Europ. J. Combinatorics (2000) **21**, 543–554
Article No. 10.1006/eujc.1999.0354
Available online at http://www.idealibrary.com on IDEAL

Finite Laguerre Near-planes of Odd Order Admitting Desarguesian Derivations

GÜNTER F. STEINKE

We introduce finite Laguerre near-planes and investigate such planes of odd order that admit a Desarguesian derivation.

1. Introduction and Results

A finite *Laguerre plane of order* n where $n \geq 2$ is an integer consists of a set P of points, a set $\mathcal{C}$ of circles and a set $\mathcal{G}$ of generators (subsets of P) such that the following four axioms are satisfied:

(P) P contains $n(n+1)$ points.
(G) $\mathcal{G}$ partitions P and each generator contains n points.
(C) Each circle intersects each generator in precisely one point.
(J) Three points, no two of which are on the same generator can be uniquely joined by a circle.

From this definition it readily follows that a Laguerre plane of order n has $n+1$ generators, that every circle contains exactly $n+1$ points and that there are n^3 circles.

All known models of finite Laguerre planes are of the following form. Let $\mathcal{O}$ be an oval in the Desarguesian projective plane $\mathcal{P}_2 = \mathrm{PG}(2, p^m)$, p a prime. Embed $\mathcal{P}_2$ into three-dimensional projective space $\mathcal{P}_3 = \mathrm{PG}(3, p^m)$ and let v be a point of $\mathcal{P}_3$ not belonging to $\mathcal{P}_2$. Then P consists of all points of the cone with base $\mathcal{O}$ and vertex v except the point v. Circles are obtained by intersecting P with planes of $\mathcal{P}_3$ not passing through v. In this way one obtains an *ovoidal Laguerre plane of order* p^m. If the oval $\mathcal{O}$ one starts off with is a conic, one obtains the *Miquelian Laguerre plane of order* p^m. All known finite Laguerre planes of odd order are Miquelian.

The *internal incidence structure* $\mathcal{A}_p$ at a point p of a Laguerre plane has the collection of all points not on the generator through p as point set and, as lines, all circles passing through p (without the point p) and all generators not passing through p. This is an affine plane, the *derived affine plane at* p. A circle K not passing through the distinguished point p induces an oval in the projective extension of the derived affine plane at p which intersects the line at infinity in the point corresponding to lines that come from generators of the Laguerre plane; in $\mathcal{A}_p$ one has a *parabolic curve*. (The derived affine planes of the Miquelian Laguerre planes are Desarguesian and the parabolic curves are parabolae whose axes are the verticals, i.e., the lines that come from generators of the Laguerre plane.) A Laguerre plane can thus be described in one derived affine plane $\mathcal{A}$ by the lines of $\mathcal{A}$ and a collection of parabolic curves. This planar description of a Laguerre plane, which is the most commonly used representation of a Laguerre plane, is then extended by the points of one generator where one has to adjoin a new point to each line and to each parabolic curve of the affine plane. It follows from [12] that every parabolic curve in a finite Desarguesian affine plane of odd order is in fact a parabola. Furthermore, using a simple counting argument, Chen and Kaerlein showed [2] that a finite Laguerre plane of odd order that admits a Desarguesian derivation is Miquelian.

0195–6698/00/040543 + 12 $35.00/0

The spatial description of an ovoidal Laguerre plane as the geometry of plane sections of an oval cone is related to the planar description in one derived plane by stereographic projection from one point of the cone onto a plane not passing through the point of projection. In this description all points of the Laguerre plane except the points on the generator through the point of projection are covered.

In this note we consider the restriction of a finite Laguerre plane of order n to one of its derived affine planes. When verifying the axioms of a Laguerre plane in such a planar representation one always has to consider special cases involving the extra points. We now ask to what extend the description in a derived affine plane determines the Laguerre plane. To our knowledge this problem has not yet been solved. To be more precise, a *Laguerre near-plane of order* $n \geq 3$ is an incidence structure of n^2 points, circles and generators satisfying the axioms (G), (C) and (J) from above. This definition extends the terminology for Minkowski planes and Möbius planes adopted in [10] and [13], respectively, see Section 4. Laguerre near-planes occur as special Laguerre semi-planes in [11] but have not been further investigated there. Also note that a Laguerre near-plane is not a restricted L1-space as defined in [18] since the restriction made in [18] on the number of points and lines in an internal incidence structure at a point is not satisfied.

Clearly, there are n generators, every circle contains exactly n points and there are n^3 circles. One obviously obtains a Laguerre near-plane of order n by deleting a generator from a Laguerre plane of order n. Conversely, it is not clear how to extend circles in order to construct a Laguerre plane from a Laguerre near-plane since all circles have the same length. Even worse, if an extension exists, it may not be unique, see Example 3.1.

Like for Laguerre planes, we have an internal incidence structure at each point of a Laguerre near-plane defined in exactly the same way, that is, the internal incidence structure $\mathcal{I}_p$ at a point p consists of all points not on the generator through p and the traces of all circles through p and all generators not passing through p. However, $\mathcal{I}_p$ is no longer an affine plane but it can be extended to an affine plane by adjoining some points, see Theorem 2.2. In view of Chen and Kaerlein's characterization [2] of Miquelian Laguerre planes of odd order in terms of a single derivation, we investigate Laguerre near-planes of odd order that contain a point whose internal incidence structure can be extended to a Desarguesian affine plane. We prove the following.

THEOREM 1. *A finite Laguerre near-plane of odd order $n \geq 7$ that admits a point whose internal incidence structure extends to a Desarguesian plane can be uniquely extended to the Miquelian Laguerre plane of order n by adjoining the points of one generator.*

Regarding Laguerre near-planes of small orders, we have the following.

THEOREM 2. *A finite Laguerre near-plane of order $n \leq 7$, $n \neq 4$, can be uniquely extended to the Miquelian Laguerre plane of order n by adjoining the points of one generator. In particular, there is no Laguerre near-plane of order 6.*

Laguerre near-planes of order 4 were studied and completely classified in [15], see Theorem 3.2 for a summary. We further have a brief look at Laguerre near-planes of even order and at the other two kinds of circle near-planes, Möbius near-planes and Minkowski near-planes, see [1] for a unifying algebraic description of Möbius, Laguerre and Minkowski planes.

2. PROOF OF THEOREMS 1 AND 2

We look at the internal incidence structure $\mathcal{I}_p$ at a point p of a Laguerre near-plane. We remove the generator through p. If $\mathcal{L}$ has order n, then we are left with $n(n-1)$ points.

Clearly, $\mathcal{L}_p$ is a linear space and there are $n+1$ lines through each point. Lines of $\mathcal{I}_p$ that come from a generator or a circle of $\mathcal{L}$ have length n or $n-1$, respectively. In particular, there is a unique line of length n through every point of $\mathcal{L}_p$. Furthermore, given a line L and a point $q \notin L$, there are either 1 or 2 lines through q that do not intersect L. Hence, we have a biaffine (or 2-affine) plane. In the notation of [7] we have found the following.

LEMMA 2.1. *The internal incidence structure $\mathcal{I}_p$ at a point p of a Laguerre near-plane is a biaffine plane of type II.*

Oehler determined all finite biaffine planes of type II, see [7, Satz 19 and Section 6].

THEOREM 2.2. *A biaffine plane of type II and order $n \geq 5$ can be obtained from an affine plane of order n by removing all points on a line, that is, from a projective plane by removing all points on two lines.*

The above theorem deals with the circles through the point p of derivation. They can be extended in at least two ways. Next we investigate circles not passing through p. Recall that a *k-arc* in a projective plane of order n is a collection of k points, no three of which are on a common line. The ovals are precisely the $(n+1)$-arcs, and the *hyperovals* are precisely the $(n+2)$-arcs. Note that hyperovals can only exist if n is even.

LEMMA 2.3. *Let p be a point of a Laguerre near-plane of order n and let $\mathcal{P}$ be a projective extension of the biaffine plane $\mathcal{I}_p$. A circle C not passing through p induces a n-arc in $\mathcal{P}$ by removing the point of C on the same generator as p and adding the infinite point of lines that come from generators of $\mathcal{L}$.*

PROOF. Let q be the point of C on the same generator as p and let (∞) be the infinite point of lines that come from generators of $\mathcal{L}$. Let $C' = C \setminus \{q\}$. It follows from axiom (J) that every line of $\mathcal{P}$ intersects C' in at most two points, i.e., C' is an $(n-1)$-arc of $\mathcal{P}$. Furthermore, lines of $\mathcal{P}$ that come from generators of $\mathcal{L}$ intersect C' in exactly one point. Thus we may add (∞) to C' and obtain an n-arc $C'' = C' \cup \{(\infty)\}$ of $\mathcal{P}$. □

In general, an n-arc in a projective plane of order n can be *complete*, that is, it is not properly included in an $(n+1)$-arc. Examples of complete 9-arcs in projective planes of order 9 can be found in [3] and [6]. For finite Desarguesian projective planes, however, we have the following, cf. [17] or [4, Section 8.6].

THEOREM 2.4. *A q-arc in a finite Desarguesian projective plane of order a prime power q can be extended to a conic by adjoining one point, if q is odd, or to a hyperoval by adjoining two points, if q is even.*

Let $\mathcal{L}$ be a finite Laguerre near-plane of odd order n and let p be a point of $\mathcal{L}$. By Lemma 2.1 and Theorem 2.2 the internal incidence structure $\mathcal{I}_p$ extends to a projective plane $\mathcal{P}$. We now assume that $\mathcal{P}$ is Desaguesian. In particular, $n = q$ is a power of an odd prime and $\mathcal{P}$ can be described over the Galois fields $\mathbb{F}_q = \mathrm{GF}(q)$ of order q. We use the representation of $\mathcal{P}$ as an affine plane plus the line at infinity, that is, $\mathcal{P}$ has point set $\mathbb{F}_q \times \mathbb{F}_q \cup \{(m) \mid m \in \mathbb{F}_q \cup \{\infty\}\}$ where (m) represents the infinite point of lines of slope m. We coordinatize $\mathcal{P}$ in such a way that the line W at infinity and the y-axis Y are the two lines that have been adjoined to $\mathcal{I}_p$ in order to obtain $\mathcal{P}$. Then the generators of $\mathcal{L}$ are represented by the vertical lines $\neq Y$. By Lemma 2.3 every circle not passing through p induces a q-arc and each such arc extends to a conic by Theorem 2.4. Since each of the q-arcs passes through $(\infty) = W \cap Y$ and has W and

Y as tangents, we find that the corresponding conics pass through (∞) and have either W or Y as a tangent. In a Desarguesian projective plane over a field $\mathbb{F}$ one readily determines these two types of conics.

LEMMA 2.5. *The conics in the Desarguesian projective plane $\mathcal{P}$ over the Galois field $\mathbb{F} = GF(q)$ that have W as a tangent are parabolae of the following form*

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_q\} \cup \{(\infty)\}$$

for $a, b, c \in \mathbb{F}_q$, $a \neq 0$.

The conics in $\mathcal{P}$ that have Y as a tangent are hyperbolae of the following form

$$\left\{\left(x, \frac{r}{x} + sx + t\right) \,\middle|\, x \in \mathbb{F}_q, x \neq 0\right\} \cup \{(\infty), (s)\}$$

for $r, s, t \in \mathbb{F}_q$, $r \neq 0$, where (s) denotes the infinite point on lines of slope s.

Correspondingly, we call circles of $\mathcal{L}$ of parabola or hyperbola type if they induce parabolae or hyperbolae, respectively, as above.

We now assume that there is a circle C whose corresponding conic in $\mathcal{P}$ is a parabola. Note that we can always make this assumption by interchanging the roles of W and Y. The next step is to make sure that every circle not passing through p is of parabola type. We begin with a special case.

LEMMA 2.6. *Let $q > 5$ and let $\mathcal{B}$ be a bundle of $q - 1$ parabolae or hyperbolae through the points $(1, 0)$ and $(u, 0)$ for some $u \in \mathbb{F}_q$, $u \neq 0, 1$. Assume that two distinct members of $\mathcal{B}$ only intersect in these two points or on $W \cup Y$. If $\mathcal{B}$ contains a parabola, then all members of $\mathcal{B}$ are parabolae.*

PROOF. The affine parts of parabolae in $\mathcal{B}$ are described by

$$y = a(x - 1)(x - u)$$

for some $a \in \mathbb{F}_q$, $a \neq 0$. Likewise, the affine parts of hyperbolae in $\mathcal{B}$ are described by

$$y = r\left(\frac{1}{x} + \frac{x}{u} - 1 - \frac{1}{u}\right) = \frac{r}{ux}(x - 1)(x - u)$$

for some $r \in \mathbb{F}_q$, $r \neq 0$.

We intersect parabolae and hyperbola of this form. For the x-coordinates of affine points of intersection one finds

$$x = 1 \qquad \text{or} \qquad x = u \qquad \text{or} \qquad x = \frac{r}{au}.$$

By assumption we must have that $\frac{r}{au} = 0$, 1 or u. This leads to

$$r = au \quad \text{or} \quad r = au^2.$$

Note that these relations must hold between any a's and r's occurring for members of $\mathcal{B}$. Hence, at most two hyperbolae are possible in $\mathcal{B}$. Since $q > 5$, there are at least three different parabolae in $\mathcal{B}$. This in turn leads to at least three different a's and thus to three different r's if there are any. As we have seen before, this is not possible. Therefore $\mathcal{B}$ cannot contain any hyperbola. $\square$

LEMMA 2.7. *For $q > 5$ all circles not passing through the point of derivation are of the same type.*

PROOF. Let C be a circle of parabola type and let $p_1, p_2 \in C$ be two points of $\mathcal{I}_p$ not on the same generator. We consider the bundle of circles in $\mathcal{L}$ through p_1 and p_2. This bundle contains $q - 1$ circle not passing through p. Furthermore, any two of these circles intersect only in p_1 and p_2. We now look at the situation induced in $\mathcal{P}$. We obtain a bundle $\mathcal{B}$ of $q - 1$ conics through p_1 and p_2 and this bundle contains a parabola (associated with the circle C). Using a collineation of $\mathcal{P}$ that fixes W and Y, we may assume that $p_1 = (1, 0)$ and $p_2 = (u, 0)$ for some $u \in \mathbb{F}_q$, $u \neq 0, 1$. Hence, all assumptions of Lemma 2.6 are satisfied and we find that all conics associated with circles through p_1 and p_2 are parabolae.

We now repeat the above argument for each of the circles through p_1 and p_2 (but not passing through p) and every pair of points on one of these circles (but none on the same generator as p). Since every circle not passing through p intersects at least one of these circles in two points, we obtain that all circles are of parabola type. □

So far we have found an extension $\mathcal{L}^*$ of the Laguerre near-plane $\mathcal{L}$. We add $Y \setminus \{(\infty)\}$ as a new generator G and each circle is extended naturally by the point of the associated parabola on Y. Clearly, the axioms (P), (G) and (C) of a Laguerre plane are satisfied in $\mathcal{L}^*$. For $\mathcal{L}^*$ to be a Laguerre plane we still have to verify axiom (J). By definition, the axiom of joining is satisfied in $\mathcal{L}$ and by construction also in $\mathcal{L}^* \setminus \{G\}$ because the latter is a Laguerre near-plane obtained from the Miquelian Laguerre plane of order q by deleting one generator. We now have to determine how the points on the generator through the point p of derivation fit into this picture.

Let $p' \neq p$ be a point on the same generator as p and let $r \in \mathcal{I}_p$. Let $\mathcal{B}$ be the bundle of circles through p' and r. There are q circles in $\mathcal{B}$ and any two of them intersect in p' and r only. We now look at the picture in the internal incidence structure $\mathcal{I}_r$ at r and its projective extension $\mathcal{P}'$. Let L_1 and L_2 be the two lines added to $\mathcal{I}_r$ in order to obtain $\mathcal{P}'$ and let $(\infty') = L_1 \cap L_2$ be the infinite point of lines in $\mathcal{P}'$ that come from generators of $\mathcal{L}$. Let L_3 be the line joining (∞') and p'. Then $\mathcal{B}$ gives us a collection of q lines passing through neither (∞') nor p such that any two of them intersect in a point of $L_1 \cup L_2 \cup L_3$. In the next lemma we study this situation for arbitrary projective planes.

LEMMA 2.8. *Let $\mathcal{P}'$ be a projective plane of order $n \geq 5$, let p_1 and p_2 be two points of $\mathcal{P}'$ and let L_1, L_2 and L_3 be three lines through p_1 such that L_3 passes through p_2. Furthermore, let $\mathcal{B}$ be a collection of n lines such that neither of them passes through p_1 nor through p_2 and such that any two of them intersect in a point of $L_1 \cup L_2 \cup L_3$. Then the lines in $\mathcal{B}$ must pass through a common point on L_3.*

PROOF. Suppose there are two lines M_1 and M_2 in $\mathcal{B}$ that intersect in a point u of L_1. Since there are n lines in $\mathcal{B}$ but only $n - 1$ points they can intersect L_3 in, there must be a line M_3 in $\mathcal{B}$ not passing through u. This line cannot intersect both M_1 and M_2 in a point of L_3 and likewise for L_2. So we may assume that $v = M_3 \cap M_1 \in L_3$ and $w = M_3 \cap M_2 \in L_2$, see Figure 1.

Since $n \geq 5$, there is a line $M_4 \in \mathcal{B} \setminus \{M_1, M_2, M_3\}$. Now M_4 cannot pass through u because otherwise it must meet M_1 in a point not on $L_1 \cup L_2 \cup L_3$. Similarly, M_4 cannot pass through v or w either. However, M_4 meets M_1 in a point of $L_1 \cup L_2 \cup L_3$. Hence M_4 must pass through $w' = M_1 \cap L_2$. One likewise finds that M_4 must pass through $u' = M_3 \cap L_1$ and $v' = M_2 \cap L_3$. In particular, M_4 is completely determined by u' and v' so that there can be at most one such line. This result contradicts $n \geq 5$.

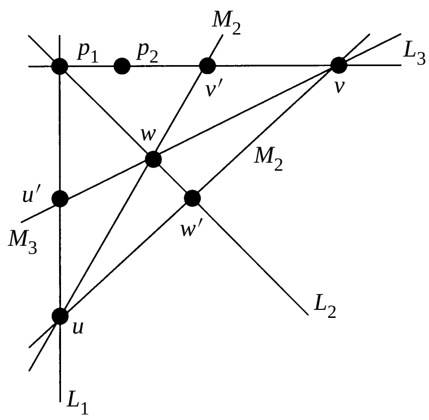

FIGURE 1.

The same argument shows that no two lines of $\mathcal{B}$ can intersect in a point of L_2. Hence any two lines of $\mathcal{B}$ must intersect in a point of L_3, that is, all these lines pass through a common point on L_3. $\square$

Note that Lemma 2.8 cannot be extended to $n = 4$ because in this case the points u', v' and w' can be collinear.

PROOF OF THEOREM 1. Let $\mathcal{L}$ be a Laguerre near-plane of order $q \geq 7$ and let p be a point of $\mathcal{L}$ whose internal incidence structure can be extended to the Desarguesian projective plane $\mathcal{P}$ of order q. Then, by Lemma 2.7, all circles not passing through p are of the same type and we can choose an extension $\mathcal{L}^*$ of $\mathcal{L}$ in such a way that the generator G through p becomes $\{\infty\} \times \mathbb{F}_q$ and circles of the extended plane are precisely the graphs of polynomials of degree at most 2.

By construction, circles through p are lines of $\mathcal{P}$, which are the graphs of polynomials of degree at most 1. Thus p can be identified with the point $(\infty, 0)$ in the Miquelian Laguerre plane of order q. As for points on G different from p, Lemma 2.8 states that the parabolae associated with the circles through $p' \in G \setminus \{p\}$ and $r \notin G$ pass through a common point in the corresponding Miquelian Laguerre plane. Hence we can label p' as (∞, a) where a denotes the leading coefficient the quadratic polynomials describing the associated parabolae. Hence, $\mathcal{L}^*$ can be identified with the Miquelian Laguerre plane of order q. $\square$

We now turn to Laguerre near-planes of small orders. We deal with orders 3 and 5 separately.

LEMMA 2.9. *A Laguerre near-plane of order 3 is obtained from the Miquelian Laguerre plane of order 3 by deleting one generator.*

PROOF. We call a triple of points *admissible* if and only if no two of the points are on the same generator. In a Laguerre near-plane of order 3, every circle corresponds to an admissible triple of points and each admissible triple of points must occur by axiom (J). This is exactly the same as in the Miquelian Laguerre plane of order 3 with one generator deleted. $\square$

LEMMA 2.10. *A Laguerre near-plane of order 5 is obtained from the Miquelian Laguerre plane of order 5 by deleting one generator.*

PROOF. It is well known that a projective plane of order 5 is Desarguesian (compare the remarks following 12.14 in [9]). Thus, we can follow the same path as in the proof of Theorem 1 except for Lemma 2.6. We take a closer look at the proof of this lemma in the case of order 5 and show that we can still obtain the result of Lemma 2.7. Adopting the notation used in the proof of Lemma 2.6 we know that all conics in the bundle $\mathcal{B}$ are parabolae if there are at least three parabolae in $\mathcal{B}$. Furthermore, given a parabola in $\mathcal{B}$ with coefficient a there are at most two hyperbolae in $\mathcal{B}$ and these hyperbolae have coefficients $r = au$ or $r = au^2$.

We therefore have two possible situations. Either all conics in $\mathcal{B}$ are parabolae—this is the case that we want to have—or there are precisely two parabolae and two hyperbolae in $\mathcal{B}$. If the parabolae have coefficients a_1 and a_2, then the coefficients r_1 and r_2 of the hyperbolae in $\mathcal{B}$ must satisfy $r_1 = a_1u$, $r_2 = a_1u^2$, $r_2 = a_2u$, $r_1 = a_2u^2$ up to relabelling. Hence, $u^2 = 1$ and thus $u = -1$. Futhermore, $a_2 = -a_1$, $r_2 = -r_1$ and $r_2 = a_1$. In particular, if $u \neq -1$, then all conics in $\mathcal{B}$ are parabolae.

Let $u = -1$ and let C be a parabola in $\mathcal{B}$ with coefficient a. We now form a second bundle $\mathcal{B}'$ consisting of the conics associated with the circles through the points $(1, 0)$ and $(v, a(v^2 - 1))$ where $v \neq 0, 1, -1$. This bundle contains the parabola C. Using a collineation of the Desarguesian plane that fixes the line W at infinity, the y-axis Y and the point $(1, 0)$, we can transform the point $(v, a(v^2 - 1))$ into a point $(v, 0)$. Therefore, as we have seen above and because $v \neq -1$, the bundle $\mathcal{B}'$ consists of parabolae only.

Let w be the fourth non-zero element $\neq 1, -1, v$ of $\mathbb{F}_5$. Then the conics in $\mathcal{B}'$ cover all but one point on the generator $G_w = \{(w, y) \mid y \in \mathbb{F}_5\}$. So if we assume that $\mathcal{B}$ contains two hyperbolae, at least one of them must intersect G_w in a point (w, h) which is also on a parabola in $\mathcal{B}$. But now the bundle consisting of the conics associated with the circles through $(1, 0)$ and (w, h) contains a parabola and a hyperbola although $w \neq -1$. This is impossible.

This shows that all circles of $\mathcal{L}$ are of parabola type and the statement now follows as in the proof of Theorem 1. $\square$

PROOF OF THEOREM 2. Let $\mathcal{L}$ be a Laguerre near-plane of order $n \leq 7$, $n \neq 4$. By Lemma 2.1 and Theorem 2.2 every internal incidence structure at a point of $\mathcal{L}$ is obtained from a projective plane or order n by removing all points on two lines. In particular, $n \neq 6$, because there are no orthogonal Latin squares of order 6 by [16] and consequently there is no projective plane of order 6. Lemmas 2.9 and 2.10 deal with order 3 and 5. Since every projective plane of order 7 is Desarguesian, see [9, Anhang 2], Theorem 1 now readily yields the desired result for $n = 7$. $\square$

3. LAGUERRE NEAR-PLANES OF EVEN ORDER

In this section we have a brief look at Laguerre near-planes of even order and give an example that shows that a Laguerre near-plane of even order may be extended in more than one way to a Laguerre plane of the same order. Furthermore, the case of order 4 provides examples of Laguerre near-planes that cannot be extended to Laguerre planes.

EXAMPLE 3.1. Consider the ovoidal Laguerre plane over an oval $\mathcal{O}$ in $\mathrm{PG}(2, 2^m)$. The tangents of $\mathcal{O}$ pass through a common point ν, the *nucleus of* $\mathcal{O}$, so that $\mathcal{O} \cup \{\nu\}$ becomes a hyperoval; cf. [5, Lemma 12.10] or [4, Section 8.1]. We can now remove any point of $\mathcal{O} \cup \{\nu\}$ and obtain again an oval. Hence, if we delete a generator from the ovoidal Laguerre over $\mathcal{O}$, we obtain a Laguerre near-plane of order 2^m. However, we can now either add the deleted generator or a generator formed from the line through the vertex and the nucleus of $\mathcal{O}$. In both cases we obtain a Laguerre plane. In general, the two Laguerre planes are not isomorphic. Substituting a point of a conic by its nucleus yields a translation oval which is not a conic unless $m \leq 2$. Hence, one extension is the Miquelian Laguerre plane whereas another extension is an ovoidal non-Miquelian Laguerre plane. In coordinates, let $\mathbb{F}_{2^m} = GF(2^m)$ be the Galois field of order 2^m. We consider the following Laguerre near-plane of order 2^m with point set $\mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, generators being the verticals $\{c\} \times \mathbb{F}_{2^m}$ for $c \in \mathbb{F}_{2^m}$ and circles being of the form

$$\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_{2^m}\}$$

for $a, b, c \in \mathbb{F}_{2^m}$. We extend this Laguerre near-plane by a generator $\{\infty\} \times \mathbb{F}_{2^m}$. A circle described by $a, b, c \in \mathbb{F}_{2^m}$ as above is adjoined the point (∞, a). This yields the Miquelian Laguerre plane of order 2^m. If we adjoin the point (∞, b) however we obtain an ovoidal non-Miquelian Laguerre plane of order 2^m if $m \geq 3$. (For $m = 1$ or 2 we obtain again the Miquelian Laguerre plane.) Explicitly, let ϕ be the permutation of $(\mathbb{F}_{2^m} \cup \{\infty\}) \times \mathbb{F}_{2^m}$ defined by $\phi(x, y) = (x^2, y)$. A circle $\{(x, ax^2 + bx + c) \mid x \in \mathbb{F}_{2^m}\} \cup \{(\infty, b)\}$ is taken under ϕ to

$$\{(u, bu^{2^{m-1}} + au + c) \mid u \in \mathbb{F}_{2^m}\} \cup \{(\infty, b)\}.$$

This is the familiar representation of the ovoidal Laguerre plane $\mathcal{L}(2^{m-1})$ over the translation oval

$$\{(x, x^{2^{m-1}}) \mid x \in \mathbb{F}_{2^m}\} \cup \{(\infty)\}$$

in the Desarguesian plane over $\mathbb{F}_{2^m}$.

The above example shows that it is possible for a Laguerre near-plane to be extended to two non-isomorphic Laguerre planes. Moreover, it is also possible that two non-isomorphic Laguerre near-planes can be extended to essentially the same Laguerre plane. To see this consider the following Laguerre near-plane of order $q = 2^m$, $m \geq 3$, whose circles are the sets

$$\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\}$$

for $a, b, c \in \mathbb{F}_q$. Adjoining the point (∞, a) to such a circle yields essentially the Laguerre plane $\mathcal{L}(2^{m-1})$ from above. The map

$$(x, y) \mapsto \begin{cases} (x^2, xy), & \text{if } x \in \mathbb{F}_q, x \neq 0, \\ (\infty, y), & \text{if } x = 0, \\ (0, y), & \text{if } x = \infty, \end{cases}$$

takes the set $\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\} \cup \{(\infty, a)\}$ to the set

$$\{(u, cu^{2^{m-1}} + bu + a) \mid u \in \mathbb{F}_q\} \cup \{(\infty, c)\}$$

so that one obtains the circles of $\mathcal{L}(2^{m-1})$. Note that $x^{q-1} = 1$ for $x \in \mathbb{F}_q, x \neq 0$. As seen above, this Laguerre plane is also the extension the Laguerre near-plane obtained from the Miquelian Laguerre plane by deleting one generator. However, the two Laguerre near-planes are not isomorphic. Under the map

$$(x, y) \mapsto \begin{cases} (x, xy), & \text{if } x \in \mathbb{F}_q, x \neq 0, \\ (0, y), & \text{if } x = 0, \end{cases}$$

the set $\{(x, ax^{q-2} + bx + c) \mid x \in \mathbb{F}_q\}$ is taken to the set

$$\{(x, bx^2 + cx + a) \mid x \in \mathbb{F}_q, x \neq 0\} \cup \{(0, c)\}$$

so that one almost has an isomorphism if it were not for the points on the generator $\{(0, y) \mid y \in \mathbb{F}_q\}$.

The two ways of extending the Laguerre near-planes in Example 3.1 also become apparent if we follow the path adopted for the proof of Theorem 1 in the case of Laguerre near-planes of odd order. Suppose we have a Laguerre near-plane $\mathcal{L}$ of even order $q = 2^m$ for some integer $m \geq 2$ such that $\mathcal{L}$ admits a Desarguesian extension $\mathcal{P}$ at one of its points. By Theorem 2.4, a circle not passing through the point p of derivation can be extended to a hyperoval in $\mathcal{P}$. Note that the situation here is completely symmetric in the two lines Y and W we had to adjoin in order to obtain the projective plane $\mathcal{P}$, that is, we can interchange the roles of Y and W. Ignoring points on W, we may add $Y \setminus \{(\infty)\}$ to $\mathcal{L}$ as a new generator of an extended incidence structure $\mathcal{L}^*$ and extend each circle by the point of intersection of its associated hyperoval with $Y \setminus \{(\infty)\}$. Alternatively, we may add $W \setminus \{(\infty)\}$ to $\mathcal{L}$ as a new generator and extend each circle by the point of intersection of its associated hyperoval with this set. As before in the odd case, the axioms (P), (G) and (C) of a Laguerre plane are satisfied in $\mathcal{L}^*$. Furthermore, the axiom of joining has only to be verified for admissible triples of points containing one point on the new generator $Y \setminus \{(\infty)\}$.

In [15] we investigated the case $q = 4$. We developed a representation of Laguerre near-planes of order 4 in terms of a single map and determined, up to isomorphism, all Laguerre near-planes of order 4. We further characterized those planes that can be extended to Laguerre planes. The results from [15] can be summarized as follows.

THEOREM 3.2. *Let $f : \mathbb{F}_4^3 \to \mathbb{F}_4$ where $\mathbb{F}_4 = \{0, 1, \omega, \omega + 1\}$, $\omega^2 = \omega + 1$, denotes the Galois field of order 4 be a map such that for each $x_0, y_0, z_0 \in \mathbb{F}_4$ the functions $x \mapsto f(x, y_0, z_0)$, $y \mapsto f(x_0, y, z_0)$ and $z \mapsto f(x_0, y_0, z)$ are permutations of $\mathbb{F}_4$. Such a map describes a Laguerre near-plane $\mathcal{L}(f)$ of order 4 as follows. The point set is $\mathbb{F}_4 \times \mathbb{F}_4$ and generators are the verticals $\{c\} \times \mathbb{F}_4$ for $c \in \mathbb{F}_4$. Circles are of the form*

$$\left\{ \left(u, (x, y, z, f(x, y, z)) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & \omega & \omega+1 & 0 \\ 1 & \omega+1 & \omega & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u^3 \\ u^2 \\ u \\ 1 \end{pmatrix} \right) \mid u \in \mathbb{F}_4 \right\}$$

for $x, y, z \in \mathbb{F}_4$. Conversely, every Laguerre near-plane of order 4 can be uniquely described in this way by such a map.

A Laguerre near-plane $\mathcal{L}(f)$ can be uniquely extended to the Miquelian Laguerre plane of order 4 by adjoining the points of one generator if and only if $f + f(0, 0, 0)$ is additive. Up to isomorphism, there are precisely five Laguerre near-planes of order 4. These planes are described by the maps

$$\begin{aligned}
f_0(x, y, z) &= x + y + z, \\
f_1(x, y, z) &= (x^2 + x)(y^2 + y) + x + y + z, \\
f_2(x, y, z) &= (x^2 + x)(y^2 + y) + (y^2 + y)(z^2 + z) + (x^2 + x)(z^2 + z) + x + y + z, \\
f_3(x, y, z) &= (x^2 + x)(y^2 + y)(z^2 + z) + x + y + z, \\
f_4(x, y, z) &= (x^2 + \omega^2 x)(y^2 + \omega y)(z^2 + \omega z) + (x^2 + \omega^2 x)(y^2 + \omega^2 y) \\
&\quad + (x^2 + \omega^2 x)(z^2 + \omega^2 z) + (y^2 + \omega y)(z^2 + \omega z) + x + y + z.
\end{aligned}$$

4. THE OTHER CIRCLE PLANES

We conclude this paper with a brief look at the other two kinds of circle planes: Möbius planes and Minkowski planes. In these cases, we have much more comprehensive results about their restrictions to derived affine planes, see [14] and [10]. To begin with, a finite *Minkowski plane of order* $n-1$ where $n \geq 3$ is an integer consists of a set P of points and a set $\mathcal{C}$ of circles such that the following four axioms are satisfied:

(P) P contains n^2 points.
(G) There are two classes $\mathcal{G}_1$ and $\mathcal{G}_2$ of generators each of which partitions P. Every generator contains n points and two generators of different classes intersect in precisely one point.
(C) Each circle intersects each generator in precisely one point.
(J) Three points, no two of which are on the same generator can be uniquely joined by a circle.

Minkowski planes of order $n-1$ correspond to sharply 3-transitive sets of permutations of a set of size n.

Deleting two intersecting generators from a Minkowski plane of order n yields an affine plane of order n and a collection of hyperbolic curves. This planar description of a Minkowski plane is the most commonly used representation of a Minkowski plane. Conversely such a model is then extended by the points of two intersecting generators. Since lines of the affine plane have n points and hyperbolic curves have $n-1$ points, it is obvious how to extend. The point of intersection of the two new generators is adjoined to all the lines of the affine plane. A hyperbolic curve is extended by two points corresponding to the two generators with which it does not have a point in common.

Again one can ask whether the planar description in a derived affine plane already determines the Minkowski plane. This question was answered by Rinaldi in [10], although the introduction of the so-called Minkowski near-planes of order n was motivated differently. More precisely, a *Minkowski near-plane of order* $n \geq 3$ is an incidence structure of points and circles satisfying the axioms (P), (G) and (J) from above but where axiom (C) is replaced by:

(C') Each circle intersects each generator in at most one point and contains either n or $n-1$ points.

Rinaldi showed that a Minkowski near-plane of order $n \geq 5$ is either a Minkowski plane of order $n-1$ or a Minkowski plane of order n with two intersecting generators deleted. Hence a *proper Minkowski near-plane of order* $n \geq 5$, that is, circles of lengths n and $n-1$ actually occur, extends to a Minkowski plane of order n. Clearly, such an extension is unique. Minkowski near-planes of order 3 and 4 were also described by Rinaldi in [10]. We summarize her results.

THEOREM 4.1. *A finite proper Minkowski near-plane of order $n \geq 5$ can be uniquely extended to a Minkowski plane of order n. The only other Minkowski near-planes of order $n \geq 5$ are the Minkowski planes of order $n-1$.*

Every Minkowski near-plane of order 3 is obtained from the Miquelian Minkowski near-plane of order 3 by deleting the points on the two generators through a particular point p and removing some (including none or all) circles not passing through p. Note that the Miquelian Minkowski plane of order 2 is obtained in the above way by removing all circles not passing through p.

Every Minkowski near-plane of order 4 is obtained from the Miquelian Minkowski plane of order 3 by converting some (including none or all) of its circles into all the 3-subsets contained in them.

A finite *Möbius* or *inversive plane of order* $n \geq 2$ is a $3-(n^2+1, n+1, 1)$ design. Explicitly, such a plane $\mathcal{M} = (P, \mathcal{C})$ consists of a set P of points and a set $\mathcal{C}$ of circles such that the following three axioms are satisfied:

(P) There are $n^2 + 1$ points.
(J) Three mutually distinct points can be uniquely joined by a circle.
(C) Each circle contains $n + 1$ points.

An *ovoidal* (or *egg-like*) *Möbius plane* is obtained as the geometry of non-trivial plane sections of an ovoid in three-dimensional projective space over some finite field. If the ovoid one starts off with is a quadric, one obtains a *Miquelian Möbius plane*.

Deleting one point from a Möbius plane of order n yields an affine plane of order n and a collection of ovals. In [14] we considered this restriction of a finite Möbius plane of order n to one of its derived affine planes and asked whether this already determines the Möbius plane. More precisely, a *Möbius near-plane of order* $n \geq 2$ is an incidence structure of points and circles satisfying the axiom (J) from above but where axioms (P) and (C) are replaced by

(P') There are n^2 points.
(C') Each circle contains $n + 1$ or n points.

The results from [14] can be summarized as follows.

THEOREM 4.2. *A finite Möbius near-plane of order $n \geq 5$ can be uniquely extended to a Möbius plane of order n by adjoining one point.*

Every Möbius near-plane of order 2 is obtained from the Miquelian Möbius plane of order 2 by deleting one point p and removing some (including none or all) circles through p. Moreover, there are, up to isomorphism, precisely eleven Möbius near-planes of order 2.

Every Möbius near-plane of order 3 is determined by a collection of 4-subsets such that these sets mutually intersect in at most two points. To obtain a Möbius near-plane one then adds all 3-subsets that are not contained in any of the 4-subsets. There are Möbius near-planes of order 3 that cannot be obtained from the Miquelian Möbius plane of order 3 by deleting one point p and replacing some circles not passing through p by the four 3-subsets contained in them.

There are Möbius near-planes of order 4 that cannot be extended to the Miquelian Möbius plane of order 4.

Olanda [8] considered finite *seminversive planes*, that is, incidence structures $(P, \mathcal{C})$ with at least two circles and at least three points on each circle such that axiom (J) is satisfied and such that for every circle $C \in \mathcal{C}$ and any two points p, q, where $p \in C$ and $q \notin C$, there are precisely one or two circles passing through q which intersect C only at p. He showed that such a seminiversive plane of order $n > 5$ is either a Möbius plane of order n or a Möbius plane of order n with one point deleted. However, the last condition in the definition of a seminversive plane is not necessarily satisfied in a Möbius near-plane and so a Möbius near-plane may not be a seminversive plane. Thus, Olanda's result [8] cannot be directly applied. However, of course, in the end the same incidence structures are obtained.

REFERENCES

1. W. Benz, *Vorlesungen über Geometrie der Algebren*, Springer, Berlin, 1973.
2. Y. Chen and G. Kaerlein, Eine Bemerkung über endliche Laguerre- und Minkowski-Ebenen, *Geom. Dedicata*, **2** (1973), 193–194.
3. R. H. F. Denniston, On arcs in projective planes of order 9, *Manuscr. Math.*, **4** (1971), 61–89.
4. J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford University Press, Oxford, 1979.

5. D. R. Hughes and F. C. Piper, *Projective Planes*, Springer, New York, 1973.
6. G. E. Martin, On arcs in a finite projective plane, *Canad. J. Math.*, **19** (1967), 376–393.
7. M. Oehler, Endliche biaffine Inzidenzebenen, *Geom. Dedicata*, **4** (1975), 419–436.
8. D. Olanda, Seminversive planes, *Ann. Discr. Math.*, **37** (1988), 311–313.
9. G. Pickert, *Projektive Ebenen*, 2nd edn, Springer, Berlin, 1975.
10. G. Rinaldi, Minkowski near-planes, *Result. Math.*, **16** (1989), 162–167.
11. G. Rinaldi, Laguerre semi-planes, *Atti Sem. Mat. Fis. Univ. Modena*, **40** (1992), 431–442.
12. B. Segre, Ovals in a finite projective plane, *Canad. J. Math.*, **7** (1955), 414–416.
13. G. F. Steinke, A remark on Benz planes of order 9, *Ars Combin.*, **34** (1992), 257–267.
14. G. F. Steinke, Finite Möbius near-planes, *Austral. J. Combin.*, **18** (1998), 77–96.
15. G. F. Steinke, Laguerre near-planes of order 4, Research Report, Department of Mathematics, University Canterbury, 1999, **175**.
16. M. G. Tarry, Le problème des 36 officiers, *Comptes rendus Ass. Franc. Avanc. Sci. natur*, **1** (1900), 122–123; **2** (1901), 170–203.
17. J. A. Thas, Complete arcs and algebraic curves in $PG(2, q)$, *J. Algebra*, **106** (1987), 441–452.
18. M. L. H. Willems and J. A. Thas, Restricted Li-spaces I, *Ars Combin.*, **13** (1982), 265–280.

Received 30 March 1999 and accepted 22 September 1999

GÜNTER F. STEINKE
Department of Mathematics and Statistics,
University of Canterbury,
Private Bag 4800,
Christchurch,
New Zealand