



# Generalized polynomials, operational identities and their applications

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## Abstract

It is shown that an appropriate combination of methods, relevant to generalized operational calculus and to special functions, can be a very useful tool to treat a large body of problems both in physics and mathematics. We discuss operational methods associated with multivariable Hermite, Laguerre, Legendre, and other polynomials to derive a wealth of identities useful in quantum mechanics, electromagnetism, optics, etc., or to derive new identities between special functions as, e.g., Mehler- or mixed-type generating functions. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The Glaisher operational rule [18]:

$$e^{\alpha \partial^2 / \partial x^2} e^{-\beta x^2} = \frac{1}{\sqrt{1 + 4\alpha\beta}} e^{-\beta x^2 / (1 + 4\alpha\beta)} \tag{1}$$

is a genuine example of how operational calculus applies to the solution of the heat equation

$$\begin{aligned} \frac{\partial}{\partial \alpha} f(x, \alpha) &= \frac{\partial^2}{\partial x^2} f(x, \alpha), \\ f(x, 0) &= e^{-\beta x^2}. \end{aligned} \tag{2}$$

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The proof of (1) is fairly straightforward and is based on the Gauss transform

$$f(x, \alpha) = \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4\alpha} f(\xi, 0) d\xi. \quad (3)$$

In this paper we will show that identities of type (1) can be extended to more complicated relations involving products of Gaussians and of Hermite or Laguerre polynomials.

The first example we consider is

$$\phi_n(x, \alpha) = e^{\alpha\partial^2/\partial x^2} (H_n(x)e^{-\beta x^2}), \quad (4)$$

where  $H_n(x)$  are the ordinary Hermite polynomials (HP), specified by the series

$$H_n(x) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r (2x)^{n-2r}}{r!(n-2r)!} \quad (5)$$

and by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt-t^2}. \quad (6)$$

By multiplying both sides of (4) by  $t^n/n!$ , by summing up over  $n$ , and by exploiting (6) and the Gauss transform, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \phi_n(x, \alpha) = \frac{1}{\sqrt{1+4\alpha\beta}} e^{2xt/(1+4\alpha\beta)-t^2(1-4\alpha/(1+4\alpha\beta))} e^{-\beta x^2/(1+4\alpha\beta)}. \quad (7)$$

By expanding the r.h.s. of Eq. (7) in terms of Kampé de Fériet polynomials (KDFP) [10]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) = e^{xt+yt^2}, \quad H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \quad (8)$$

and by equating the like  $t$  power terms, between both sides of Eq. (7), we end up with

$$\phi_n(x, \alpha) = \frac{1}{\sqrt{1+4\alpha\beta}} e^{-\beta x^2/(1+4\alpha\beta)} H_n\left(\frac{2x}{1+4\alpha\beta}, \frac{4\alpha(1-\beta)-1}{1+4\alpha\beta}\right). \quad (9)$$

This last identity is an example of how two-variable HP of the KDF type, can be employed within the framework of an operational solution of the heat equation.

The use of the  $H_n(x, y)$  is not strictly necessary for the solution of Eq. (9), which can be rewritten, albeit in a more involuted form, in terms of ordinary HP only. It is therefore worth considering a case in which the use of generalized HP forms is crucial. The problem we will discuss is

$$\phi_{n,m}(x, \alpha) = e^{\alpha\partial^2/\partial x^2} H_n(\gamma x + \delta) H_m(\tau x + \lambda) e^{-\beta x^2}. \quad (10)$$

The procedure we follow is the same as before: we multiply indeed both sides of (9) by  $t^n/n!$  and by  $u^m/m!$ , sum up over the two indices and eventually get

$$\phi_{n,m}(x, \alpha) = \frac{1}{\sqrt{1+4\alpha\beta}} e^{-\beta x^2/(1+4\alpha\beta)} H_{n,m}(A, B; C, D | F),$$

$$A = 2 \left( \delta + \frac{x\gamma}{1+4\alpha\beta} \right), \quad B = \frac{4\alpha(\gamma^2 - \beta) - 1}{1+4\alpha\beta},$$

$$C = 2 \left( \lambda + \frac{x\tau}{1 + 4\alpha\beta} \right), \quad D = \frac{4\alpha(\tau^2 - \beta) - 1}{1 + 4\alpha\beta},$$

$$F = \frac{8\alpha\gamma\tau}{1 + 4\alpha\beta}, \tag{11}$$

where we have denoted by  $H_{n,m}$  the two-index, four-variable, one-parameter Hermite polynomials specified by [3]

$$H_{n,m}(A, B; C, D | F) = n!m! \sum_{r=0}^{\min(m,n)} \frac{F^r H_{n-r}(A, B) H_{m-r}(C, D)}{r!(n-r)!(m-r)!},$$

$$\sum_{n,m=0}^{\infty} \frac{t^n u^m}{n! m!} H_{n,m}(A, B; C, D | F) = e^{At+Br^2+Cu+Du^2+Ftu}. \tag{12}$$

The previous examples show that it is fairly simple to use the wealth of HP like forms to state nontrivial generalizations of the operational rule (1), useful to treat problems involving (e.g.) Fokker–Plank and Schrödinger-type equations.

Let us, furthermore, note that a consequence of the Gauss transform is the following important identity:

$$e^{y\partial^2/\partial x^2}(x^n) = H_n(x, y), \tag{13}$$

which can be easily proved by using the already exploited method of generating functions.

Two-variable Laguerre polynomials have been introduced in [1] and are specified by the series

$$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{r=0}^n \frac{y^{n-r} \Gamma(n + \alpha + 1)(-x)^r}{(n-r)!r!\Gamma(r + \alpha + 1)} \tag{14}$$

and by the generating function

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n^{(\alpha)}(x, y) = \frac{1}{(1-yt)^{\alpha+1}} e^{-xt/(1-yt)}. \tag{15}$$

According to this last identity and Eq. (1), we can also conclude that

$$e^{\alpha\partial^2/\partial x^2} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} x^{2n} = \sum_{n=0}^{\infty} (-\beta)^n \mathcal{L}_n^{(-1/2)}(-x^2, 4\alpha), \tag{16}$$

which, on account of identity (13), yields

$$H_{2n}(x, \alpha) = n! \mathcal{L}_n^{(-1/2)}(-x^2, 4\alpha), \tag{17}$$

as a further example of the usefulness of the method we are discussing.

The points, we have just touched on, are essentially introductory examples aimed at providing the feeling of how nonstandard polynomials and operational rules can be combined to treat problems of different nature, both in pure and applied mathematics. In the following sections we will treat more deeply these topics and discuss various problems whose solution is greatly simplified by the use of the methods based on an appropriate embedding of operational rules and of the properties of the generalized special functions.

## 2. Operational rules and generating functions

The Burchnell rule [8] is a fairly interesting relation, involving differential operators, which may be exploited in different contexts. Below we present some generalizations of this identity, along with some applications.

The following relation, reducing to the original Burchnell identity for  $\beta = 2$  and  $\alpha = -1$ :

$$T_n(x; \alpha, \beta) = \left( \alpha \frac{d}{dx} + \beta x \right)^n = \sum_{s=0}^n \binom{n}{s} \alpha^s H_{n-s} \left( \beta x, \frac{1}{2} \alpha \beta \right) \frac{d^s}{dx^s} \tag{18}$$

can be proved by exploiting the generating function method and elementary decoupling theorems; we get indeed

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n(x; \alpha, \beta) = e^{zt \, d/dx + \beta tx}. \tag{19}$$

The exponential on the r.h.s. of Eq. (19) contains noncommuting operators, therefore the use of the Weyl decoupling identity [15]

$$e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-k/2}, \tag{20}$$

if  $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} - \hat{B}\hat{A} = k, k \in C$

yields

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n(x; \alpha, \beta) = e^{zt \, d/dx + \beta tx} = e^{\beta tx + (1/2)\alpha\beta t^2} e^{zt \, d/dx} = \sum_{r=0}^{\infty} \frac{t^r}{r!} H_r \left( \beta x, \frac{1}{2} \alpha \beta \right) \sum_{s=0}^{\infty} \frac{(\alpha t)^s}{s!} \frac{d^s}{dx^s}, \tag{21}$$

which, after rearranging the summations and comparing the coefficients with the same  $t$ -powers, provides Eq. (18).

Let us now consider the possibility of extending identities of the above type. Before discussing this aspect of the problem, we introduce the three-variable Hermite polynomials and a more general decoupling theorem.

(a) The polynomials  $H_n(x, y, z)$  are specified by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z) = e^{xt + yt^2 + zt^3},$$

$$H_n(x, y, z) = n! \sum_{r=0}^{[n/3]} \frac{z^r H_{n-3r}(x, y)}{r!(n-3r)!}. \tag{22}$$

(b) If  $\hat{A}, \hat{B}$  are noncommuting operators such that

$$[\hat{A}, \hat{B}] = m \hat{A}^{1/2}, \quad m \in C \tag{23}$$

then, the following decoupling identity holds [8]:

$$e^{\hat{A} + \hat{B}} = e^{m^2/12 - (m/2)\hat{A}^{1/2} + \hat{A}} e^{\hat{B}}. \tag{24}$$

The use of the relations given above and the procedure, which we have outlined for the proof of the previous example, allow to state the identity

$$\left(x^2 + \alpha \frac{d}{dx}\right)^n = \sum_{r=0}^n \binom{n}{r} \alpha^r H_{n-r} \left(x^2, \alpha x, \frac{\alpha^2}{3}\right) \frac{d^r}{dx^r}, \tag{25}$$

which is a further example of the close interplay between operational rules and nonstandard forms of special functions.

The case we are going to describe is more general than the examples we have given so far and involves polynomials with many variables and many indices.

We consider indeed the case [6]

$$T_{m,n}(x, y; a, b, c) = \left(ax + by - \frac{\partial}{dx}\right)^m \left(bx + cy - \frac{\partial}{dy}\right)^n, \tag{26}$$

where

$$a, c > 0, \quad \Delta = ac - b^2 > 0. \tag{27}$$

We can adopt a slight extension of the previous procedure to get

$$\begin{aligned} \sum_{(m,n)=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} T_{m,n}(x, y; a, b, c) &= e^{t(ax+by-\partial/\partial x)} e^{u(bx+cy-\partial/\partial y)} \\ &= e^{(ax+by)t+(bx+cy)u-(1/2)(at^2+2but+cu^2)} e^{-t\partial/\partial x} e^{-u\partial/\partial y}. \end{aligned} \tag{28}$$

By recalling that [3]

$$e^{(ax+by)t+(bx+cy)u-(1/2)(at^2+2but+cu^2)} = \sum_{(m,n)=0}^{\infty} \frac{t^m}{m!} \frac{u^n}{n!} H_{m,n}(x, y), \tag{29}$$

we end up with the identity

$$T_{m,n}(x, y; a, b, c) = \sum_{r=0}^m \sum_{s=0}^n \binom{m}{r} \binom{n}{s} (-1)^{r+s} H_{m-r,n-s}(x, y) \frac{\partial^{r+s}}{\partial x^r \partial y^s}, \tag{30}$$

which is a two-variable two-index generalization of the one-variable one-index Burchnall identity.

It is worth noting that the  $H_{m,n}(x, y)$  belong to the same family of the polynomials given in (12), and can be explicitly rewritten as

$$H_{m,n}(x, y) = H_{m,n}(ax + by, -\frac{1}{2}a; bx + cy, -\frac{1}{2}c \mid -b). \tag{31}$$

In addition, by generalizing the operational rule (13), we write

$$e^{-(1/2\Delta)(c\partial^2/\partial x^2 - 2b\partial^2/\partial x\partial y + a\partial^2/\partial y^2)} [(ax + by)^m (bx + cy)^n] = H_{m,n}(x, y). \tag{32}$$

This fairly important result has been exploited in many different applications. We note that the operator appearing in the argument of the exponential can be viewed as a generalized Laplacian.

Relations of the above type can, therefore, be utilized in dealing with multivariable Fokker–Plank equations.

Just to give an example, we note that the operational rules and the formalism we have developed can be exploited to evaluate integrals of the type:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{\infty} dy e^{-(1/2)(ax^2+2bxy+cy^2)}(x + \alpha)^m(y + \beta)^n = \frac{2\pi}{\sqrt{\Delta}} H_{m,n} \left( \alpha, \frac{c}{2\Delta}; \beta, \frac{a}{2\Delta} \middle| -\frac{b}{\Delta} \right). \tag{33}$$

This last result is general enough and can be exploited in different contexts involving, e.g., coupled harmonic oscillator problems.

An interesting by-product of the present formalism is the derivation of Mehler-type generating functions. We consider indeed the following example:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y) H_n(z, w) &= e^{y\partial^2/\partial x^2 + w\partial^2/\partial z^2} (e^{tx}) \\ &= \frac{1}{\sqrt{1 - 4t^2 yw}} e^{(xzt + t^2(wx^2 + yz^2))/(1 - 4t^2 yw)}, \quad |t| < \frac{1}{2\sqrt{yw}}, \end{aligned} \tag{34}$$

which is a fairly straightforward consequence of identity (13) and of the Gauss transform.

### 3. Operational formalism and mixed generating functions

In the previous section, we have dealt with ordinary generating functions. Here, we will refer to mixed generating functions (MGF) for cases involving sums of the type

$$S(x, y; t) = \sum_n t^n f_n(x + ny) \tag{35}$$

or their suitable generalizations.

The theory of MGF has been pioneered by Carlitz [2] and Srivastava [5], who employed the Lagrange expansion as the essential tool to develop a unifying point of view on the problem and to derive families of MGF in a fairly direct way (see also [18]). In the following we will consider generalization of the results quoted (e.g.) in [18] (and the references therein) by employing the two-variable Lagrange expansion [14]

$$\begin{aligned} \frac{h[u(s, t), v(s, t), s, t]}{\Delta(u(s, t), v(s, t), s, t)} &= \sum_{m,n=0}^{\infty} \frac{s^m t^n}{m! n!} \left\{ \frac{\partial^{m+n}}{(\partial x)^m (\partial y)^n} [h(x, y, s, t)(f(x, y))^m (g(x, y))^n] \right\}_{x=y=0} \\ u(s, t) &= sf(u(s, t), v(s, t)), v(s, t) = tg(u(s, t), v(s, t)), \\ \Delta(x, y, s, t) &= \left(1 - s \frac{\partial}{\partial x} f(x, y)\right) \left(1 - t \frac{\partial}{\partial y} g(x, y)\right) - ts \left[ \frac{\partial}{\partial y} f(x, y) \right] \left[ \frac{\partial}{\partial x} g(x, y) \right], \end{aligned} \tag{36}$$

on the basis of the above expansion, we can easily obtain the following mixed generating function (MGF) [4]:

$$\sum_{m,n=0}^{\infty} \frac{u^m t^n}{m! n!} S_{m,n}(x + mw, y + nz) = \frac{e^{\psi_1(ax+by) + \psi_2(bx+cy)}}{1 - aw\psi_1 - cz\psi_2 + \Delta w z \psi_1 \psi_2}$$

$$S_{m,n}(x, y) = (ax + by)^m (bx + cy)^n, \quad \Delta = ac - b^2,$$

$$\psi_1 = ue^{(a\psi_1 + b\psi_2)w}, \quad \psi_2 = te^{(b\psi_1 + c\psi_2)z}. \tag{37}$$

It is quite natural to apply Eq. (32) and derive from Eq. (37) the MGF for two-variable two-index Hermite polynomials

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{t^n}{n!} H_{m,n}(x + mw, y + nz) = \frac{e^{\psi_1(ax+by) + \psi_2(bx+cy) - \frac{1}{2}(a\psi_1^2 + 2b\psi_1\psi_2 + c\psi_2^2)}}{1 - aw\psi_1 - cz\psi_2 + \Delta w z \psi_1 \psi_2}. \tag{38}$$

To provide a further example, we also consider the polynomials  $h_{m,n}(x, y; \tau)$  specified by

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} h_{m,n}(x, y; \tau) = e^{xt + yu + \tau uv},$$

$$h_{m,n}(x, y; \tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s x^{m-s} y^{n-s}}{s!(m-s)!(n-s)!} \tag{39}$$

and by the operational rule

$$e^{\tau \partial^2 / \partial x \partial y} (x^m y^n) = h_{m,n}(x, y; \tau). \tag{40}$$

The combination of Eqs. (36) and (40) yields the following MGF:

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} h_{m,n}(x + \alpha m + \beta n, y + \gamma m + \delta n; \tau) = \frac{e^{x\psi_1 + y\psi_2 + \tau\psi_1\psi_2}}{1 - \alpha\psi_1 - \delta\psi_2 + E\psi_1\psi_2},$$

$$E = \alpha\delta - \beta\gamma,$$

$$\psi_1 = ue^{\alpha\psi_1 + \beta\psi_2}, \quad \psi_2 = ve^{\gamma\psi_1 + \delta\psi_2} \tag{41}$$

and

$$\sum_{m,n=0}^{\infty} \frac{u^m}{m!} \frac{v^n}{n!} (x + \alpha m + \beta n)^{(m+n)/2} h_{m,n} \left( \sqrt{x + \alpha m + \beta n}, \sqrt{x + \alpha m + \beta n}; \tau \right)$$

$$= \frac{e^{(\psi_1 + \psi_2 + \tau\psi_1\psi_2)x}}{1 - \alpha\psi_1 - \beta\psi_2 - \tau(\alpha + \beta)\psi_1\psi_2},$$

$$\psi_1 = ue^{\alpha(\psi_1 + \psi_2 + \tau\psi_1\psi_2)}, \quad \psi_2 = ve^{\beta(\psi_1 + \psi_2 + \tau\psi_1\psi_2)}. \tag{42}$$

This last relation generalizes an analogous result due to Srivastava [17,18] and is valid for the ordinary Hermite polynomials.

So far we have discussed the case of Hermite polynomials and associated generalized forms. We will discuss MGF associated with  $H_{m,n}(x, y)$  polynomials in the concluding section. Here we will consider the use of operational methods to derive new families of MGF involving Laguerre polynomials. To this aim, we remind that the operational rules defining the associated Laguerre polynomials yield [12]

$$L_n^{(m)}(x) = \left(1 - \frac{d}{dx}\right)^m (1 - \mathcal{D}_x^{-1})^n, \tag{43}$$

where  $\mathcal{D}_x^{-1}$  is the inverse of the derivative operator. By recalling that

$$\mathcal{D}_x^{-n} = \frac{x^n}{n!}, \quad e^{\lambda d/dx} f(x) = f(x + \lambda), \tag{44}$$

we can use Eq. (43) to derive the following generating function involving both indices  $m, n$ :

$$\sum_{(m,n)=0}^{\infty} \frac{u^m t^n}{m! n!} L_n^{(m)}(x) = e^{u(1-d/dx)} e^{t(1-\mathcal{D}_x^{-1})} = e^{u+t} C_0[(x-u)t], \quad C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}. \tag{45}$$

The same procedure can be exploited to prove that

$$\sum_{(m,n)=0}^{\infty} \frac{u^m t^n}{m! n!} L_n^{(m)}(x, y, z) = e^{zu+yt} C_0((x-u)t), \tag{46}$$

$$L_n^{(m)}(x, y, z) = \sum_{q=0}^m \sum_{r=0}^n \binom{m}{q} \binom{n}{r} (z-1)^q (y-1)^r L_{n-r}^{(m-q)}(x),$$

by combining Eq. (46) and the two-variable Lagrange expansion, we are able to derive the following MGF:

$$\begin{aligned} \sum_{(m,n)=0}^{\infty} \frac{u^m t^n}{m! n!} L_n^{(m)}(x, y + \alpha m + \beta n, z + \gamma m + \delta n) \\ = \frac{e^{z\psi_1 + y\psi_2}}{1 - \gamma\psi_1 - \beta\psi_2 + \Delta\psi_1\psi_2} C_0[(x - \psi_1)\psi_2], \quad \Delta = \gamma\beta - \alpha\delta, \end{aligned} \tag{47}$$

$$\psi_1 = ue^{\gamma\psi_1 + \alpha\psi_2}, \quad \psi_2 = te^{\delta\psi_1 + \beta\psi_2}.$$

The results obtained so far will be complemented by further considerations developed in the concluding section where, among other things, we will extend the methods we have developed to the derivation of higher-order generating functions.

### 4. Higher-order generating functions

Higher-order generating functions (HOGF) involve sums of the type

$$S_j(x, t) = \sum_n t^{jn} f_{jn}(x), \quad j \in \mathcal{N} \tag{48}$$

and have been recently discussed in [7,13,16]. Their importance stems from their use in the theory of squeezed states in quantum optics and in classical electromagnetism. Initially they were proved for Hermite polynomials only [13,16]. Subsequently, they have been extended to all the families of special functions possessing a generating function [7].

The derivation of HOGF is greatly simplified by the use of operational methods. It is indeed well known that

$$\sum_{n=0}^{\infty} \frac{(tx)^{jn}}{(jn)!} = \frac{1}{j} \sum_{k=1}^j e^{x\tau_k}, \quad \tau_k = te^{2\pi ik/j}, \tag{49}$$



which, according to Eq. (13), yields

$$\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} H_{jn}(x, y) = \frac{1}{j} \sum_{k=1}^j e^{x\tau_k + y\tau_k^2}. \tag{50}$$

The above result can be further generalized to get HOMGF. From Eqs. (49) and (13) and from the Lagrange expansion, we find that

$$\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} H_{jn}(x + jnz, y) = \frac{1}{j} \sum_{k=1}^j \frac{e^{xw_k + yw_k^2}}{1 - zw_k}, \quad w_k = \tau_k e^{zw_k}. \tag{51}$$

Just to provide a further example aimed at stressing the effectiveness of the operational methods, we note that from Eqs. (32) and (49) we are able to derive the two-variable HOGF as follows [9]:

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{u^{km}}{(km)!} \frac{v^{jn}}{(jn)!} H_{km,jn}(x, y) = \frac{1}{kj} \sum_{r=1}^k \sum_{l=1}^j e^{z^T \hat{M} W_{r,l} - \frac{1}{2} W_{r,l}^T \hat{M} W_{r,l}}, \tag{52}$$

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad W_{r,l} = \begin{pmatrix} ue^{i2\pi r/k} \\ ve^{i2\pi l/j} \end{pmatrix}.$$

We have used a matrix notation to have a more compact form; note that the superscript T denotes transpose. The extension to the mixed case can be considered trivial and left out for the sake of conciseness. For further comments, the reader is referred to [9].

A noticeable example involving the use of operational rules is provided by sums of the type

$$S_{(m)} = \sum_{n=0}^{\infty} \frac{a_n}{[n/m]!}. \tag{53}$$

We note indeed that [11]

$$\sum_{n=0}^{\infty} \frac{x^n}{[n/m]} = \frac{x^m - 1}{x - 1} e^{x^m}, \tag{54}$$

which, according to Eq. (9), yields

$$e^{y\partial^2/\partial x^2} \sum_{n=0}^{\infty} \frac{t^n}{[n/2]!} x^n = \frac{[1 + xt - 4yt^2]}{(1 - 4yt^2)^{\frac{3}{2}}} e^{(xt)^2/(1-4yt^2)}. \tag{55}$$

The case with  $m > 2$  deserves particular care. We limit ourselves to  $m = 3$ , but the consideration we develop is general. We recall the following identity [8]:

$$e^{y\partial^m/\partial x^m} (f(x)g(x)) = f\left(x + my \frac{\partial^{m-1}}{\partial x^{m-1}}\right) e^{y\partial^m/\partial x^m} (g(x)). \tag{56}$$

It is, therefore, evident that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{[n/3]!} H_n(x, y) &= e^{y\partial^2/\partial x^2} \sum_{n=0}^{\infty} \frac{x^n}{[n/3]!} = e^{y\partial^2/\partial x^2} [(x^2 + x + 1)e^{x^3}] \\ &= \left[ \left(x + 2y \frac{\partial}{\partial x}\right)^2 + \left(x + 2y \frac{\partial}{\partial x}\right) + 1 \right] ({}_H e_3(x, y)). \end{aligned} \tag{57}$$

The function  ${}_H e_3(x, y)$  is a Hermite-based exponential function defined as

$$({}_H e_3(x, y)) = e^{y\partial^2/\partial x^2}(e^{x^3}) = e^{y\partial^2/\partial x^2} \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{H_{3n}(x, y)}{n!}, \tag{58}$$

it satisfies the differential equation

$$\left[ 12y^2 \frac{\partial^2}{\partial x^2} + (12xy - 1) \frac{\partial}{\partial x} + 3(x^2 + 2y) \right] ({}_H e_3(x, y)) = 0 \tag{59}$$

and the role played by Hermite-based functions will be discussed it in the next section.

### 5. Concluding remarks

The introductory remarks of this paper have essentially been devoted to the heat equation (2). In the following we will consider the fairly straightforward generalization:

$$\begin{aligned} \frac{\partial}{\partial \alpha} f(x, \alpha) &= \frac{\partial^m}{\partial x^m} f(x, \alpha), \\ f(x, 0) &= g(x). \end{aligned} \tag{60}$$

If  $g(x) = x^n$ , the natural solutions of Eq. (60) are

$$\begin{aligned} f(x, \alpha) &= H_n^{(m)}(x, \alpha), \\ H_n^{(m)}(x, \alpha) &= n! \sum_{r=0}^{[n/m]} \frac{\alpha^r x^{n-mr}}{r!(n-mr)!}. \end{aligned} \tag{61}$$

The polynomials  $H_n^{(m)}(x, \alpha)$  are further Hermite generalized form [8,10] and are specified by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x, \alpha) = e^{xt+\alpha t^m}. \tag{62}$$

If  $f(x)$  is not a polynomial, but can be expressed through any polynomial expansion, namely

$$g(x) = \sum_n g_n x^n, \tag{63}$$

we can conclude

$$\begin{aligned} f(x, \alpha) &= {}_H g_m(x, \alpha), \\ {}_H g_m(x, \alpha) &= \sum_n g_n H_n^{(m)}(x, \alpha). \end{aligned} \tag{64}$$

Functions of the type  $({}_H g_m(x, \alpha))$  are called *Hermite-based functions* [5] and are obtained, e.g., from the ordinary functions, after replacing the ordinary monomial with the corresponding Hermite

polynomial ( $x^n \mapsto H_n^{(m)}(x, y)$ ), in their Taylor expansion. The importance of this class of functions stems from the fact that the HP are quasi-monomials under the action of the multiplication ( $\hat{M}$ ) and derivative ( $\hat{P}$ ) operators

$$\hat{M} = x + my \frac{\partial^{m-1}}{\partial x^{m-1}},$$

$$\hat{P} = \frac{\partial}{\partial x}. \quad (65)$$

We find indeed

$$\hat{M}H_n^{(m)}(x, y) = H_{n+1}^{(m)}(x, y),$$

$$\hat{P}H_n^{(m)}(x, y) = nH_{n-1}^{(m)}(x, y). \quad (66)$$

The remarks clarify the role and the meaning of the function ( ${}_H e_3(x, y)$ ) introduced in Eq. (57).

It is quite natural to ask whether other families of polynomials satisfy properties of quasi-monomiality. A further elementary example is provided by the two-variable Laguerre polynomials [12] specified by the generating function

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n(x, y) = \frac{1}{1-yt} e^{-xt/(1-yt)},$$

$$\mathcal{L}_n(x, 1) = L_n(x), \quad (67)$$

which behaves as quasi-monomials under the action of

$$\hat{M} = y - \mathcal{D}_x^{-1},$$

$$\hat{P} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}. \quad (68)$$

Most of the properties of  $\mathcal{L}_n(x, y)$  as well as of  $L_n(x)$  can be derived from monomiality conditions of type (66). A remarkable aspect is the possibility of constructing all the quasi-monomials from the  $n$ th-power of the multiplication operator. In the case of Laguerre polynomials, we get the operational identity

$$\mathcal{L}_n(x, y) = (y - \mathcal{D}_x^{-1})^n = n! \sum_{r=0}^n \frac{(-1)^r y^{n-r} x^r}{(r!)^2 (n-r)!}. \quad (69)$$

A further example of quasi-monomials is provided by the Legendre polynomials or better by a generalized class of Legendre's. We consider indeed the case

$${}_2L_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{y^{n-2r} x^r}{(r!)^2 (n-2r)!}. \quad (70)$$

This class of polynomials shares analogies with Laguerre and Hermite. According to the previous discussion, we can write (70) in the operational form

$${}_2L_n(x, y) = H_n(y, \mathcal{D}_x^{-1}). \quad (71)$$

which suggests that this class of polynomials behaves as quasi-monomials under the action of

$$\hat{M} = y + 2D_x^{-1} \frac{\partial}{\partial y},$$

$$\hat{P} = \frac{\partial}{\partial y}. \quad (72)$$

It is fairly straightforward to exploit the monomiality recurrences to infer the properties of  ${}_2L_n(x, y)$ , a few of which are listed below:

$$\frac{\partial^2}{\partial y^2}({}_2L_n(x, y)) = \frac{\partial}{\partial x} x \frac{\partial}{\partial x}({}_2L_n(x, y)),$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}({}_2L_n(x, y)) = e^{yt} C_0(-xt^2),$$

$$\sum_{n=0}^{\infty} t^n({}_2L_n(x, y)) = \frac{1}{[1 + (y^2 - 4x)t^2 - 2yt]^{1/2}}, \quad (73)$$

where  $C_0(x)$  is defined in Eq. (45). Furthermore, it is worth noting that

$${}_2L_n(-\frac{1}{4}(1 - y^2)), \quad y = P_n(y), \quad (74)$$

where  $P_n(y)$  are the ordinary Legendre polynomials [1].

In this paper we have covered a large number of topics. We have seen that methods based on operational identities may provide powerful tools to deal with the possibilities offered by generalized forms of ordinary polynomials. Further progress in this direction can be made by exploiting the wealth of identities associated with the theory of generalized shift operators [8]. Within such a context, new polynomial structures emerge with wide possibilities of applications in physics and engineering.

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