < 0,\ a_{r_n} - \arg g(r_n) < v < b_{r_n} - \arg g(r_n)\}$, where $a_{r_n} < \arg g(r_n) < b_{r_n}$. We infer that each $b_{r_n} - a_{r_n} < \pi + \epsilon$, or else either $|\tau_n - (-1 + i\pi/2)| \geqslant \epsilon/2$ or $|\tau_n' - (-1 - i\pi/2)| \geqslant \epsilon/2$ and we have a contradiction.

It follows that for $\rho \leqslant r < 1$, $b_r - a_r \leqslant \pi + \epsilon$, and consequently $\lim_{r\to 1^-}(b_r - a_r) \leqslant \pi$ since ϵ is arbitrary. Let $a = \lim_{r\to 1} a_r$ and $b = \lim_{r\to 1} b_r$; then a and b exist and satisfy $b - a \leqslant \pi$. With $T = \{u + iv\colon\ -\infty < u < \infty,\ a < v < b\}$, we show that $\log g(\mathbf{D}) \subset T$. Obviously, $\log g(\mathbf{D}) \cap T \neq \emptyset$. Suppose $\tau, \tau' \in \log g(\mathbf{D})$, say $\Im\tau < \Im\tau'$, $\tau \in T$ and $\tau' \notin \overline{T}$; $\overline{T}$ is the closure of T. Then there exists a Jordan arc in $\log g(\mathbf{D})$ connecting τ and τ'. Using the restricted horizontal convexity of $\log g(\mathbf{D})$ once again, we can find a horizontal semi-strip $\{s + it\colon\ s < s_0,\ \Im\tau < t < \Im\tau'\}$ for some s_0, that lies in $\log g(\mathbf{D})$. This yields a contradiction and $\log g(\mathbf{D}) \subset T$.

Therefore, g is a univalent function with $\Re(e^{i\beta} g) > 0$ for some real β. Since $0 < \alpha \leqslant 1$ and $f = g^{2\alpha}$, the function f is univalent in $\mathbf{D}$.

Therefore $f \in S_0^*$. This completes the proof of Main theorem. □

We combine Main theorem and Theorem 1 [2, Theorem 3.2] as follows.

Theorem 3. *Let f be an analytic function of* $\mathbf{D}$ *with angular limit zero at* 1. *Then $f \in \mathcal{S}_0^*$ if and only if there exists $\omega \in \mathcal{B}(\alpha)$, $\alpha \in (0, 1]$, such that* (1) *holds.*

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