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Existence of strong solutions and global attractors for the coupled suspension bridge equations ☆

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ABSTRACT

In this paper, we show the existence of the strong solutions for the coupled suspension bridge equations. Furthermore, existence of the strong global attractors is investigated using a new semigroup scheme. Since the solutions of the coupled equation have no higher regularity and the semigroup associated with the solutions is not continuous in the strong Hilbert space, the results are new and appear to be optimal.

1. Introduction

In the paper, we consider the following system which describes the vibrating beam equation coupled with a vibrating string equation

$$\begin{cases} u_{tt} + \alpha u_{xxxx} + \delta_1 u_t + k(u-v)^+ + f_B(u) = h_B, & \text{in } [0,L]\times\mathbb{R}^+, \\ v_{tt} - \beta v_{xx} + \delta_2 v_t - k(u-v)^+ + f_S(v) = h_S, & \text{in } [0,L]\times\mathbb{R}^+ \end{cases} \tag{1.1}$$

with the simply supported boundary conditions at both ends

$$\begin{gathered} u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \geqslant 0, \\ v(0,t) = v(L,t) = 0, \quad t \geqslant 0, \end{gathered} \tag{1.2}$$

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and the initial-value conditions

$$\begin{aligned} &u(x,0)=u_0, \qquad u_t(x,0)=u_1,\\ &v(x,0)=v_0, \qquad v_t(x,0)=v_1, \quad x\in[0,L], \end{aligned} \tag{1.3}$$

where the first equation of (1.1) represents the vibration of the road bed in the vertical direction and the second equation describes that of the main cable from which the road bed is suspended by the tie cables (see [10]). $k>0$ denotes the spring constant of the ties, $\alpha>0$ and $\beta>0$ are the flexural rigidity of the structure and coefficient of tensile strength of the cable, respectively. $\delta_1,\delta_2>0$ are constants, $h_B, h_S \in L^2(0,L)$.

We assume that the nonlinear functions $f_B \in C^3(\mathbb{R},\mathbb{R})$ and $f_S \in C^2(\mathbb{R},\mathbb{R})$ satisfy the following conditions:

$$\text{(F1)} \quad \liminf_{|\tau|\to\infty}\frac{f_B(\tau)}{\tau}\geqslant\delta, \qquad \liminf_{|\tau|\to\infty}\frac{f_S(\tau)}{\tau}\geqslant\delta;$$

$$\text{(F2)} \quad \big|f_B(\tau)\big|, \big|f_S(\tau)\big| \leqslant C_0\big(1+|\tau|^p\big), \quad \forall p\geqslant 1,$$

for any $\tau\in\mathbb{R}$, where C_0, δ are positive constants.

Just for the first equation in (1.1), it is originally in [1] introduced by Lazer and McKenna as the new problems in fields of nonlinear analysis. Lately, similar models have been studied by many authors, but most of them have only concentrated on the existence of solutions, see [9,10], while the existence of the global attractors for the suspension bridge equations are most of our concern. In [11], we firstly obtained the global attractors of the weak solutions for the suspension bridge equations. Recently, the existence of the strong solutions and the strong global attractors have also been achieved in [2].

For the coupled suspension bridge equations, Ahmed and Harbi studied (1.1) in [10], and pointed out that the system is conservative and asymptotically stable with respect to the rest state for $k>0$, and showed that the Cauchy problem of system (1.1) has at least one weak solution. In 2004, G. Litcanu and J. Malik have also proposed and studied the similar models in [5,6]. G. Holubová and A. Matas in [4] considered the initial–boundary value problem for the more general nonlinear string–beam system and obtained the existence and uniqueness of the weak solution by the Faedo–Galerkin method. In [12], we firstly proved the existence of the global attractors of the weak solutions for (1.1). To the best of our knowledge, however, the existence of the strong solutions and the strong global attractors for (1.1) are still not studied, it is just our interest in this paper. For proper k, which will be given in Section 3, we will firstly establish the existence of strong solutions based on the standard Faedo–Galerkin methods, then discuss the compact attractors of the strong solutions by making use of the condition (C) introduced in [3,13] and combining with the techniques of energy estimates. Especially, our results are hard to be improved because (1.1) have no higher regularity and the solution semigroup associated with (1.1) is not continuous in a strong Hilbert space. So our results appear to be optimal. On the other hand, the following assumptions, namely, there exists $C>0$ such that

$$\liminf_{|\tau|\to\infty}\frac{\tau f_B(\tau)-CF_B(\tau)}{\tau^2}\geqslant 0, \qquad \liminf_{|\tau|\to\infty}\frac{\tau f_S(\tau)-CF_S(\tau)}{\tau^2}\geqslant 0,$$

where $F_B(\tau)=\int_0^\tau f_B(r)\,dr$, $F_S(\tau)=\int_0^\tau f_S(r)\,dr$, used in [2,11,13,14] for nonlinearity of usual wave equations can be deleted in this paper.

2. Preliminaries

With the usual notation, we denote

$$Y_0 = L^2(0,L), \qquad Y_1 = H_0^1(0,L), \qquad Y_2 = D(A) = H^2(0,L) \cap H_0^1(0,L),$$
$$Y_3 = D(A^2) = \{u \in H^2(0,L) \mid A^2 u \in L^2(0,L)\},$$

where $A = -\frac{\partial^2}{\partial x^2}$, $A^2 = \frac{\partial^4}{\partial x^4}$. And we introduce some spaces V_1, V_2 which are used throughout the paper, that is

$$V_1 = Y_0 \times Y_0, \qquad V_2 = Y_2 \times Y_1,$$

and endow space V_1 with the usual scalar product and norm, $(\cdot,\cdot)$, $|\cdot|$, namely, for any $u = (u^1, u^2)$, $v = (v^1, v^2)$, denote

$$(u, v) = \int_0^L (u^1 v^1 + u^2 v^2)\, dx, \qquad |u|^2 = |u^1|_{L^2}^2 + |u^2|_{L^2}^2.$$

Especially, $(\cdot,\cdot)$ and $|\cdot|$ also denote the scalar product and the norm of $L^2(0,L)$. We can also define the scalar product $((\cdot,\cdot))$ and norm $\|\cdot\|$ in V_2, i.e.

$$((u, v)) = \int_0^L (u_{xx}^1 v_{xx}^1 + u_x^2 v_x^2)\, dx, \qquad \|u\|^2 = |u_{xx}^1|_{L^2}^2 + |u_x^2|_{L^2}^2.$$

Especially, $((\cdot,\cdot))$ and $\|\cdot\|$ also denote the scalar product and the norm of Y_1, and $|Au|$ be the norm of Y_2. In addition, we write $V_3 = Y_3 \times Y_2$, and denote $|A^2 u|$ as the norm of Y_3. Moreover, we have $V_3 \subset V_2 \subset V_1 = V_1^* \subset V_2^* \subset V_3^*$, where V_1^*, V_2^* and V_3^* are the dual of V_1, V_2 and V_3, respectively, and each space is dense in the next one and the injections are continuous.

Let λ_1 be the first eigenvalue of $-v_{xx} = \lambda v$, $x \in [0,L]$; $v(0) = v(L) = 0$, the corresponding eigenfunction $\phi_1(x)$ is positive on $[0,L]$. It is easy to know that λ_1^2 is the first eigenvalue of $u_{xxxx} = \lambda u$, $x \in [0,L]$; $u(0) = u(L) = u_{xx}(0) = u_{xx}(L) = 0$. Choosing $\lambda = \min\{\lambda_1, \lambda_1^2\}$, by the Poincaré inequality, we have

$$\|u\|^2 \geqslant \lambda |u|^2, \quad \forall u \in V_2. \tag{2.1}$$

Next we iterate some notations and abstract theorems in [3,13], which are important for getting our main results.

Definition 2.1. Let X be a Banach space, $\{S(t)\}_{t\geqslant 0}$ be a family operator on X. We say that $\{S(t)\}_{t\geqslant 0}$ is a norm-to-weak continuous semigroup on X, if $\{S(t)\}_{t\geqslant 0}$ satisfies

(i) $S(0) = \mathrm{Id}$ (the identity);
(ii) $S(t)S(s) = S(t+s)$, $\forall t, s \geqslant 0$;
(iii) $S(t_n)x_n \rightharpoonup S(t)x$, if $t_n \to t$, $x_n \to x$ in X.

Definition 2.2. A C^0 semigroup $\{S(t)\}_{t\geqslant 0}$ in a Banach space X is said to satisfy the condition (C) if for any $\varepsilon > 0$ and for any bounded set B of X, there exists $t(B) > 0$ and a finite dimensional subspace X_1 of X, such that $\{\|PS(t)x\|_X,\ x \in B,\ t \geqslant t(B)\}$ is bounded and

$$\big\|(I-P)S(t)x\big\|_X < \varepsilon, \quad t \geqslant t(B),\ x \in B,$$

where $P : X \to X_1$ is a bounded projector.

Theorem 2.3. *Let X be a Banach space and $\{S(t)\}_{t\geqslant 0}$ be a norm-to-weak continuous semigroup on X. Then $\{S(t)\}_{t\geqslant 0}$ has a global attractor in the topology of X, if the following conditions hold:*

(i) $\{S(t)\}_{t\geqslant 0}$ *has a bounded absorbing set* B_0; *and*
(ii) $\{S(t)\}_{t\geqslant 0}$ *satisfies the condition* (C).

Theorem 2.4. *(See [14].) Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If a function φ belongs to $L^\infty(0, T; X)$ and is weakly continuous with values in Y, then φ is weakly continuous with values in X.*

For simplicity, we introduce two symbols E_1 and E_2:

$$E_1 = V_2 \times V_1, \qquad E_2 = V_3 \times V_2.$$

Theorem 2.5. *(See [4,10–12].) Suppose that $k > 0$, $\alpha, \beta, \delta_1, \delta_2 > 0$ and* (F1)–(F2) *hold. If $h_B, h_S \in L^2(0, L)$, $(u_0, v_0, u_1, v_1) \in E_1$, then for any given $T > 0$, there exists a unique solution (u, v) of* (1.1)–(1.3) *such that*

$$u \in C\big([0,T], Y_2\big), \qquad u_t \in C\big([0,T], Y_0\big),$$
$$v \in C\big([0,T], Y_1\big), \qquad v_t \in C\big([0,T], Y_0\big).$$

Furthermore, $\{u_0, v_0, u_1, v_1\} \to \{u(t), v(t), u_t(t), v_t(t)\}$ is continuous in E_1. Consequently, it admits to define a C^0 semigroup

$$S(t) : \{u_0, v_0, u_1, v_1\} \to \big\{u(t), v(t), u_t(t), v_t(t)\big\}, \quad t \in \mathbb{R}^+,$$

and it maps E_1 into itself.

Finally, from (F1) and the definition of F_B, F_S we know that there exist two positive constants K_1, K_2 and $\eta = \eta(\delta) > 0$ such that

$$f_B(\tau)\tau + \eta\tau^2 + K_1 \geqslant 0, \qquad f_S(\tau)\tau + \eta\tau^2 + K_1 \geqslant 0, \quad \forall \tau \in \mathbb{R}, \tag{2.2}$$

and

$$F_B(\tau) + \eta\tau^2 + K_2 \geqslant 0, \qquad F_S(\tau) + \eta\tau^2 + K_2 \geqslant 0, \quad \forall \tau \in \mathbb{R}. \tag{2.3}$$

3. A priori estimates

3.1. A priori estimates in E_1

Choose $0 < \varepsilon < \varepsilon'$ be fixed in the course of the proof, where

$$\varepsilon' = \min\left\{\frac{\alpha\lambda}{2\delta_1}, \frac{\delta_1}{4}, \frac{\beta\lambda}{2\delta_2}, \frac{\delta_2}{4}\right\}. \tag{3.1}$$

Taking the scalar product in $L^2(0,L)$ of the first and the second equation of (1.1) with $\phi = u_t + \varepsilon u$ and $\psi = v_t + \varepsilon v$, respectively, then adding them, this yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\alpha|Au|^2 + |\phi|^2 + \beta\|v\|^2 + |\psi|^2\big) + \varepsilon\alpha|Au|^2 + (\delta_1 - \varepsilon)|\phi|^2 \\
&\quad - \varepsilon(\delta_1 - \varepsilon)(u,\phi) + \beta\varepsilon\|v\|^2 + (\delta_2 - \varepsilon)|\psi|^2 - \varepsilon(\delta_2 - \varepsilon)(v,\psi) \\
&\quad + k\big((u-v)^+, \phi - \psi\big) + \big(f_B(u),\phi\big) + \big(f_S(v),\psi\big) \\
&= (h_B,\phi) + (h_S,\psi).
\end{aligned} \tag{3.2}$$

In line with (2.1), (3.1), the Hölder and Young inequalities, we conclude

$$\begin{aligned}
&\varepsilon\alpha|Au|^2 + (\delta_1 - \varepsilon)|\phi|^2 - \varepsilon(\delta_1 - \varepsilon)(u,\phi) + \beta\varepsilon\|v\|^2 + (\delta_2 - \varepsilon)|\psi|^2 - \varepsilon(\delta_2 - \varepsilon)(v,\psi) \\
&\quad \geqslant \frac{\varepsilon\alpha}{2}|Au|^2 + \frac{\varepsilon\beta}{2}\|v\|^2 + \frac{\delta_1}{2}|\phi|^2 + \frac{\delta_2}{2}|\psi|^2.
\end{aligned} \tag{3.3}$$

In addition,

$$k\big((u-v)^+, \phi - \psi\big) = \frac{1}{2}\frac{d}{dt}k\big|(u-v)^+\big|^2 + \varepsilon k\big|(u-v)^+\big|^2, \tag{3.4}$$

and

$$\begin{aligned}
\big(f_B(u),\phi\big) + \big(f_S(v),\psi\big) &= \frac{d}{dt}\left(\int_0^L F_B(u)\,dx + \int_0^L F_S(v)\,dx\right) \\
&\quad + \varepsilon\int_0^L f_B(u)u\,dx + \varepsilon\int_0^L f_S(v)v\,dx,
\end{aligned} \tag{3.5}$$

and

$$(h_B,\phi) + (h_S,\psi) = \frac{d}{dt}\left(\int_0^L h_B u\,dx + \int_0^L h_S v\,dx\right) + \varepsilon\int_0^L h_B u\,dx + \varepsilon\int_0^L h_S v\,dx. \tag{3.6}$$

Consequently, collecting with (3.2)–(3.6), there holds

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg[\alpha|Au|^2 + \beta\|v\|^2 + |\phi|^2 + |\psi|^2 + k\big|(u-v)^+\big|^2 \\
&\quad + 2\int_0^L F_B(u)\,dx + 2\int_0^L F_S(v)\,dx - 2\int_0^L h_B u\,dx - 2\int_0^L h_S v\,dx\Bigg] \\
&\quad + \frac{\varepsilon\alpha}{2}|Au|^2 + \frac{\varepsilon\beta}{2}\|v\|^2 + \frac{\delta_1}{2}|\phi|^2 + \frac{\delta_2}{2}|\psi|^2 + \varepsilon k\big|(u-v)^+\big|^2 \\
&\quad + \varepsilon\int_0^L f_B(u)u\,dx + \varepsilon\int_0^L f_S(v)v\,dx - \varepsilon\int_0^L h_B u\,dx - \varepsilon\int_0^L h_S v\,dx \\
&\leqslant 0.
\end{aligned} \tag{3.7}$$

Provided that $\varepsilon_0 = \min\{\varepsilon, \delta_1, \delta_2\}$, let

$$E(t) = \alpha|Au|^2 + \beta\|v\|^2 + |\phi|^2 + |\psi|^2 + k\left|(u-v)^+\right|^2 + 2\int_0^L F_B(u)\,dx + 2\int_0^L F_S(v)\,dx - 2\int_0^L h_B u\,dx - 2\int_0^L h_S v\,dx \tag{3.8}$$

and

$$I(t) = \alpha|Au|^2 + \beta\|v\|^2 + |\phi|^2 + |\psi|^2 + k\left|(u-v)^+\right|^2 + 2\int_0^L f_B(u)u\,dx + 2\int_0^L f_S(v)v\,dx - 2\int_0^L h_B u\,dx - 2\int_0^L h_S v\,dx. \tag{3.9}$$

We have

$$\frac{d}{dt}E(t) + \varepsilon_0 I(t) \leqslant 0$$

which implies

$$E(t) \leqslant -\varepsilon_0 \int_0^t I(\tau)\,d\tau + E(0), \tag{3.10}$$

where

$$E(0) = \alpha|Au_0|^2 + \beta\|v_0\|^2 + |u_1 + \varepsilon u_0|^2 + |v_1 + \varepsilon v_0|^2 + k\left|(u_0 - v_0)^+\right|^2 + 2\int_0^L F_B(u_0)\,dx + 2\int_0^L F_S(v_0)\,dx - 2\int_0^L h_B u_0\,dx - 2\int_0^L h_S v_0\,dx. \tag{3.11}$$

Noticing that (2.2)–(2.3) and (3.8)–(3.9), and using the compact Sobolev embedding theorem we get

$$\begin{aligned} E(t) &\geqslant \alpha|Au|^2 + \beta\|v\|^2 + |\phi|^2 + |\psi|^2 - (k + 2\eta + \varepsilon_0)|u|^2 - (k + 2\eta + \varepsilon_0)|v|^2 - M_1 \\ &\geqslant \left(\alpha - \frac{k + 2\eta + \varepsilon_0}{\lambda}\right)|Au|^2 + \left(\beta - \frac{k + 2\eta + \varepsilon_0}{\lambda}\right)\|v\|^2 + |\phi|^2 + |\psi|^2 - M_1, \end{aligned} \tag{3.12}$$

where $M_1 = 4K_2 L + \frac{|h_B|^2 + |h_S|^2}{\varepsilon_0}$. Similarly

$$\begin{aligned} I(t) &= \alpha|Au|^2 + \beta\|v\|^2 + |\phi|^2 + |\psi|^2 + k\left|(u-v)^+\right|^2 \\ &\quad + 2\int_0^L f_B(u)u\,dx + 2\int_0^L f_S(v)v\,dx - 2\int_0^L h_B u\,dx - 2\int_0^L h_S v\,dx \\ &\geqslant \left(\alpha - \frac{k + 2\eta + \varepsilon_0}{\lambda}\right)|Au|^2 + \left(\beta - \frac{k + 2\eta + \varepsilon_0}{\lambda}\right)\|v\|^2 + |\phi|^2 + |\psi|^2 - M_2, \end{aligned} \tag{3.13}$$

where $M_2 = 4K_1L + \frac{|h_B|^2+|h_S|^2}{\varepsilon_0}$. Therefore, let $\frac{k+2\eta}{\lambda} < \min\{\alpha, \beta\}$ and $0 < \varepsilon_0 < \alpha\lambda - k - 2\eta$, we have

$$\alpha - \frac{k+2\eta+\varepsilon_0}{\lambda} > 0, \qquad \beta - \frac{k+2\eta+\varepsilon_0}{\lambda} > 0. \tag{3.14}$$

Associated with (3.12)–(3.14), there exists a positive constant C_1 such that

$$E(t) \geqslant C_1\big(|Au|^2 + \|v\|^2 + |\phi|^2 + |\psi|^2\big) - M_1, \tag{3.15}$$

$$I(t) \geqslant C_1\big(|Au|^2 + \|v\|^2 + |\phi|^2 + |\psi|^2\big) - M_2. \tag{3.16}$$

So we deduce from (3.15)–(3.16) and (3.10) that

$$\begin{aligned} &C_1\big(|Au|^2 + \|v\|^2 + |\phi|^2 + |\psi|^2\big) - M_1 \\ &\quad \leqslant -\varepsilon_0 \int_0^t \big[C_1\big(|Au|^2 + \|v\|^2 + |\phi|^2 + |\psi|^2\big) - M_2\big]\,dt + E(0). \end{aligned} \tag{3.17}$$

Thus, for any $K > \frac{M_2}{C_1}$, there exists $t_0 = t_0(B)$ such that

$$\big|Au(t_0)\big|^2 + \big\|v(t_0)\big\|^2 + \big|\phi(t_0)\big|^2 + \big|\psi(t_0)\big|^2 \leqslant K. \tag{3.18}$$

As a results, if u, v are the solution of the system (1.1)–(1.3), let $B_0 = \bigcup_{t\geqslant 0} S(t)B_1$, where

$$B_1 = \big\{(u_0, u_1, v_0, v_1)^T \in E_0\colon\ |Au_0|^2 + \|v_0\|^2 + |\phi_0|^2 + |\psi_0|^2 \leqslant K\big\},$$

then B_0 is a bounded absorbing set of $\{S(t)\}_{t\geqslant 0}$.

On the other hand, from the above discussion, there exists a positive constant μ_1 such that

$$\big|Au(t)\big|^2 + \big\|v(t)\big\|^2 + \big|\phi(t)\big|^2 + \big|\psi(t)\big|^2 \leqslant \mu_1^2, \quad \forall t \geqslant t_0. \tag{3.19}$$

3.2. *A priori estimates in E_2*

Choose $0 < \varepsilon < 1$. Taking the scalar product in $L^2(0, L)$ of the first and second equation of (1.1) with $A^2\phi = A^2u_t + \varepsilon A^2u$ and $A\psi = Av_t + \varepsilon Av$, respectively, then after computation, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\alpha\big|A^2u\big|^2 + \beta|Av|^2 + |A\phi|^2 + \|\psi\|^2\big) \\ &\quad + \varepsilon\alpha\big|A^2u\big|^2 + (\delta_1 - \varepsilon)|A\phi|^2 - \varepsilon(\delta_1 - \varepsilon)\big(A^2u, \phi\big) + \varepsilon\beta|Av|^2 \\ &\quad + (\delta_2 - \varepsilon)\|\psi\|^2 - \varepsilon(\delta_2 - \varepsilon)(Av, \psi) + \big(k(u-v)^+, A^2\phi\big) \\ &\quad - \big(k(u-v)^+, A\psi\big) + \big(f_B(u), A^2\phi\big) + \big(f_S(v), A\psi\big) \\ &= \big(h_B, A^2\phi\big) + (h_S, A\psi). \end{aligned} \tag{3.20}$$

Thanks to the Hölder inequalities, Young inequalities and (3.19), we obtain

$$
\begin{aligned}
\big(k(u-v)^+, A^2\phi\big) &= \frac{d}{dt}\big(k(u-v)^+, A^2u\big) - \big(k\big((u-v)^+\big)_t, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) \\
&\geqslant \frac{d}{dt}\big(k(u-v)^+, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) - k\big|\big((u-v)^+\big)_t\big| \cdot \big|A^2u\big| \\
&\geqslant \frac{d}{dt}\big(k(u-v)^+, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) - k\big|(u-v)_t\big| \cdot \big|A^2u\big| \\
&\geqslant \frac{d}{dt}\big(k(u-v)^+, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) - k\big(|u_t| + |v_t|\big) \cdot \big|A^2u\big| \\
&\geqslant \frac{d}{dt}\big(k(u-v)^+, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) - 2k\mu_1\big|A^2u\big| \\
&\geqslant \frac{d}{dt}\big(k(u-v)^+, A^2u\big) + \varepsilon\big(k(u-v)^+, A^2u\big) \\
&\quad - \frac{\varepsilon\alpha}{2}\big|A^2u\big|^2 - \frac{2k^2\mu_1^2}{\varepsilon\alpha}, \quad t \geqslant t_0,
\end{aligned} \tag{3.21}
$$

and

$$
\begin{aligned}
-\big(k(u-v)^+, A\psi\big) &= k\big(\big((u-v)^+\big)_x, \psi_x\big) \geqslant -k\big\|(u-v)^+\big\| \cdot \|\psi\| \\
&\geqslant -2k\mu_1\|\psi\| \geqslant -\frac{\delta_2}{2}\|\psi\|^2 - \frac{2k^2\mu_1^2}{\delta_2}, \quad t \geqslant t_0.
\end{aligned} \tag{3.22}
$$

On the other hand, we know that $f_B(u)$, $f_B'(u)$, $f_S(v)$, $f_S'(v)$ are uniformly bounded in L^∞ due to (F2), (3.19) and the Sobolev embedding theorem, that is, there exists a constant $M > 0$, such that

$$
\big|f_B(u)\big|_{L^\infty} \leqslant M, \qquad \big|f_B'(u)\big|_{L^\infty} \leqslant M, \qquad \big|f_S(v)\big|_{L^\infty} \leqslant M, \qquad \big|f_S'(v)\big|_{L^\infty} \leqslant M. \tag{3.23}
$$

Therefore, in line with the Hölder inequality, Cauchy inequality and (3.19) again, it follows that

$$
\begin{aligned}
&\big(f_B(u), A^2\phi\big) + \big(f_S(v), A\psi\big) \\
&\quad = \frac{d}{dt}\big(f_B(u), A^2u\big) + \varepsilon\big(f_B(u), A^2u\big) - \big(f_B'(u)u_t, A^2u\big) - \big(f_S'(v)v_x, \psi_x\big) \\
&\quad \geqslant \frac{d}{dt}\big(f_B(u), A^2u\big) + \varepsilon\big(f_B(u), A^2u\big) - \int_0^L \big|f_B'(u)\big| \cdot |u_t| \cdot \big|A^2u\big|\,dx - \int_0^L \big|f_S'(v)\big| \cdot |v_x| \cdot |\psi_x|\,dx \\
&\quad \geqslant \frac{d}{dt}\big(f_B(u), A^2u\big) + \varepsilon\big(f_B(u), A^2u\big) - M\mu_1\big|A^2u\big| - M\mu_1\|\psi\| \\
&\quad \geqslant \frac{d}{dt}\big(f_B(u), A^2u\big) + \varepsilon\big(f_B(u), A^2u\big) - \frac{\varepsilon\alpha}{4}\big|A^2u\big|^2 - \frac{\delta_2}{4}\|\psi\|^2 - \frac{M^2\mu_1^2}{\varepsilon\alpha} - \frac{M^2\mu_1^2}{\delta_2}, \quad t \geqslant t_0,
\end{aligned} \tag{3.24}
$$

and

$$
\big(h_B, A^2\phi\big) + (h_S, A\psi) = \frac{d}{dt}\big(h_B, A^2u\big) + \varepsilon\big(h_B, A^2u\big) + \frac{d}{dt}(h_S, Av) + \varepsilon(h_S, Av). \tag{3.25}
$$

Thus, collecting (3.21)–(3.22) and (3.24)–(3.25), from (3.20) yields

$$\begin{aligned}&\frac{d}{dt}\big[\alpha\big|A^2u\big|^2+\beta|Av|^2+|A\phi|^2+\|\psi\|^2+2\big(k(u-v)^+,A^2u\big)\\&\quad+2\big(f_B(u),A^2u\big)-2\big(h_B,A^2u\big)-2(h_S,Av)\big]\\&\quad+\frac{\varepsilon\alpha}{2}\big|A^2u\big|^2+2(\delta_1-\varepsilon)|A\phi|^2-2\varepsilon(\delta_1-\varepsilon)\big(A^2u,\phi\big)+2\varepsilon\beta|Av|^2\\&\quad+2\left(\frac{\delta_2}{4}-\varepsilon\right)\|\psi\|^2-2\varepsilon(\delta_2-\varepsilon)(Av,\psi)+2\varepsilon\big(k(u-v)^+,A^2u\big)\\&\quad+2\varepsilon\big(f_B(u),A^2u\big)-2\varepsilon\big(h_B,A^2u\big)-2\varepsilon(h_S,Av)\\&\leqslant C,\quad t\geqslant t_0,\end{aligned}\tag{3.26}$$

where

$$C=2\mu_1^2\big(2k^2+M^2\big)\left(\frac{1}{\varepsilon\alpha}+\frac{1}{\delta_2}\right).$$

Furthermore, by (2.1) and the Young inequality, we obtain

$$\begin{aligned}&\frac{\varepsilon\alpha}{2}\big|A^2u\big|^2+2(\delta_1-\varepsilon)|A\phi|^2-2\varepsilon(\delta_1-\varepsilon)\big(A^2u,\phi\big)\\&\quad\geqslant\frac{\varepsilon\alpha}{2}\big|A^2u\big|^2+2(\delta_1-\varepsilon)\|\phi\|^2-\frac{2\varepsilon\delta_1}{\sqrt{\lambda}}\big|A^2u\big|\cdot\|\phi\|\\&\quad\geqslant\frac{\varepsilon\alpha}{4}\big|A^2u\big|^2+2\left(\delta_1-\varepsilon-\frac{\delta_1^2\varepsilon}{\lambda\alpha}\right)|A\phi|^2,\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}&2\varepsilon\beta|Av|^2+2\left(\frac{\delta_2}{4}-\varepsilon\right)\|\psi\|^2-2\varepsilon(\delta_2-\varepsilon)(Av,\psi)\\&\quad\geqslant2\varepsilon\beta|Av|^2+\left(\frac{\delta_2}{2}-2\varepsilon\right)\|\psi\|^2-\frac{2\delta_2\varepsilon}{\sqrt{\lambda}}|Av|\cdot\|\psi\|\\&\quad\geqslant\varepsilon\beta|Av|^2+\left(\frac{\delta_2}{2}-2\varepsilon-\frac{\delta_2^2\varepsilon}{\lambda\beta}\right)\|\psi\|^2.\end{aligned}\tag{3.28}$$

Therefore, choosing ε small enough, such that

$$\delta_1-\left(1+\frac{\delta_1^2}{\lambda\alpha}\right)\varepsilon>\frac{\delta_1}{2},\qquad\frac{\delta_2}{2}-\left(2+\frac{\delta_2^2}{\lambda\beta}\right)\varepsilon>\frac{\delta_2}{4},\tag{3.29}$$

and taking $\varepsilon_0=\min\{\frac{\varepsilon}{4},\delta_1,\frac{\delta_2}{4}\}$, and together with (3.26)–(3.28) we have

$$\begin{aligned}&\frac{d}{dt}\Big[\alpha\Big|A^2u\Big|^2+\beta|Av|^2+|A\phi|^2+\|\psi\|^2+2\big(k(u-v)^+,A^2u\big)\\&\quad+2\big(f_B(u),A^2u\big)-2\big(h_B,A^2u\big)-2(h_S,Av)\Big]\\&\quad+\varepsilon_0\Big[\alpha\Big|A^2u\Big|^2+\beta|Av|^2+|A\phi|^2+\|\psi\|^2+2\big(k(u-v)^+,A^2u\big)\\&\quad+2\big(f_B(u),A^2u\big)-2\big(h_B,A^2u\big)-2(h_S,Av)\Big]\\&\leqslant C.\end{aligned}\tag{3.30}$$

On the other hand, by the Hölder inequality, the Sobolev embedding theorem and (3.19), it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha}{2}\left|A^2u\right|^2+2\big(k(u-v)^+,A^2u\big)\right)\\
&=\frac{d}{dt}\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2-\frac{4k^2}{\alpha}\int_\Omega\left|(u-v)^+\right|\cdot\left|\big((u-v)^+\big)_t\right|dx\\
&\geqslant\frac{d}{dt}\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2-\frac{4k^2}{\alpha}\left|(u-v)\right|\cdot\left|(u-v)_t\right|\\
&\geqslant\frac{d}{dt}\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2-\frac{16k^2\mu_1^2}{\alpha},\quad t\geqslant t_0,
\end{aligned}\tag{3.31}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha}{2}\left|A^2u\right|^2+2\big(f_B(u),A^2u\big)-2\big(h_B,A^2u\big)\right)\\
&\geqslant\frac{d}{dt}\left|\sqrt{\frac{\alpha}{2}}A^2u+\sqrt{\frac{2}{\alpha}}f_B(u)-\sqrt{\frac{2}{\alpha}}h_B\right|^2\\
&\quad-\frac{4}{\alpha}\int_\Omega\left|f_B(u)\right|\cdot\left|f_B'(u)\right|\cdot|u_t|\,dx+2\int_\Omega f_B'(u)u_t\cdot h_B\,dx\\
&\geqslant\frac{d}{dt}\left|\sqrt{\frac{\alpha}{2}}A^2u+\sqrt{\frac{2}{\alpha}}f_B(u)-\sqrt{\frac{2}{\alpha}}h_B\right|^2-\frac{4M^2\mu_1}{\alpha}-2M\mu_1|h_B|,\quad t\geqslant t_0.
\end{aligned}\tag{3.32}$$

Therefore, integrating with (3.31)–(3.32), we get from (3.30)

$$\begin{aligned}
&\frac{d}{dt}\Bigg(\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2+\left|\sqrt{\frac{\alpha}{2}}A^2u+\sqrt{\frac{2}{\alpha}}f_B(u)-\sqrt{\frac{2}{\alpha}}h_B\right|^2\\
&\quad+\left|\sqrt{\beta}Av-\frac{1}{\sqrt{\beta}}h_S\right|^2+|A\phi|^2+\|\psi\|^2\Bigg)\\
&\quad+\varepsilon_0\Bigg(\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2+\left|\sqrt{\frac{\alpha}{2}}A^2u+\sqrt{\frac{2}{\alpha}}f_B(u)-\sqrt{\frac{2}{\alpha}}h_B\right|^2\\
&\quad+\left|\sqrt{\beta}Av-\frac{1}{\sqrt{\beta}}h_S\right|^2+|A\phi|^2+\|\psi\|^2\Bigg)\\
&\leqslant\check{C},\quad t\geqslant t_0,
\end{aligned}\tag{3.33}$$

where

$$\check{C}=C+\frac{16k^2\mu_1^2}{\alpha}(1+\varepsilon_0)+\frac{2M^2}{\alpha}(\varepsilon_0+2\mu_1)+2M|h_B|\left(\frac{2\varepsilon_0}{\alpha}+\mu_1\right)+\frac{2\varepsilon_0}{\alpha}|h_B|^2+\frac{\varepsilon_0}{\beta}|h_S|^2.$$

Thus, denote

$$y(t)=\left|\sqrt{\frac{\alpha}{2}}A^2u+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u-v)^+\right|^2+\left|\sqrt{\frac{\alpha}{2}}A^2u+\sqrt{\frac{2}{\alpha}}f_B(u)-\sqrt{\frac{2}{\alpha}}h_B\right|^2$$
$$+\left|\sqrt{\beta}Av-\frac{1}{\sqrt{\beta}}h_S\right|^2+|A\phi|^2+\|\psi\|^2,$$

we have $\frac{d}{dt}y(t)+\varepsilon_0 y(t)\leqslant \check{C}$, $t\geqslant t_0(B)$. By the Gronwall lemma, we conclude that

$$y(t)\leqslant y(t_0)\exp\bigl(-\varepsilon_0(t-t_0)\bigr)+\frac{\check{C}}{\varepsilon_0},\quad t\geqslant t_0. \tag{3.34}$$

Now, if $B\subset B_{E_2}(p_0,\rho)$, the ball of E_2, centered at p_0 of radius ρ, then it follows from (3.34) that there exists a constant $R_1>0$, such that

$$\sup_{(u(t_0),v(t_0),u_t(t_0),v_t(t_0))\in B} y(t_0)\leqslant R_1^2.$$

Provided

$$t_1-t_0\geqslant\frac{1}{\varepsilon_0}\log R_1^2,$$

then

$$y(t)\leqslant\mu_2^2,\quad t\geqslant t_1, \tag{3.35}$$

where $\mu_2^2=1+\frac{\check{C}}{\varepsilon_0}$.

4. Existence of the strong solutions

Theorem 4.1. *Suppose that $k>0$ and $\frac{k+2\eta}{\lambda}<\min\{\alpha,\beta\}$, $\alpha,\beta,\delta_1,\delta_2>0$, f_B, f_S satisfy* (F1)–(F2)*, $f_B(0)=f_S(0)=0$, $h_B,h_S\in L^2(0,L)$. Then for any given $T>0$, the initial–boundary value problem* (1.1)–(1.3) *has a unique solution (u,v) with*

$$u\in L^\infty(0,T;Y_3),\qquad u_t\in L^\infty(0,T;Y_2),$$
$$v\in L^\infty(0,T;Y_2),\qquad v_t\in L^\infty(0,T;Y_1)$$

for $(u_0,v_0,u_1,v_1)\in E_2$. Moreover, (u,v,u_t,v_t) are weakly continuous functions from $[0,T]$ to E_2, where λ, η are given by (2.1)–(2.3)*.*

Proof. The principle of the proof is classical. Assume that there exists an orthonormal basis of $Y_3\times Y_2$ consisting of eigenvectors (ω_i,χ_j) of $A^2\times A$ in $Y_3\times Y_2$, simultaneously they are also orthonormal basis of $Y_2\times Y_1$. The corresponding eigenvalues are (ν_i,λ_j), $i,j=1,2,\dots,$ satisfying

$$A^2\omega_i=\nu_i\omega_i,\qquad A\chi_j=\lambda_j\chi_j,\quad \forall i,j\in\mathbb{N}.$$

Now we prove the existence of strong solutions by the standard Faedo–Galerkin schemes in [7,8,14].

For each m, n, by the basic theory of ordinary differential equations, there exists an approximate solution (u_m,v_n) of the form

$$u_m(t)=\sum_{i=1}^{m}u_{mi}\omega_i,\qquad v_n(t)=\sum_{j=1}^{n}v_{nj}\chi_j$$

satisfying

$$\begin{cases}\dfrac{d^2u_m}{dt^2}+\delta_1\dfrac{du_m}{dt}+\alpha A^2u_m+k(u_m-v_n)^++P_mf_B(u_m)=(h_B)_m,\\ \dfrac{d^2v_n}{dt^2}+\delta_2\dfrac{dv_n}{dt}-\beta Av_n-k(u_m-v_n)^++Q_nf_S(v_n)=(h_S)_n,\\ u_m(0)=P_mu_0,\quad u_m'(0)=P_mu_1,\\ v_n(0)=Q_nv_0,\quad v_n'(0)=Q_nv_1,\end{cases}\tag{4.1}$$

where $P_m:Y_3\to V_1^m$ is the orthogonal projector in V_1^m, $Q_n:Y_2\to V_2^n$ is the orthogonal projector in V_2^n, $(h_B)_m=P_mh_B$, $(h_S)_n=Q_nh_S$, and

$$V_1^m=\operatorname{span}\{\omega_1,\omega_2,\dots,\omega_m\},\qquad V_2^n=\operatorname{span}\{\chi_1,\chi_2,\dots,\chi_n\}.$$

Existence and uniqueness results for ODEs imply that we have a unique solution (u_m,v_n) of (4.1), at least on some short time interval $[0,T_m]$. We can extend this time interval to infinity if we know that the (u_m,v_n) are bounded.

Fixed $0<\varepsilon<1$. Taking the scalar product in $L^2(0,L)$ of the first and second equation of (4.1) with $A^2\phi_m=A^2u_m'+\varepsilon A^2u_m$ and $A\psi_n=Av_n'+\varepsilon Av_n$, respectively, and noting that $(P_mf_B(u_m),A^2\phi_m)=(f_B(u_m),P_mA^2\phi_m)=(f_B(u_m),A^2\phi_m)$ and $(Q_nf_S(v_n),A\psi_n)=(f_S(v_n),A\psi_n)$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\alpha\big|A^2u_m\big|^2+\beta|Av_n|^2+|A\phi_m|^2+\|\psi_n\|^2\big)+\varepsilon\alpha\big|A^2u_m\big|^2\\&\quad+(\delta_1-\varepsilon)|A\phi_m|^2-\varepsilon(\delta_1-\varepsilon)\big(A^2u_m,\phi_m\big)+\varepsilon\beta|Av_n|^2\\&\quad+(\delta_2-\varepsilon)\|\psi_n\|^2-\varepsilon(\delta_2-\varepsilon)(Av_n,\psi_n)+\big(k(u_m-v_n)^+,A^2\phi_m\big)\\&\quad-\big(k(u_m-v_n)^+,A\psi_n\big)+\big(f_B(u_m),A^2\phi_m\big)+\big(f_S(v_n),A\psi_n\big)\\&=\big((h_B)_m,A^2\phi_m\big)+\big((h_S)_n,A\psi_n\big).\end{aligned}\tag{4.2}$$

Like the estimates of (3.21)–(3.30), we have

$$\begin{aligned}&\frac{d}{dt}\Big[\alpha\big|A^2u_m\big|^2+\beta|Av_n|^2+|A\phi_m|^2+\|\psi_n\|^2+2\big(k(u_m-v_n)^+,A^2u_m\big)\\&\quad+2\big(f_B(u_m),A^2u_m\big)-2\big((h_B)_m,A^2u_m\big)-2\big((h_S)_n,Av_n\big)\Big]\\&\quad+\varepsilon_0\Big[\alpha\big|A^2u_m\big|^2+\beta|Av_n|^2+|A\phi_m|^2+\|\psi_n\|^2+2\big(k(u_m-v_n)^+,A^2u_m\big)\\&\quad+2\big(f_B(u_m),A^2u_m\big)-2\big((h_B)_m,A^2u_m\big)-2\big((h_S)_n,Av_n\big)\Big]\\&\leqslant C,\end{aligned}\tag{4.3}$$

where C and ε_0 are given by (3.30). Exploiting (3.31)–(3.32) again, the following inequality is true:

$$\begin{aligned}
&\frac{d}{dt}\left(\left|\sqrt{\frac{\alpha}{2}}A^2u_m+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u_m-v_n)^+\right|^2+\left|\sqrt{\frac{\alpha}{2}}A^2u_m+\sqrt{\frac{2}{\alpha}}f_B(u_m)-\sqrt{\frac{2}{\alpha}}(h_B)_m\right|^2\right.\\
&\qquad\left.+\left|\sqrt{\beta}Av_n-\frac{1}{\sqrt{\beta}}(h_S)_n\right|^2+|A\phi_m|^2+\|\psi_n\|^2\right)\\
&\qquad+\varepsilon_0\left(\left|\sqrt{\frac{\alpha}{2}}A^2u_m+\frac{\sqrt{2}k}{\sqrt{\alpha}}(u_m-v_n)^+\right|^2+\left|\sqrt{\frac{\alpha}{2}}A^2u_m+\sqrt{\frac{2}{\alpha}}f_B(u_m)-\sqrt{\frac{2}{\alpha}}(h_B)_m\right|^2\right.\\
&\qquad\left.+\left|\sqrt{\beta}Av_n-\frac{1}{\sqrt{\beta}}(h_S)_n\right|^2+|A\phi_m|^2+\|\psi_n\|^2\right)\\
&\leqslant \check{C},\quad t\geqslant t_0,
\end{aligned}\tag{4.4}$$

where $\check{C}$ is given by (3.33).

Consequently, by the Gronwall lemma we easily infer from (4.4) that $\{u_m\}$, $\{u_m'\}$ and $\{v_n\}$, $\{v_n'\}$ remain in a bounded set of $L^\infty(0,T;Y_3)$, $L^\infty(0,T;Y_2)$ and $L^\infty(0,T;Y_2)$, $L^\infty(0,T;Y_1)$, respectively, as $m,n\to\infty$. By means of (4.1) again we know that

$$\begin{cases}
\dfrac{d^2u_m}{dt^2}=-\delta_1\dfrac{du_m}{dt}-\alpha A^2u_m-k(u_m-v_n)^+-P_mf_B(u_m)+(h_B)_m,\\
\dfrac{d^2v_n}{dt^2}=-\delta_2\dfrac{dv_n}{dt}+\beta Av_n+k(u_m-v_n)^+-Q_nf_S(v_n)+(h_S)_n.
\end{cases}$$

Therefore $\{u_m''\}$, $\{v_n''\}$ are uniformly bounded in $L^\infty(0,T;Y_0)$ and $L^\infty(0,T;Y_0)$, respectively. Thus we can extract subsequences, still denoted as $\{u_m\}$, $\{v_n\}$, such that

$$\begin{aligned}
&u_m\rightharpoonup u\quad\text{star in } L^\infty(0,T;Y_3),\\
&u_m'\rightharpoonup u'\quad\text{star in } L^\infty(0,T;Y_2),\\
&u_m''\rightharpoonup u''\quad\text{star in } L^\infty(0,T;Y_0),\\
&(u_m-v_n)^+\rightharpoonup(u-v)^+\quad\text{star in } L^\infty(0,T;Y_0),\ \text{as } m,n\to\infty.
\end{aligned}$$

Since $L^\infty(0,T;Y_3)\subset L^2(0,T;Y_3)$, $L^\infty(0,T;Y_2)\subset L^2(0,T;Y_2)$, and $L^\infty(0,T;Y_0)\subset L^2(0,T;Y_0)$, it follows that

$$\begin{aligned}
&u_m\rightharpoonup u\quad\text{in } L^2(0,T;Y_3),\\
&u_m'\rightharpoonup u'\quad\text{in } L^2(0,T;Y_2),\\
&u_m''\rightharpoonup u''\quad\text{in } L^2(0,T;Y_0),\\
&(u_m-v_n)^+\rightharpoonup(u-v)^+\quad\text{in } L^2(0,T;Y_0),\ \text{as } m,n\to\infty.
\end{aligned}$$

The similar process leads to

$$\begin{aligned}
&v_n\rightharpoonup v\quad\text{in } L^2(0,T;Y_1),\\
&v_n'\rightharpoonup v'\quad\text{in } L^2(0,T;Y_1),\\
&v_n''\rightharpoonup v''\quad\text{in } L^2(0,T;Y_0),\ \text{as } m,n\to\infty.
\end{aligned}$$

On the other hand, by (F2) and the continuity of $f_B(u)$, $f_S(v)$ we have

$$f_B(u_m) \to f_B(u) \quad \text{in } L^2(0,T;Y_0),$$
$$f_S(v_n) \to f_S(v) \quad \text{in } L^2(0,T;Y_0), \text{ as } m,n \to \infty.$$

Moreover, $(h_B)_m \to h_B$ in $L^2(0,T;Y_0)$, $(h_S)_n \to h_S$ in $L^2(0,T;Y_0)$. It is then easy to pass to the limit in (4.1) and we conclude that (u,v) is a solution of (4.1) such that

$$u \in L^\infty(0,T;Y_3), \qquad u' \in L^\infty(0,T;Y_2),$$
$$v \in L^\infty(0,T;Y_1), \qquad v' \in L^\infty(0,T;Y_1).$$

Furthermore, making use of Theorems 2.4 and 2.5 we know that (u,v,u',v') are weakly continuous functions from $[0,T]$ to E_2.

Finally, uniqueness is followed from Refs. [4,10], since any two strong solutions would both be weak solutions. □

Thus Theorem 4.1 holds and we also obtain a bounded absorbing set for the solution semigroup $\{S(t)\}_{t\geqslant 0}$ of (1.1)–(1.3) in E_2. This is the following results:

Theorem 4.2. *Suppose that $k>0$ and $\frac{k+2\eta}{\lambda} < \min\{\alpha,\beta\}$, $\alpha,\beta,\delta_1,\delta_2>0$ and* (F1)–(F2) *hold. Then there exists a bounded absorbing set in E_2 for the semigroup $\{S(t)\}_{t\geqslant 0}$, where λ and η are given by* (2.1)–(2.3)*.*

5. Global attractors in E_2

In order to obtain our main results, we need the following compactness results and the norm-to-weak continuity of semigroup.

Lemma 5.1. *Assume that* (F1) *and* (F2) *hold, $f_B(0)=f_S(0)=0$, and $(f_B,f_S): Y_3\times Y_2 \to Y_2\times Y_1$ are defined by*

$$\big(\big(f_B(u),\varphi\big)\big) = \int_0^L \frac{\partial^2 f_B(u)}{\partial x^2}\cdot \varphi_{xx}\,dx, \qquad \big(\big(f_S(v),\psi\big)\big) = \int_0^L \frac{\partial f_S(v)}{\partial x}\cdot \psi_x\,dx,$$

$\forall (u,v)\in Y_3\times Y_2$, $(\varphi,\psi)\in Y_2\times Y_1$. Then (f_B,f_S) are continuous compact.

Proof. Assume that $\{u_n\}$ and $\{v_n\}$ are bounded in Y_3 and Y_2, respectively, and let $\{u_n\}$ converge weakly to u_0 in Y_3, $\{v_n\}$ converge weakly to v_0 in Y_2. By the Sobolev embedding theorem, we know that

$$\{u_n\} \text{ is bounded and converges to } u_0 \text{ in } L^p(0,L),\ W^{1,p}(0,L),\ W^{2,p}(0,L),\ \forall p\geqslant 1;$$
$$\{v_n\} \text{ is bounded and converges to } v_0 \text{ in } W^{1,p}(0,L),\ \forall p\geqslant 1. \tag{5.1}$$

Write $u_n - u_0 = \omega_n$, $v_n - v_0 = \chi_n$.

By (F2), (3.19) and the Sobolev embedding theorem we show that $f_B'(u)$, $f_B''(u)$, $f_B'''(u)$, $f_S'(v)$, $f_S''(v)$ are uniformly bounded in L^∞, that is, there exists a constant $M>0$, such that

$$\big|f_B'(u)\big|_{L^\infty} \leqslant M, \qquad \big|f_B''(u)\big|_{L^\infty} \leqslant M, \qquad \big|f_B'''(u)\big|_{L^\infty} \leqslant M; \tag{5.2}$$
$$\big|f_S'(v)\big|_{L^\infty} \leqslant M, \qquad \big|f_S''(v)\big|_{L^\infty} \leqslant M. \tag{5.3}$$

Since there exists constant $0 < \theta < 1$, such that

$$\begin{aligned}
&\left(\int_0^L \left|\frac{\partial^2}{\partial x^2}\big(f_B(u_n)-f_B(u_0)\big)\right|^2 dx\right)^{\frac{1}{2}} \\
&\leqslant \left(\int_0^L \left|f_B''(u_0+\theta\omega_n)\cdot\frac{\partial^2(u_0+\theta\omega_n)}{\partial x^2}\cdot\omega_n\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad+\left(\int_0^L \left|f_B'''(u_0+\theta\omega_n)\left(\frac{\partial(u_0+\theta\omega_n)}{\partial x}\right)^2\cdot\omega_n\right|^2 dx\right)^{\frac{1}{2}}+\left(\int_0^L \left|f_B'(u_0+\theta\omega_n)\cdot\frac{\partial^2\omega_n}{\partial x^2}\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad+2\left(\int_0^L \left|f_B''(u_0+\theta\omega_n)\cdot\frac{\partial(u_0+\theta\omega_n)}{\partial x}\cdot\frac{\partial\omega_n}{\partial x}\right|^2 dx\right)^{\frac{1}{2}} \\
&\leqslant M\left(\int_0^L \left|\frac{\partial^2(u_0+\theta\omega_n)}{\partial x^2}\cdot\omega_n\right|^2 dx\right)^{\frac{1}{2}}+M\left(\int_0^L \left|\left(\frac{\partial(u_0+\theta\omega_n)}{\partial x}\right)^2\cdot\omega_n\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad+M\left(\int_0^L \left|\frac{\partial^2\omega_n}{\partial x^2}\right|^2 dx\right)^{\frac{1}{2}}+2M\left(\int_0^L \left|\frac{\partial(u_0+\theta\omega_n)}{\partial x}\cdot\frac{\partial\omega_n}{\partial x}\right|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we achieve due to (5.1)

$$\lim_{n\to\infty}\left(\int_0^L \left|\frac{\partial^2}{\partial x^2}\big(f_B(u_n)-f_B(u_0)\big)\right|^2 dx\right)^{\frac{1}{2}}=0.$$

Like the above estimates, we also get

$$\lim_{n\to\infty}\left(\int_0^L \left|\frac{\partial}{\partial x}\big(f_S(v_n)-f_S(v_0)\big)\right|^2 dx\right)^{\frac{1}{2}}=0.$$

The proof is complete. □

Lemma 5.2. *(See [2].) Let $g(u,u_t)=f_B'(u)u_t$, and* (F1), (F2) *hold, $f_B(0)=0$. Then $g: Y_3\times Y_2\to Y_0$ is continuous compact.*

Lemma 5.3. *(See [2].) The semigroup $\{S(t)\}_{t\geqslant 0}$ associated with* (1.1)–(1.3) *is norm-to-weak continuous in E_2.*

By Theorem 2.3, if the solution semigroup $\{S(t)\}_{t\geqslant 0}$ associated with the problem (1.1)–(1.3) is norm-to-weak continuous semigroup from strong topology to weak topology in E_2, then we can conclude the following results:

Theorem 5.4. *Suppose that $k>0$ and $\frac{k+2\eta}{\lambda}<\min\{\alpha,\beta\}$, $\alpha,\beta,\delta_1,\delta_2>0$, $h_B,h_S\in L^2(0,L)$, and conditions* (F1)–(F2) *hold. $\{S(t)\}_{t\geqslant 0}$ is a norm-to-weak continuous semigroup generated by the solution of the system* (1.1)–(1.3)*. Then the solution semigroup $\{S(t)\}_{t\geqslant 0}$ has a global attractor $\mathcal{A}$ in E_2, it attracts all bounded subsets of E_2 in the norm of E_2, where λ, η are given by* (2.1)–(2.3)*.*

Proof. Applying Theorems 2.3 and 4.2, we only need to prove that the condition (C) holds in E_2.

Similar to Theorem 4.1, let $\{\omega_i\}_{i=1}^{\infty}$ be an orthonormal basis of Y_3 which consists of eigenvectors of A^2, the corresponding eigenvalues are denoted by

$$0 < \nu_1 < \nu_2 \leqslant \nu_3 \leqslant \cdots, \quad \nu_i \to \infty, \text{ as } i \to \infty,$$

with $A^2\omega_i = \nu_i\omega_i$, $\forall i \in \mathbb{N}$. And we write $V_m = \mathrm{span}\{\omega_1, \omega_2, \ldots, \omega_m\}$.

In addition, let $\{\chi_j\}_{j=1}^{\infty}$ be an orthonormal basis of Y_2 which consists of eigenvectors of A, the corresponding eigenvalue are denoted by

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots, \quad \lambda_j \to \infty, \text{ as } j \to \infty,$$

with $A\chi_j = \lambda_j\chi_j$, $\forall j \in \mathbb{N}$. And we write $G_n = \mathrm{span}\{\chi_1, \chi_2, \ldots, \chi_n\}$.

Since $h_B, h_S \in L^2(0, L)$, $(f_B, f_S) : Y_3 \times Y_2 \to Y_2 \times Y_1$ are compact continuously verified by Lemma 5.1, then for any $\varepsilon > 0$, there exists $N > 0$, such that

$$\begin{aligned}
\big|(I - P_m)h_B\big|_{Y_0} &\leqslant \frac{\varepsilon}{4}, \\
\big|(I - P_m)f_B(u)\big|_{Y_2} &\leqslant \frac{\varepsilon}{4}, \quad \forall u \in B_{Y_3}(0, \mu_2), \\
\big|(I - Q_n)h_S\big|_{Y_0} &\leqslant \frac{\varepsilon}{4}, \\
\big|(I - Q_n)f_S(v)\big|_{Y_1} &\leqslant \frac{\varepsilon}{4}, \quad \forall v \in B_{Y_1}(0, \mu_2),
\end{aligned} \tag{5.4}$$

for $m, n \geqslant N$, where $P_m : Y_2 \to V_m$ and $Q_n : Y_1 \to G_n$ are orthogonal projector, μ_2 is given by (3.35). For any $(u, v; u_t, v_t) \in E_2$, we divide into

$$(u, v; u_t, v_t) = (u_1, v_1; u_{1t}, v_{1t}) + (u_2, v_2; u_{2t}, v_{2t}),$$

where $(u_1, v_1; u_{1t}, v_{1t}) = (P_m u, Q_n v; P_m u_t, Q_n v_t)$.

Taking the scalar product in $L^2(0, L)$ of the first and second equation of (1.1) with $A^2\phi_2 = A^2u_{2t} + \sigma A^2u_2$ and $A\psi_2 = Av_{2t} + \sigma Av_2$, respectively, after a computation, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\alpha\big|A^2u_2\big|^2 + \beta|Av_2|^2 + |A\phi_2|^2 + \|\psi_2\|^2\big) + \sigma\alpha\big|A^2u_2\big|^2 \\
&\quad + (\delta_1 - \sigma)|A\phi_2|^2 - \sigma(\delta_1 - \sigma)\big(A^2u_2, \phi_2\big) + \sigma\beta|Av_2|^2 \\
&\quad + (\delta_2 - \sigma)\|\psi_2\|^2 - \sigma(\delta_2 - \sigma)(Av_2, \psi_2) + \big(k\big((u - v)^+\big)_2, A^2\phi_2\big) \\
&\quad - \big(k\big((u - v)^+\big)_2, A\psi_2\big) + \big(f_B(u), A^2\phi_2\big) + \big(f_S(v), A\psi_2\big) \\
&= \big(h_B, A^2\phi_2\big) + (h_S, A\psi_2).
\end{aligned} \tag{5.5}$$

Now we deal with each of the terms one by one on the left-hand side. According to the Hölder inequality, Poincaré inequality and Young inequality, (3.19) and (3.35), we obtain

$$\begin{aligned}
&\big(k\big((u - v)^+\big)_2, A^2\phi_2\big) \\
&= \frac{d}{dt}\big(k\big((u - v)^+\big)_2, A^2u_2\big) - \big(k\big((u - v)^+\big)_{2t}, A^2u_2\big) + \sigma\big(k\big((u - v)^+\big)_2, A^2u_2\big)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{d}{dt}\big(\big(k(u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - k\big|\big((u-v)^+\big)_{2t}\big| \cdot \big|A^2u_2\big| \\
&\geqslant \frac{d}{dt}\big(k\big((u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - k\big|(u-v)_{2t}\big| \cdot \big|A^2u_2\big| \\
&\geqslant \frac{d}{dt}\big(k\big((u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - k\big(|u_{2t}| + |v_{2t}|\big) \cdot \big|A^2u_2\big| \\
&\geqslant \frac{d}{dt}\big(k\big((u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - k\left(\frac{\|u_{2t}\|}{\nu_{m+1}} + \frac{\|v_{2t}\|}{\lambda_{n+1}}\right)\big|A^2u_2\big| \\
&\geqslant \frac{d}{dt}\big(k\big((u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - k\mu_2\left(\frac{1}{\nu_{m+1}} + \frac{1}{\lambda_{n+1}}\right)\big|A^2u_2\big| \\
&\geqslant \frac{d}{dt}\big(k\big((u-v)^+\big)_2, A^2u_2\big) + \sigma\big(k\big((u-v)^+\big)_2, A^2u_2\big) - \frac{\sigma\alpha}{4}\big|A^2u_2\big|^2 \\
&\quad - \frac{2k^2\mu_2^2}{\sigma\alpha}\left(\frac{1}{\nu_{m+1}^2} + \frac{1}{\lambda_{n+1}^2}\right), \quad t \geqslant t_1,
\end{aligned} \tag{5.6}
$$

and

$$
\begin{aligned}
-\big(k\big((u-v)^+\big)_2, A\psi_2\big) &= k\big(\big((u-v)^+\big)_{2x}, \psi_{2x}\big) \geqslant -k\big|\big((u-v)^+\big)_{2x}\big| \cdot |\psi_{2x}| \\
&\geqslant -k\big|(u-v)_{2x}\big| \cdot |\psi_{2x}| \geqslant -k\big(|u_{2x}| + |v_{2x}|\big) \cdot |\psi_{2x}| \\
&\geqslant -k\left(\frac{1}{\nu_{m+1}}\|u_2\| + \frac{1}{\lambda_{n+1}}|Av_2|\right) \cdot \|\psi_2\| \\
&\geqslant -k\left(\frac{\mu_1}{\nu_{m+1}} + \frac{\mu_2}{\lambda_{n+1}}\right) \cdot \|\psi_2\| \\
&\geqslant -\frac{\delta_2}{4}\|\psi_2\|^2 - \frac{2k^2}{\delta_2}\left(\frac{\mu_1^2}{\nu_{m+1}^2} + \frac{\mu_2^2}{\lambda_{n+1}^2}\right), \quad t \geqslant t_1.
\end{aligned} \tag{5.7}
$$

Moreover, due to (5.4) there holds

$$
\begin{aligned}
&\big(h_B, A^2\phi_2\big) + (h_S, A\psi_2) \\
&\quad \leqslant \frac{d}{dt}\big((h_B)_2, A^2u_2\big) + \frac{d}{dt}\big((h_S)_2, Av_2\big) + \frac{\sigma\varepsilon}{4}\big|A^2u_2\big| + \frac{\sigma\varepsilon}{4}|Av_2| \\
&\quad \leqslant \frac{d}{dt}\big((h_B)_2, A^2u_2\big) + \frac{d}{dt}\big((h_S)_2, Av_2\big) + \frac{\sigma\alpha}{4}\big|A^2u_2\big|^2 + \frac{\sigma\beta}{4}|Av_2|^2 + \left(\frac{\sigma}{16\alpha} + \frac{\sigma}{16\beta}\right)\varepsilon^2,
\end{aligned} \tag{5.8}
$$

where $(h_B)_2 = (I - P_m)h_B$, $(h_S)_2 = (I - Q_n)h_S$. On the other hand,

$$
\big(f_B(u), A^2\phi_2\big) = \frac{d}{dt}\big(\big(f_B(u)\big)_2, A^2u_2\big) - \big(\big(f_B'(u)u_t\big)_2, A^2u_2\big) + \sigma\big(\big(f_B(u)\big)_2, A^2u_2\big). \tag{5.9}
$$

Using Lemma 5.2, it follows that

$$
\big|\big(f_B'(u)u_t\big)_2\big| < \frac{\varepsilon}{4}, \quad \forall (u, u_t) \in B_{E_2}\big((0,0), \mu_2\big). \tag{5.10}
$$

As a results, by (5.4), (5.9) and (5.10) we achieve

$$\begin{aligned}
&\big(f_B(u), A^2\phi_2\big) + \big(f_S(v), A\psi_2\big)\\
&\quad\geqslant \frac{d}{dt}\big(\big(f_B(u)\big)_2, A^2u_2\big) - \big(\big(f_B'(u)u_t\big)_2, A^2u_2\big) + \sigma\big(\big(f_B(u)\big)_2, A^2u_2\big) - \frac{\varepsilon}{4}\|\psi_2\|\\
&\quad\geqslant \frac{d}{dt}\big(\big(f_B(u)\big)_2, A^2u_2\big) + \sigma\big(\big(f_B(u)\big)_2, A^2u_2\big) - \frac{\varepsilon}{4}\big|A^2u_2\big| - \frac{\varepsilon}{4}\|\psi_2\|\\
&\quad\geqslant \frac{d}{dt}\big(\big(f_B(u)\big)_2, A^2u_2\big) + \sigma\big(\big(f_B(u)\big)_2, A^2u_2\big)\\
&\qquad - \frac{\sigma\alpha}{4}\big|A^2u_2\big|^2 - \frac{\delta_2}{4}\|\psi_2\|^2 - \left(\frac{1}{16\sigma\alpha} + \frac{1}{16\delta_2}\right)\varepsilon^2.
\end{aligned}\tag{5.11}$$

Together with (5.5)–(5.8) and (5.11) it leads to

$$\begin{aligned}
&\frac{d}{dt}\big[\alpha\big|A^2u_2\big|^2 + \beta|Av_2|^2 + |A\phi_2|^2 + \|\psi_2\|^2 + 2k\big(\big((u-v)^+\big)_2, A^2u_2\big)\\
&\qquad + 2\big(\big(f_B(u)\big)_2, A^2u_2\big) - 2\big((h_B)_2, A^2u_2\big) - 2\big((h_S)_2, Av_2\big)\big]\\
&\qquad + \frac{\sigma\alpha}{2}\big|A^2u_2\big|^2 + 2(\delta_1-\sigma)|A\phi_2|^2 - 2\sigma(\delta_1-\sigma)\big(A^2u_2, \phi_2\big)\\
&\qquad + \frac{3\sigma\beta}{2}|Av_2|^2 + (\delta_2 - 2\sigma)\|\psi_2\|^2 - 2\sigma(\delta_2-\sigma)(Av_2, \psi_2)\\
&\qquad + 2\sigma k\big(\big((u-v)^+\big)_2, A^2u_2\big) + 2\sigma\big(\big(f_B(u)\big)_2, A^2u_2\big)\\
&\quad\leqslant C\varepsilon^2, \quad t \geqslant t_1,\ \text{as } m, n \to \infty,
\end{aligned}\tag{5.12}$$

where

$$C = \frac{4k^2\mu_2^2}{\sigma\alpha} + \frac{4k^2(\mu_1^2+\mu_2^2)}{\delta_2} + \frac{1}{16\sigma\alpha} + \frac{1}{16\delta_2} + \left(\frac{1}{16\alpha} + \frac{1}{16\beta}\right)\sigma.$$

In line with the Poincaré and Hölder inequalities again, we obtain

$$\begin{aligned}
&\frac{\sigma\alpha}{2}\big|A^2u_2\big|^2 + 2(\delta_1-\sigma)|A\phi_2|^2 - 2\sigma(\delta_1-\sigma)\big(A^2u_2, \phi_2\big)\\
&\quad\geqslant \frac{\sigma\alpha}{2}\big|A^2u_2\big|^2 + 2(\delta_1-\sigma)|A\phi_2|^2 - \frac{2\sigma\delta_1}{\nu_1}\big|A^2u_2\big| \cdot \|\phi_2\|\\
&\quad\geqslant \frac{\sigma\alpha}{4}\big|A^2u_2\big|^2 + 2\left(\delta_1 - \sigma - \frac{\delta_1^2\sigma}{\alpha\nu_1^2}\right)|A\phi_2|^2,
\end{aligned}\tag{5.13}$$

and

$$\begin{aligned}
&\frac{3\sigma\beta}{2}|Av_2|^2 + (\delta_2-2\sigma)\|\psi_2\|^2 - 2\sigma(\delta_2-\sigma)(Av_2, \psi_2)\\
&\quad\geqslant \frac{3\sigma\beta}{2}|Av_2|^2 + (\delta_2-2\sigma)\|\psi_2\|^2 - \frac{2\sigma\delta_2}{\lambda_1}|Av_2| \cdot \|\psi_2\|\\
&\quad\geqslant \sigma\beta|Av_2|^2 + \left(\delta_2 - 2\sigma - \frac{2\delta_2^2\sigma}{\beta\lambda_1^2}\right)\|\psi_2\|^2.
\end{aligned}\tag{5.14}$$

Provided that σ is small enough, such that

$$\delta_1 - \sigma - \frac{\delta_1^2\sigma}{\alpha\nu_1^2} > \frac{\delta_1}{4}, \qquad \delta_2 - 2\sigma - \frac{2\delta_2^2\sigma}{\beta\lambda_1^2} > \frac{\delta_2}{2},$$

then combining with (5.13)–(5.14), we conclude from (5.12) that

$$\begin{aligned}
&\frac{d}{dt}\big[\alpha\big|A^2u_2\big|^2 + \beta|Av_2|^2 + |A\phi_2|^2 + \|\psi_2\|^2 + 2k\big(\big((u-v)^+\big)_2, A^2u_2\big) \\
&\quad + 2\big(\big(f_B(u)\big)_2, A^2u_2\big) - 2\big((h_B)_2, A^2u_2\big) - 2\big((h_S)_2, Av_2\big)\big] \\
&\quad + \frac{\sigma\alpha}{4}\big|A^2u_2\big|^2 + \frac{\delta_1}{2}|A\phi_2|^2 + \sigma\beta|Av_2|^2 + \frac{\delta_2}{2}\|\psi_2\|^2 \\
&\quad + 2\sigma\big(\big(f_B(u)\big)_2, A^2u_2\big) + 2\sigma k\big(\big((u-v)^+\big)_2, A^2u_2\big) \\
&< C\varepsilon^2, \quad t \geqslant t_1.
\end{aligned}$$

Furthermore, take $\rho = \min\{\frac{\sigma}{4}, \frac{\delta_1}{2}, \frac{\delta_2}{2}\}$, it follows that

$$\begin{aligned}
&\frac{d}{dt}\big[\alpha\big|A^2u_2\big|^2 + \beta|Av_2|^2 + |A\phi_2|^2 + \|\psi_2\|^2 + 2k\big(\big((u-v)^+\big)_2, A^2u_2\big) \\
&\quad + 2\big(\big(f_B(u)\big)_2, A^2u_2\big) - 2\big((h_B)_2, A^2u_2\big) - 2\big((h_S)_2, Av_2\big)\big] \\
&\quad + \rho\big[\alpha\big|A^2u_2\big|^2 + \beta|Av_2|^2 + |A\phi_2|^2 + \|\psi_2\|^2 \\
&\quad + 2k\big(\big((u-v)^+\big)_2, A^2u_2\big) + 2\big(\big(f_B(u)\big)_2, A^2u_2\big)\big] \\
&< C\varepsilon^2, \quad t \geqslant t_1.
\end{aligned} \tag{5.15}$$

Exploiting the Sobolev compact embedding inequality, integrating with (3.19) and (3.35), we achieve

$$\big|\big((u-v)^+\big)_2\big| < \varepsilon, \qquad \big|\big((u-v)^+\big)_{2t}\big| < \varepsilon.$$

In addition, we find

$$\begin{aligned}
&\frac{d}{dt}\big(\alpha\big|A^2u_2\big|^2 + 2k\big(\big((u-v)^+\big)_2, A^2u_2\big) + 2\big(\big(f_B(u)\big)_2, A^2u_2\big) - 2\big((h_B)_2, A^2u_2\big)\big) \\
&= \frac{d}{dt}\left|\sqrt{\alpha}A^2u_2 + \frac{k}{\sqrt{\alpha}}\big((u-v)^+\big)_2 + \frac{1}{\sqrt{\alpha}}\big(f_B(u)\big)_2 - \frac{1}{\sqrt{\alpha}}(h_B)_2\right|^2 \\
&\quad - \frac{d}{dt}\left[\frac{k^2}{\alpha}\big|\big((u-v)^+\big)_2\big|^2 + \frac{1}{\alpha}\big|\big(f_B(u)\big)_2\big|^2 + \frac{2k}{\alpha}\big(\big((u-v)^+\big)_2, \big(f_B(u)\big)_2\big)\right. \\
&\quad \left. - \frac{2k}{\alpha}\big(\big((u-v)^+\big)_2, (h_B)_2\big) - \frac{2}{\alpha}\big(\big(f_B(u)\big)_2, (h_B)_2\big)\right].
\end{aligned}$$

Therefore, due to Lemmas 5.1 and 5.2, together with (5.4), (5.15) we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\left|\sqrt{\alpha}A^2u_2 + \frac{k}{\sqrt{\alpha}}\big((u-v)^+\big)_2 + \frac{1}{\sqrt{\alpha}}\big(f_B(u)\big)_2 - \frac{1}{\sqrt{\alpha}}(h_B)_2\right|^2\right. \\
&\quad \left. + \left|\sqrt{\beta}Av_2 - \frac{1}{\sqrt{\beta}}(h_S)_2\right|^2 + \|\phi_2\|^2 + \|\psi_2\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\rho\bigg(\bigg|\sqrt{\alpha}A^2u_2+\frac{k}{\sqrt{\alpha}}\big((u-v)^+\big)_2+\frac{1}{\sqrt{\alpha}}\big(f_B(u)\big)_2-\frac{1}{\sqrt{\alpha}}(h_B)_2\bigg|^2\\
&+\bigg|\sqrt{\beta}Av_2-\frac{1}{\sqrt{\beta}}(h_S)_2\bigg|^2+\|\phi_2\|^2+\|\psi_2\|^2\bigg)\\
\leqslant{}& C\varepsilon^2+\frac{k^2}{\alpha}\big|\big((u-v)^+\big)_2\big|\cdot\big|\big((u-v)^+\big)_{2t}\big|+\frac{2}{\alpha}\big|\big(f_B(u)\big)_2\big|\cdot\big|\big(f_B'(u)u_t\big)_2\big|\\
&+\frac{2k}{\alpha}\big|\big((u-v)^+\big)_{2t}\big|\cdot\big|\big(f_B(u)\big)_2\big|+\frac{2k}{\alpha}\big|\big((u-v)^+\big)_2\big|\cdot\big|\big(f_B'(u)u_t\big)_2\big|\\
&+\frac{2k}{\alpha}\big|\big((u-v)^+\big)_{2t}\big|\cdot\big|(h_B)_2\big|+\frac{2}{\alpha}\big|\big(f_B'(u)u_t\big)_2\big|\cdot\big|(h_B)_2\big|\\
&+\frac{\rho k^2}{\alpha}\big|\big((u-v)^+\big)_2\big|^2+\frac{\rho}{\alpha}\big|\big(f_B(u)\big)_2\big|^2+\frac{\rho}{\alpha}\big|(h_S)_2\big|^2+\frac{\rho}{\beta}\big|(h_S)_2\big|^2\\
&+\frac{2\rho k}{\alpha}\big|\big((u-v)^+\big)_2\big|\cdot\big|\big(f_B(u)\big)_2\big|+\frac{2\rho k}{\alpha}\big|\big((u-v)^+\big)_2\big|\cdot\big|(h_B)_2\big|+\frac{2\rho}{\alpha}\big|\big(f_B(u)\big)_2\big|\cdot\big|(h_B)_2\big|\\
<{}&\tilde{C}\varepsilon^2,\quad t\geqslant t_1,
\end{aligned}
$$

where

$$\tilde{C}=C+\frac{(1+\rho)k^2+(6+4\rho)k+4+4\rho}{\alpha}+\frac{\rho}{\beta}.$$

We denote

$$
\begin{aligned}
Y(t)={}&\bigg|\sqrt{\alpha}A^2u_2+\frac{k}{\sqrt{\alpha}}\big((u-v)^+\big)_2+\frac{1}{\sqrt{\alpha}}\big(f_B(u)\big)_2-\frac{1}{\sqrt{\alpha}}(h_B)_2\bigg|^2\\
&+\bigg|\sqrt{\beta}Av_2-\frac{1}{\sqrt{\beta}}(h_S)_2\bigg|^2+|A\phi_2|^2+\|\psi_2\|^2,
\end{aligned}
$$

then

$$\frac{d}{dt}Y(t)+\rho Y(t)<\tilde{C}\varepsilon^2,\quad t\geqslant t_1.$$

By the Gronwall lemma

$$Y(t)\leqslant Y(t_1)\exp\big(-\rho(t-t_1)\big)+\frac{\tilde{C}\varepsilon^2}{\rho}.$$

Taking $t_2-t_1\geqslant\frac{1}{\rho}\log\frac{\mu_2^2}{\varepsilon^2}$, it follows that

$$Y(t)\leqslant\bigg(1+\frac{\tilde{C}}{\rho}\bigg)\varepsilon^2,\quad t\geqslant t_2.$$

So we complete the proof. □

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