Results in Physics 4 (2014) 187–188

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

Microarticle

Testing general relativity using a bouncing ball

Shiuan-Ni Liang, Boon Leong Lan *

School of Engineering, Monash University, 46150 Bandar Sunway, Selangor, Malaysia

A R T I C L E I N F O

Article history:
Received 10 July 2014
Accepted 11 August 2014
Available online 18 August 2014

Keywords:
Bouncing ball
General relativity
Newtonian approximation

A B S T R A C T

In a recent article (Liang and Lan, (2011)), we showed that the trajectories predicted by general-relativistic and Newtonian mechanics from the same parameters and initial conditions for a low-speed weak-gravity bouncing ball system will rapidly disagree completely if the trajectories are chaotic. Here, we determine how accurate the parameters and initial conditions of the system must be known so that the two different calculated chaotic trajectories are sufficiently accurate for an empirical test.

Introduction

The standard practice [1,2] in the field of nonlinear dynamics and chaos in physics and engineering is to use Newtonian mechanics, instead of general-relativistic mechanics, to study the trajectory of a low-speed weak-gravity dynamical system (low-speed means that the speed of the system is much less than the speed of light, and weak-gravity means that the gravitational potential is much less than the square of the speed of light). This is because it is conventionally believed [3–5] that, in general, if the speed of the dynamical system is low and gravity plays a role in the dynamics but is weak, the trajectory predicted by general-relativistic mechanics is well approximated by the trajectory predicted by Newtonian mechanics for the *same* parameters and initial conditions.

However, it was recently shown [6,7] numerically for a bouncing ball system that, although the speed of the ball is low and the gravitational field is weak, the Newtonian trajectory rapidly disagrees completely with the general-relativistic trajectory if the trajectories are chaotic. This result raises a very interesting and crucial fundamental question: When the two theories, Newtonian and general-relativistic mechanics, predict completely different chaotic trajectories for a low-speed weak-gravity dynamical system, which trajectory prediction is empirically correct?

The chaotic bouncing ball, which we studied numerically in [6,7], is realizable [8] in the laboratory to test the different predictions of the two theories. In this paper, we address the important pragmatic question of precisely how accurate we need to know the parameters and initial conditions of a low-speed weak-gravity bouncing ball system so that sufficiently accurate Newtonian and general-relativistic chaotic trajectories can be calculated for comparison with the measured trajectory. Details of the bouncing ball system are given next, followed by the presentation and discussion of our finding.

Bouncing ball

In the bouncing ball system [7,8], a vertically-bouncing ball undergoes repeated impacts with a table which oscillates vertically in a sinusoidal fashion with amplitude A and angular frequency ω. The impact between the ball and the table is instantaneous and inelastic, where the coefficient of restitution α $(0 \leqslant \alpha < 1)$ measures the energy lost of the ball at each impact. The table is not affected by the impact because the table's mass is much larger than the ball's mass. In between impacts, the ball moves vertically in a uniform gravitational field since the distance traveled is much less than the Earth's radius. We will use the ball's velocity v and position y just after each impact to describe the motion of the bouncing ball.

In the Newtonian framework, the dynamics of the bouncing ball is exactly described by the following map [7,8]:

$$A[\sin(\theta_k)+1]+v_k\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right]-\frac{1}{2}g\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right]^2-A[\sin(\theta_{k+1})+1] = 0 \tag{1}$$

$$v_{k+1}=(1+\alpha)\omega A\cos(\theta_{k+1})-\alpha\left\{v_k-g\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right]\right\} \tag{2}$$

where v_k and θ_k are, respectively, the ball's velocity and the table's phase just after the kth impact, $g = GM/R^2$, M and R are, respectively, the mass and radius of the earth, and G is the gravitational constant.

In the general-relativistic framework, the dynamics of the bouncing ball is exactly described by the following map [7]:

* Corresponding author.

http://dx.doi.org/10.1016/j.rinp.2014.08.004
2211-3797/

$$A[\sin(\theta_k)+1]-A[\sin(\theta_{k+1})+1]-\frac{c^2}{2g}\left\{1-\frac{2g[R_{TLP}+A[\sin(\theta_k)+1]]}{c^2}\right\}\times\left\{1-\left\{\frac{1}{2}\left[(1+\beta_k)e^{-\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}+(1-\beta_k)e^{\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}\right]\right\}^{-2}\right\}=0 \quad (3)$$

$$v_{k+1}=\frac{-c^2\alpha\left(\frac{v'_{k+1}-u_{k+1}}{c^2-v'_{k+1}u_{k+1}}\right)+u_{k+1}}{1-\alpha u_{k+1}\left(\frac{v'_{k+1}-u_{k+1}}{c^2-v'_{k+1}u_{k+1}}\right)} \quad (4)$$

where the constant R_{TLP} is the distance between the table's lowest position and the center of the earth, c is the speed of light, $\beta_k = v_k/c$, $u_{k+1} = A\omega\cos(\theta_{k+1})$ is the table's velocity just after the $(k+1)$th impact, and

$$v'_{k+1}=c\left[\frac{(1+\beta_k)e^{-\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}-(1-\beta_k)e^{\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}}{(1+\beta_k)e^{-\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}+(1-\beta_k)e^{\frac{g}{c}\left(\frac{\theta_{k+1}-\theta_k}{\omega}\right)}}\right]$$

is the ball's velocity just before the $(k+1)$th impact.

The ball's vertical position (which is also the table's position) just after each impact can be calculated using the table's phase just after each impact via the simple relationship $y = A[\sin(\theta)+1]$. The impact phase maps, Eqs. (1) and (3), which are implicit algebraic equations for θ_{k+1}, must be solved numerically by finding the zero of the function on the left side of the equation, given θ_k and v_k. We used Brent's method for this purpose. Numerical accuracy of the solutions was carefully checked by varying the tolerances used in finding the zeroes. For a given set of parameters and initial conditions, the trajectory (Newtonian or general relativistic) is calculated twice with the maps, first in quadruple precision (35 significant figures) with a tolerance of 10^{-30} for the zeroes, then in quadruple precision with a smaller tolerance of 10^{-32} for the zeroes, to determine its accuracy. The accurate digits of the trajectory are the common digits of the 10^{-30}-tolerance trajectory and the 10^{-32}-tolerance trajectory. For example, if the position is 0.1234... in the 10^{-30}-tolerance calculation and 0.1239... in the 10^{-32}-tolerance calculation, the position is accurate to three significant figures, i.e., 0.123.

Results and discussion

We will now compare and discuss the results for two cases. For both cases, we used $g = 981$ cm/s^2, $c = 3\times10^{10}$ cm/s, and $R_{TLP} = 6.4\times10^8$ cm (mean radius of the Earth).

In Case 1, for parameters $\omega = (2\pi)\times 60$ Hz, $A = 0.012$ cm and $\alpha = 0.5$, initial conditions $y_0 = 0.02022$ cm and $v_0 = 8.17001$ cm/s, both the Newtonian and general-relativistic trajectories are chaotic. The magnitude of the difference between the two trajectories, which is exactly zero initially, grows exponentially with time (measured by impacts) causing the two trajectories to disagree completely after 55 impacts [7]. The Newtonian and general-relativistic trajectories after 55 impacts are:

	Position (cm)	Velocity (cm/s)
Newtonian	0.021708438663706886 9712430	8.03536779410858668 4123
General relativistic	0.0219472840845825046725434 0152550	7.8650447286064452986054 3925156

Only the common digits of the 10^{-30}-tolerance and the 10^{-32}-tolerance calculations are shown above for each quantity in each theory.

In Case 2, all the parameters and initial conditions differ from those in Case 1 by 10^{-13}. In this case, the Newtonian and general-relativistic trajectories are also chaotic. The trajectories after 55 impacts are:

	Position (cm)	Velocity (cm/s)
Newtonian	0.0217044307152067058 3226151	8.03812036336526892 257903
General relativistic	0.021944037676348321062985 73952785	7.86744686246560234743 196250478

Only the common digits of the 10^{-30}-tolerance and the 10^{-32}-tolerance calculations are shown above for each quantity in each theory.

Let us assume that the parameters and initial conditions in Case 1 are the *actual* ones, and the parameters and initial conditions in Case 2 are the *measured* ones. In other words, the measurement error is 10^{-13} for all the parameters and initial conditions. Comparison of the trajectories in Case 2 with the trajectories in Case 1 shows that the trajectories calculated using the measured parameters and initial conditions in Case 2 are only accurate to four and three significant figures, respectively, for the position and velocity after 55 impacts:

	Position (cm)	Velocity (cm/s)
Newtonian	0.02170	8.03
General relativistic	0.02194	7.86

However, if the measurement uncertainty is greater than 10^{-13} (e.g., 10^{-12}, 10^{-11}, ...) for all the parameters and initial conditions, the trajectories calculated using the measured parameters and initial conditions bear no resemblance to the trajectories calculated using the actual parameters and initial conditions in Case 1 after 55 impacts.

Conclusion

The example in the previous section shows that the parameters and the initial position and velocity of a low-speed weak-gravity bouncing ball system must be known to very high accuracies – measurement uncertainties are at most 10^{-13} – so that sufficiently accurate Newtonian and general-relativistic chaotic trajectories can be calculated in order to test which of the two trajectories is empirically correct when they disagree. Whether such accuracies can be achieved experimentally remains to be studied.

References

[1] Moon FC. Chaotic and fractal dynamics: an introduction for applied scientists and engineers. Wiley-Interscience; 1992.
[2] Strogatz SH. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry and engineering. Perseus Books; 2001.
[3] Einstein A. Relativity: the special and the general theory. Random House; 1961.
[4] Lapidus IR. Am J Phys 1972;40:1509–10.
[5] Wald RM. General relativity. University of Chicago Press; 1984.
[6] Liang SN, Lan BL. Results Phys 2011;1:36–9.
[7] Liang SN, Lan BL. PLOS One 2012;7(4):e34720.
[8] Tufillaro NB, Abbott T, Reilly J. An experimental approach to nonlinear dynamics and chaos. Addison-Wesley; 1992.