The spectrum for quasigroups with cyclic automorphisms and additional symmetries

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Abstract

We determine necessary and sufficient conditions for the existence of a quasigroup of order \( n \) having an automorphism consisting of a single cycle of length \( m \) and \( n - m \) fixed points, and having any combination of the additional properties of being idempotent, unipotent, commutative, semi-symmetric or totally symmetric. Quasigroups with such additional properties and symmetries are equivalent to various classes of triple systems.

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1. Introduction

A quasigroup \((Q, \circ)\) is a set \( Q \) together with a binary operation \( \circ \) on \( Q \) such that for every \( a, b \in Q \), there exists a unique \( x \) satisfying the equation \( a \circ x = b \) and a unique \( y \) satisfying the equation \( y \circ a = b \). A quasigroup \((Q, \circ)\) of order \( n \) may be thought of as a set of \( n^2 \) triples \((a, b, c)\) (where \( a, b, c \in Q \) and \( a \circ b = c \)) such that no two triples agree in more than one coordinate. We shall be concerned only with finite quasigroups. For brevity, a quasigroup \((Q, \circ)\), of order \( n \), having an automorphism consisting of a single cycle of length \( m \) and \( f = n - m \) fixed points, will be called an \( f \)-type quasigroup. In this paper, we shall always take the underlying set of such a quasigroup to be \( \mathbb{Z}_m \cup \{\infty_1, \infty_2, \ldots, \infty_f\} \).

A Latin square of order \( n \) is an \( n \) by \( n \) matrix in which each row and column is a permutation of some (fixed) symbol set of size \( n \). Clearly, \((Q, \circ)\) is a quasigroup if and only if the matrix \( Q_{i,j} = i \circ j \) with rows and columns indexed by \( Q \) is a Latin square. The Latin squares corresponding to \( f \)-type quasigroups were studied by Franklin in [11,12]. Franklin coined the term diagonally cyclic for those Latin squares corresponding to 0-type quasigroups and the term bordered diagonally cyclic for those corresponding to \( f \)-type quasigroups with \( f > 0 \). These are apt descriptions since the automorphisms imply that the elements occur in cyclic order in the diagonals (parallel to the main diagonal) of the Cayley table of a 0-type quasigroup (when the rows and columns are indexed in the natural order). The Cayley
table of an $f$-type quasigroup of order $n$ has an $m$ by $m$ body with a similar structure, except that $\infty_i$ occurs on one of the body's diagonals for $i = 1, 2, \ldots, f$, and there are additional rows and columns, the border, indexed by $\infty_1, \infty_2, \ldots, \infty_f$. The elements in rows $0, 1, \ldots, m - 1$ of columns $\infty_1, \infty_2, \ldots, \infty_f$ also occur in cyclic order, as do the elements in columns $0, 1, \ldots, m - 1$ of rows $\infty_1, \infty_2, \ldots, \infty_f$, and there is an $f \times f$ subsquare in the cells having both row and column index in $\{\infty_1, \infty_2, \ldots, \infty_f\}$. Note that this last condition implies that there are no $f$-type quasigroups of order $n$ for $f < n < 2f$. For further results and a survey of papers on diagonally and bordered diagonally cyclic Latin squares see [22]. The examples below show a 0-type quasigroup of order 7 (on the left) and a 1-type quasigroup of order 8 (on the right). Explicit examples of 2-type and 4-type quasigroups are given in Section 5.

\[
\begin{array}{cccccccc}
0 & 3 & 6 & 1 & 5 & 4 & 2 & \infty_1 \\
3 & 1 & 4 & 0 & 2 & 6 & 5 & \infty_1 \\
6 & 4 & 2 & 5 & 1 & 3 & 0 & 0 \\
1 & 0 & 5 & 3 & 6 & 2 & 4 & \infty_1 \\
5 & 2 & 1 & 6 & 4 & 0 & 3 & 6 \\
4 & 6 & 3 & 2 & 0 & 5 & 1 & \infty_1 \\
2 & 5 & 0 & 4 & 3 & 1 & 6 & \infty_1 \\
\end{array}
\]

In this paper we are interested in $f$-type quasigroups with additional symmetries and properties. A quasigroup $(Q, \circ)$ is idempotent if it satisfies the identity $x \circ x = x$. A quasigroup $(Q, \circ)$ is unipotent if there exists a $u \in Q$ such that $a \circ u = u$ for all $a \in Q$. A quasigroup $(Q, \circ)$ is commutative if it satisfies the identity $x \circ y = y \circ x$. A quasigroup $(Q, \circ)$ is semi-symmetric if it satisfies the identity $y \circ (x \circ y) = x$. Finally, a quasigroup $(Q, \circ)$ is totally symmetric if it satisfies both identities $x \circ y = y \circ x$ and $y \circ (x \circ y) = x$; that is, if it is both commutative and semi-symmetric. In the above examples, the 0-type quasigroup of order 7 is idempotent and totally symmetric, whilst the 1-type quasigroup of order 8 is unipotent and semi-symmetric.

Associated with each quasigroup $(Q, \circ)$ there are six conjugate quasigroups, one for each permutation in $S_3$. The conjugate corresponding to the identity permutation is the quasigroup itself. The conjugates are defined by taking all the equations $a \circ b = c$ which define $(Q, \circ)$ and uniformly permuting the positions of the variables $a, b, c$. For example, the transpose $(Q, *)$ of $(Q, \circ)$ is defined by $b \ast a = c$ if and only if $a \circ b = c$, while the translate $(Q, \cdot)$ of $(Q, \circ)$ is defined by $c \cdot a = b$ if and only if $a \circ b = c$. It should be obvious that a quasigroup is commutative precisely when it equals its transpose. It is also not hard to check that a quasigroup is semi-symmetric if and only if it equals its translate. In that case, it also equals the translate of its translate so that, in fact, three conjugates are equal. Moreover, a quasigroup is totally symmetric if and only if all six of its conjugates are equal.

The only possibilities for the number of equal conjugates are 1, 2, 3 or 6. In the case when 2 conjugates are equal there is always some conjugate of the quasigroup which is commutative. Also, if any three conjugates of a quasigroup are equal then the quasigroup must be semi-symmetric. So in studying commutative, semi-symmetric and totally symmetric quasigroups we are essentially studying all possible conjugate symmetries, that is, situations in which conjugates are equal.

The spectrum for a class $\mathcal{C}$ of quasigroups is the set of integers $n$ for which there is a quasigroup of order $n$ in $\mathcal{C}$. The goal of this paper is to summarise the known results and, where necessary, complete the spectrum for $f$-type quasigroups, for all non-negative integers $f$, which have any combination of the additional properties of being idempotent, unipotent, commutative, semi-symmetric or totally symmetric.

These classes of quasigroups have well-known equivalences with various combinatorial designs including triple systems and 1-factorisations of complete and complete bipartite graphs. Designs with underlying set $\mathbb{Z}_m$ and having the permutation $a \rightarrow a + 1$ as an automorphism are called cyclic designs and designs with underlying set $\mathbb{Z}_m \cup \{\infty\}$ and having the permutation $a \rightarrow a + 1$ as an automorphism are called $l$-rotational. Hence, many of the 0-type and 1-type quasigroup spectra in which we are interested have already been determined under the guise of cyclic and 1-rotational designs.

2. Main results

We begin this section with a well-known theorem, and then we summarise the main results which will be proved in later sections.
**Theorem 2.1.** There are no 0-type quasigroups of even order.

This simple result has been proved in a number of different guises. It is used (perhaps implicitly) in the results we quote in later sections to establish other spectra. See [22] for a proof and a further discussion of its history.

It was mentioned previously that there are no $f$-type quasigroups of order $n$ for $f < n < 2f$. The following theorem settles the case $n = f$ and is easily established from well-known results such as the existence of Steiner and Mendelsohn triple systems. For example, unipotent totally symmetric quasigroups are equivalent to *Steiner loops*, or *sloops*, and a sloop of order $n + 1$ exists if and only if there exists a Steiner triple system of order $n$. The equivalence is essentially obtained using the method of prolongation and contraction described in the next section. Results used to establish the theorem can be found in [17,3,14] and [9]. Note that throughout this paper, “arbitrary” will be shorthand for “possibly idempotent, possibly unipotent and possibly neither”.

**Theorem 2.2.** For $n < 2f$, the only $f$-type quasigroups satisfy $n = f$ and the spectra for these quasigroups are as indicated in the following table.

<table>
<thead>
<tr>
<th>$f = n$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>all</td>
<td>all</td>
<td>all</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all</td>
<td>all odd</td>
<td>0,1 mod 3 except 6</td>
<td>1,3 mod 6</td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>all even</td>
<td>1,2 mod 3 except 7</td>
<td>2,4 mod 6</td>
</tr>
</tbody>
</table>

**Theorem 2.2** will be used later to establish the spectra for $f$-type quasigroups having a non-trivial body. The following theorem, which will be proven in subsequent sections, gives the spectrum for the case $n \geq 2f$.

**Theorem 2.3.** There are no idempotent $f$-type quasigroups of order $n = 2f$. Otherwise, for $n \geq 2f$ there exists an $f$-type quasigroup of order $n$ having the additional properties indicated if and only if $n$ satisfies the conditions given in the following tables.

<table>
<thead>
<tr>
<th>$f = 0$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arbitrary</td>
<td>all odd</td>
<td>all odd</td>
<td>1,3 mod 6</td>
<td>1,3 mod 6</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all odd</td>
<td>all odd</td>
<td>1,3 mod 6 except 9</td>
<td>1,3 mod 6 except 9</td>
</tr>
<tr>
<td>Unipotent</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f = 1$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>all</td>
<td>1,2,3,4,5 mod 6 except 10</td>
<td>3,9 mod 24</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all</td>
<td>all odd</td>
<td>1,3,4 mod 6 except 10</td>
<td>3,9 mod 24</td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>all even</td>
<td>2,4 mod 6 except 10</td>
<td>2,4 mod 6 except 10</td>
</tr>
<tr>
<td>$f \equiv 2 \mod 3$</td>
<td>No specified conjugate symmetry</td>
<td>Commutative</td>
<td>Semi-symmetric</td>
<td>Totally symmetric</td>
</tr>
<tr>
<td>-------------------</td>
<td>---------------------------------</td>
<td>-------------</td>
<td>----------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>all even</td>
<td>1,2 mod 3</td>
<td>4,10 mod 24</td>
</tr>
<tr>
<td>Idempotent</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>all even</td>
<td>2,4,5 mod 6</td>
<td>4,10 mod 24</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>except 11</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f \equiv 0 \mod 3$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f \geq 3$</td>
<td></td>
<td>$\emptyset$</td>
<td>0,1 mod 3</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>0,1 mod 3</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>except when</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$n = 12$ or $f = 6$</td>
<td></td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f \equiv 1 \mod 3$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f \geq 4$</td>
<td></td>
<td>$\emptyset$</td>
<td>all</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>$\emptyset$</td>
<td>1,2 mod 3</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>1,2 mod 3</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>except when</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$n = 13$ or $f = 7$</td>
<td></td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>1,2 mod 3</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f \equiv 2 \mod 3$</th>
<th>No specified conjugate symmetry</th>
<th>Commutative</th>
<th>Semi-symmetric</th>
<th>Totally symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f \geq 5$</td>
<td></td>
<td>$\emptyset$</td>
<td>1,2 mod 3</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Arbitrary</td>
<td>all</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Unipotent</td>
<td>all</td>
<td>$\emptyset$</td>
<td>1,2 mod 3</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

### 3. Prolongation and contraction

In this section we observe a connection between $f$-type quasigroups of order $n$ and unipotent $(f+1)$-type quasigroups of order $n+1$. This connection is well known, but we include it here for completeness and so that it is stated in the most convenient form for our purposes.

**Theorem 3.1.** There exists an idempotent, commutative $f$-type quasigroup of order $n$ if and only if there exists a unipotent, commutative $(f+1)$-type quasigroup of order $n+1$. The same sentence is true if both occurrences of “commutative” are replaced by “semi-symmetric”, or both by “totally symmetric”.

**Proof.** Given an idempotent $f$-type quasigroup $(Q, \circ)$ of order $n$, we define a binary operation $*$ on $Q \cup \{\infty_{f+1}\}$ by $a * a = \infty_{f+1}$ for all $a \in Q \cup \{\infty_{f+1}\}$, $a * \infty_{f+1} = \infty_{f+1} * a = a$ for all $a \in Q$, and $a * b = a \circ b$ for distinct $a, b \in Q$. Then $(Q \cup \{\infty_{f+1}\}, *)$ is a unipotent $(f+1)$-type quasigroup of order $n+1$. Moreover, if $(Q, \circ)$ is commutative, semi-symmetric or totally symmetric, then $(Q \cup \{\infty_{f+1}\}, *)$ is commutative, semi-symmetric or totally symmetric respectively.
Conversely, suppose we are given a unipotent \((f + 1)\)-type quasigroup \((Q \cup \{\infty_{f+1}\}, \cdot)\) of order \(n + 1\) which is commutative, semi-symmetric or totally symmetric. Clearly, any such quasigroup satisfies the conditions \(a \cdot a = \infty_i\) and \(a \cdot \infty_i = \infty_i \cdot a\) for some \(i \in \{1, 2, \ldots, f + 1\}\) and for all \(a \in Q\). Applying an isomorphism which interchanges \(\infty_i\) and \(\infty_{f+1}\) preserves the properties of commutativity, semi-symmetry and total-symmetry, and yields a unipotent \((f + 1)\)-type quasigroup \((Q \cup \{\infty_{f+1}\}, \cdot)\) in which \(a \cdot a = \infty_{f+1}\) and \(a \cdot \infty_{f+1} = \infty_{f+1} \cdot a\) for all \(a \in Q\).

Now define a binary operation \(\circ\) on \(Q\) by \(a \circ a = a \cdot \infty_{f+1}\) for all \(a \in Q\) and \(a \circ b = a \cdot b\) for distinct \(a, b \in Q\). Then \((Q, \circ)\) is an \(f\)-type quasigroup in which \(\{a \circ a : a \in Q\} = Q\). Then \((Q, \circ)\) is an \(f\)-type quasigroup which is commutative, semi-symmetric or totally symmetric if \((Q \cup \{\infty_{f+1}\}, \cdot)\) is commutative, semi-symmetric or totally symmetric respectively. Since \(a \cdot a = \infty_{f+1}\) for all \(a \in Q\), if \((Q \cup \{\infty_{f+1}\}, \cdot)\) is semi-symmetric or totally symmetric then we have \(a \cdot \infty_{f+1} = \infty_{f+1} \cdot a\) for all \(a \in Q\), and so \((Q, \circ)\) is idempotent. Finally, if \((Q \cup \{\infty_{f+1}\}, \cdot)\) is commutative we can relabel symbols such that each \(i \in \mathbb{Z}_m\) \((m = n - f)\) is relabeled \(i + k\) for some fixed \(k \in \mathbb{Z}_m\) and such that the resulting quasigroup is idempotent. This relabeling preserves commutativity and yields the required idempotent \(f\)-type quasigroup. 

The process of constructing the \((f + 1)\)-type quasigroup \((Q \cup \{\infty_{f+1}\}, \cdot)\) from the quasigroup \((Q, \circ)\) as given in the proof of the preceding theorem is known as prolongation or extension (see, for example, [9]). Prolongation can be defined more generally, but in this paper we will always use it in the restricted sense just defined. The reverse of prolongation is called contraction or compression [9].

4. Commutative quasigroups

In this section we prove the results in Theorem 2.3 pertaining to commutative \(f\)-type quasigroups. The following result describes on which diagonals of a commutative \(f\)-type quasigroup the fixed points may occur.

**Theorem 4.1.** Let \((Q, \circ)\) be a commutative \(f\)-type quasigroup of order \(m + f\). If \(m\) is odd, then \(f \in \{0, 1\}\) and when \(f = 1\), \(a \circ a = \infty_1\) for all \(a \in Q\). If \(m = 2k\) is even, then \(f \in \{1, 2\}\). If \(m = 2k\) and \(f = 1\), then \(a \circ (a + k) = \infty_1\) for all \(a \in Q\). If \(m = 2k\) and \(f = 2\), then either

(i) \(a \circ a = \infty_1\) and \(a \circ (a + k) = \infty_2\) for all \(a \in \mathbb{Z}_m\), or

(ii) \(a \circ a = \infty_2\) and \(a \circ (a + k) = \infty_1\) for all \(a \in \mathbb{Z}_m\).

**Proof.** Suppose \(0 \circ d = \infty_i\) for some \(d \in \mathbb{Z}_m\) and some fixed point \(\infty_i\). Then since \((Q, \circ)\) is \(f\)-type, \(d \circ (d + d) = \infty_i\), and since \((Q, \circ)\) is commutative, \(d \circ 0 = \infty_i\). Hence \(d + d = 0\) and so \(2d \equiv 0 (\text{mod } m)\). If \(m\) is odd, this implies \(d = 0\) and so we have \(f \in \{0, 1\}\) and \(a \circ a = \infty_1\) for all \(a \in Q\) when \(f = 1\). If \(m = 2k\) is even, then we have \(d = 0\) or \(d = k\) and so \(f \in \{0, 1, 2\}\). Theorem 2.1 excludes the case \(f = 0\). Simple counting arguments give that unipotent commutative quasigroups are necessarily of even order. Hence when \(m = 2k\) and \(f = 1\), we have \(d \neq 0\) and thus \(a \circ (a + k) = \infty_1\) for all \(a \in Q\). Finally, if \(f = 2\) then either (i) or (ii) must hold. 

A **starter** in \(\mathbb{Z}_{2k+1}\) is a set \(S = \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_k, b_k\}\) such that

(i) \(a_1, b_1, a_2, b_2, \ldots, a_k, b_k\) are all the non-zero elements of \(\mathbb{Z}_{2k+1}\);

(ii) \(\pm(a_1 - b_1), \pm(a_2 - b_2), \ldots, \pm(a_k - b_k)\) are all the non-zero elements of \(\mathbb{Z}_{2k+1}\).

An **even starter** in \(\mathbb{Z}_{2k}\) is a set \(E = \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_{k-1}, b_{k-1}\}\) such that

(i) \(a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}\) are distinct non-zero elements of \(\mathbb{Z}_{2k}\);

(ii) \(\pm(a_1 - b_1), \pm(a_2 - b_2), \ldots, \pm(a_{k-1} - b_{k-1})\) are all the non-zero elements except \(k\) of \(\mathbb{Z}_{2k}\).

It is immediate from (i) that there is a unique non-zero element of \(\mathbb{Z}_{2k}\) which is not present in \(E\). Readers not familiar with starters and even starters may wish to consult [10].

**Theorem 4.2.** There is a natural bijection between idempotent, commutative 0-type quasigroups of order \(2k + 1\) and starters in \(\mathbb{Z}_{2k+1}\).

**Proof.** We give a bijection which maps each starter \(S\) in \(\mathbb{Z}_{2k+1}\) to an idempotent commutative 0-type quasigroup \((\mathbb{Z}_{2k+1}, \circ)\) of order \(2k + 1\). Suppose \(S\) is a starter as defined above. For each pair \([a, b]\) in \(S\) and each \(c \in \mathbb{Z}_{2k+1}\), we put \((a + c) \circ (b + c) = (b + c) \circ (a + c) = c\) and \(c \circ c = c\). Then \((\mathbb{Z}_{2k+1}, \circ)\) is an idempotent, commutative 0-type quasigroup of order \(2k + 1\).
Conversely, suppose that \((\mathbb{Z}_{2k+1}, \circ)\) is an idempotent, commutative 0-type quasigroup of order \(2k + 1\). If we let
\[
S = \{\{a, b\} : a \circ b = 0 \text{ and } a, b \in \mathbb{Z}_{2k+1} \setminus \{0\}\},
\]
then \(S\) is a starter in \(\mathbb{Z}_{2k+1}\). \(\square\)

The corresponding result for 1-type quasigroups is:

**Theorem 4.3.** There is a natural bijection between idempotent, commutative 1-type quasigroups of order \(2k + 1\) and even starters in \(\mathbb{Z}_{2k}\).

**Proof.** The proof is similar to that of the previous result. Let \(E\) be an even starter and let \(e\) be the unique non-zero element of \(\mathbb{Z}_{2k}\) that is not present in \(E\). For each pair \(\{a, b\}\) in \(E\) and for each \(c \in \mathbb{Z}_{2k}\) we define
\[c + a \circ (c + b) = (c + b) \circ (c + a) = c, c \circ c = c, \infty_1 \circ (c + e) = (c + e) \circ \infty_1 = c\] and \(c \circ (c + k) = \infty_1\). Finally, let \(\infty_1 \circ \infty_1 = \infty_1\). Then \((\mathbb{Z}_{2k} \cup \{\infty_1\}, \circ)\) is an idempotent, commutative 1-type quasigroup of order \(2k + 1\).

Conversely, suppose that \((\mathbb{Z}_{2k} \cup \{\infty_1\}, \circ)\) is an idempotent commutative 1-type quasigroup of order \(2k + 1\). If we let
\[
E = \{\{a, b\} : a \circ b = 0 \text{ and } a, b \in \mathbb{Z}_{2k} \setminus \{0\}\},
\]
then \(E\) is an even starter in \(\mathbb{Z}_{2k}\). Note that by Theorem 4.1, we have \(c \circ (c + k) = (c + k) \circ c = \infty_1\) for all \(c \in \mathbb{Z}_{2k}\), and so we do not get any pairs \(\{a, b\}\) with \(\pm(a - b) = k\). \(\square\)

The three results just proved enable us to settle the existence spectra for commutative \(f\)-type quasigroups. Starters exist in \(\mathbb{Z}_{2k+1}\) for every positive integer \(k\) (one can take, for example, the well-known patterned starter). This gives us idempotent, commutative 0-type quasigroups of all odd orders, and hence by prolongation, unipotent, commutative 1-type quasigroups of all even orders. Given that Anderson [1] showed that even starters exist in \(\mathbb{Z}_{2k}\) for all positive integers \(k\), idempotent commutative 1-type quasigroups of all odd orders, and hence (by prolongation) unipotent commutative 2-type quasigroups of all even orders, can be constructed from Theorem 4.3. These constructions yield all the possible orders. To see this, note that Theorem 2.1 rules out 0-type quasigroups of even order, that simple counting arguments rule out idempotent, commutative quasigroups of even order and unipotent, commutative quasigroups of odd order, and that Theorem 4.1 rules out idempotent, commutative 2-type quasigroups, commutative 2-type quasigroups of odd order, and commutative \(f\)-type quasigroups for \(f \geq 3\).

5. Semi-symmetric quasigroups

In this section we prove the results in Theorem 2.3 pertaining to semi-symmetric \(f\)-type quasigroups.

A Mendelsohn triple system of order \(n\) is a pair \((V, B)\) where \(V\) is an \(n\)-set, and \(B\) is a collection of cyclically ordered (or directed) triples of elements of \(V\) such that each ordered pair of distinct elements in \(V\) occurs in exactly one triple of \(B\) (the triple \((a, b, c)\) contains the three ordered pairs \((a, b), (b, c)\) and \((c, a))\). We note the well-known equivalence between Mendelsohn triple systems and idempotent, semi-symmetric quasigroups. For distinct \(a\) and \(b\), \(a \circ b = c\) if and only if \(c\) is the third element of the triple containing the ordered pair \((a, b)\) (and \(a \circ a = a\) for all \(a \in V\)).

Consider a triple \((a, b, c)\) of a semi-symmetric \(f\)-type quasigroup \((Q, \circ)\), where \(a, b\) and \(c\) are distinct and \(a, b, c \in \mathbb{Z}_m\). Because \((Q, \circ)\) is semi-symmetric and has an automorphism consisting of a single cycle of length \(m\), it follows that for each \(i \in \mathbb{Z}_m\),
\[
(a + i, b + i, c + i), (b + i, c + i, a + i), (c + i, a + i, b + i)
\]
are three triples of \((Q, \circ)\). Unless \(m \equiv 0(\text{mod } 3)\) and \(a \equiv b \equiv c(\text{mod } m/3)\), these \(3m\) triples are distinct and yield three distinct diagonals in the body of \((Q, \circ)\). If \(m \equiv 0(\text{mod } 3)\) and \(a \equiv b \equiv c(\text{mod } m/3)\), then there are only \(m\) distinct triples and one diagonal results. This observation yields the necessary conditions on the order of semi-symmetric \(f\)-type quasigroups stated in the following lemma. The additional necessary conditions for the idempotent and unipotent cases follow directly from Theorem 2.2.
Lemma 5.1. An idempotent, semi-symmetric \( f \)-type quasigroup of order \( n \) exists only if \( f \equiv 0, 1(\text{mod } 3) \), \( f \neq 6 \) and \( n \equiv 0, 1(\text{mod } 3) \). A unipotent, semi-symmetric \( f \)-type quasigroup of order \( n \) exists only if \( f \equiv 1, 2(\text{mod } 3) \), \( f \neq 7 \) and \( n \equiv 1, 2(\text{mod } 3) \). An arbitrary, semi-symmetric \( f \)-type quasigroup of order \( n \) exists only if \( f \equiv 0(\text{mod } 3) \) and \( n \equiv 0, 1(\text{mod } 3) \), or \( f \equiv 1(\text{mod } 3) \), or \( f \equiv 2(\text{mod } 3) \) and \( n \equiv 1, 2(\text{mod } 3) \).

Lemma 5.2. Let \((M, o)\) be an arbitrary, semi-symmetric \( f \)-type quasigroup of order \( n = m + f \), where \( m \geq f + 3 \). Then there exists an arbitrary, semi-symmetric \((f + 3)\)-type quasigroup of order \( n + 3 \). Furthermore, if \((M, o)\) is idempotent and \( m \geq f + 4 \), and there exists an idempotent, semi-symmetric quasigroup of order \( f + 3 \), then there exists an idempotent, semi-symmetric \((f + 3)\)-type quasigroup of order \( n + 3 \).

Proof. Either

(i) \((M, o)\) contains a triple of the form \((0, a, b)\) where the orbit of \((0, a, b)\) under the permutation \((0, 1, \ldots, m - 1)\) has length \( m \), or

(ii) \((M, o)\) contains the three triples \((0, 0, 0), (0, m/3, 2m/3)\) and \((0, 2m/3, m/3)\).

Note that if \((M, o)\) is idempotent and \( m \geq f + 4 \), then \((M, o)\) satisfies condition (i). Let \( M^\infty = \{\infty_1, \infty_2, \ldots, \infty_{f + 3}\} \). If \( f \equiv 0, 1(\text{mod } 3) \) and \( f \neq 3 \), let \((M^\infty, *)\) be an idempotent, semi-symmetric quasigroup of order \( f + 3 \). Otherwise, let \((M^\infty, *)\) be an arbitrary, semi-symmetric quasigroup of order \( f + 3 \).

If condition (i) holds, we define a binary operation \(*\) on \( M \cup \{\infty_{f + 1}, \infty_{f + 2}, \infty_{f + 3}\} \) as follows:

\( x * y = x \circ y \) for each \( x, y \in \mathbb{Z}_m \), \((x, y) \not\in \{(i, i + a), (i + b, i), (i + a, i + b) : i \in \mathbb{Z}_m\}\);

\( x * \infty_i = x \circ \infty_i \), \( \infty_i * x = \infty_i \circ x \) for each \( x \in \mathbb{Z}_m \), \( i \in \{1, 2, \ldots, f\} \);

\( \infty_i * \infty_k = \infty_j * \infty_k \) for all \( j, k \in \{1, 2, \ldots, f + 3\} \);

\( i * (i + a) = \infty_{f + 1}, \infty_{f + 2} * i = i + a, (i + a) * \infty_{f + 1} = i \) for each \( i \in \mathbb{Z}_m \);

\( (i + b) * i = \infty_{f + 2}, \infty_{f + 3} * (i + b) = i, i * \infty_{f + 2} = i + b \) for each \( i \in \mathbb{Z}_m \);

\( (i + a) * (i + b) = \infty_{f + 3}, \infty_{f + 3} * (i + a) = i + b, (i + b) * \infty_{f + 3} = i + a \) for each \( i \in \mathbb{Z}_m \).

Then \((M \cup \{\infty_{f + 1}, \infty_{f + 2}, \infty_{f + 3}\}, *)\) is an arbitrary, semi-symmetric \((f + 3)\)-type quasigroup of order \( n + 3 \).

Note that if \((M, o)\) is idempotent, \( m \geq f + 4 \) and \( f \equiv 0, 1(\text{mod } 3) \) with \( f \neq 3 \), then \((M \cup \{\infty_{f + 1}, \infty_{f + 2}, \infty_{f + 3}\}, *)\) is idempotent.

If condition (ii) holds, we define a binary operation \(*\) on \( M \cup \{\infty_{f + 1}, \infty_{f + 2}, \infty_{f + 3}\} \) as follows:

\( x * y = x \circ y \) for each \( x, y \in \mathbb{Z}_m \), \((x, y) \not\in \{(i, i + a), (i + b, i), (i + a, i + b) : i \in \mathbb{Z}_m\}\);

\( x * \infty_i = x \circ \infty_i \), \( \infty_i * x = \infty_i \circ x \) for each \( x \in \mathbb{Z}_m \), \( i \in \{1, 2, \ldots, f\} \);

\( \infty_i * \infty_k = \infty_j * \infty_k \) for all \( j, k \in \{1, 2, \ldots, f + 3\} \);

\( i * i = \infty_{f + 1}, \infty_{f + 2} * i = i, i * \infty_{f + 1} = i \) for each \( i \in \mathbb{Z}_m \);

\( (i + m/3) * (i + 2m/3) = \infty_{f + 3}, \infty_{f + 3} * (i + m/3) = i + 2m/3, (i + 2m/3) * \infty_{f + 3} = i + m/3 \) for each \( i \in \mathbb{Z}_m \).

Then \((M \cup \{\infty_{f + 1}, \infty_{f + 2}, \infty_{f + 3}\}, *)\) is an arbitrary, semi-symmetric \((f + 3)\)-type quasigroup of order \( n + 3 \). □

We require the following definitions and results in the subsections which follow.

Definition. Suppose \( S \) is a collection \((a_1, b_1), (a_2, b_2), \ldots, (a_x, b_x)\) of \( x \) ordered pairs of positive integers with \( b_i - a_i = i \) for \( 1 \leq i \leq x \).

- If \([a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x] = \{1, 2, \ldots, 2x\}\) then \( S \) is a Skolem sequence of order \( x \).

- If \([a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x] = \{1, 2, \ldots, 2x + 1\} \setminus \{2x\}\) then \( S \) is a hooked Skolem sequence of order \( x \).

- If \(k \in \{1, 2, \ldots, 2x + 2\}\) and \([a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x] = \{1, 2, \ldots, 2x + 1\} \setminus \{k\}\) then \( S \) is a \( k \)-extended Skolem sequence of order \( x \).

- If \(k \in \{1, 2, \ldots, 2x + 2\}\) \setminus \{2x + 1\} and \([a_1, a_2, \ldots, a_x, b_1, b_2, \ldots, b_x] = \{1, 2, \ldots, 2x + 2\} \setminus \{2x + 1, k\}\) then \( S \) is a hooked \( k \)-extended Skolem sequence of order \( x \).

The existence problem for Skolem sequences and their variants defined above has been settled; see [2,15,18,21].

Theorem 5.3. • There exists a Skolem sequence of order \( x \) if and only if \( x \equiv 0, 1(\text{mod } 4)\) [21].

• There exists a hooked Skolem sequence of order \( x \) if and only if \( x \equiv 2, 3(\text{mod } 4)\) [18].
• There exists a $k$-extended Skolem sequence of order $x$ if and only if $x \equiv 0, 1(\text{mod } 4)$ and $k$ is odd, or $x \equiv 2, 3(\text{mod } 4)$ and $k$ is even [2].

• There exists a hooked $k$-extended Skolem sequence of order $x$ if and only if $x \equiv 2, 3(\text{mod } 4)$ and $k$ is odd or $x \equiv 0, 1(\text{mod } 4)$ and $k$ is even and $(k, x) \neq (2, 1)$ [15].

5.1. Idempotent and unipotent semi-symmetric $f$-type quasigroups

To determine the spectra for idempotent and unipotent semi-symmetric $f$-type quasigroups, we need some results on the existence of such quasigroups for specific small values of $f$.

Lemma 5.4. For all $n \equiv 1(\text{mod } 6)$ with $n \geq 7$, there exists an idempotent, semi-symmetric 3-type quasigroup of order $n$.

Proof. Let $n = 6x + 7$ and let $\mathcal{M}_\infty$ be a Mendelsohn triple system of order 3. If $x \equiv 0, 1(\text{mod } 4)$ then let $S$ be a Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x + 1, \infty_1), (t, t + 3x + 2, \infty_2), (t, t + 3x + 3, \infty_3)$$

for each $t \in \mathbb{Z}_{6x+4}$. If $x \equiv 2, 3(\text{mod } 4)$ then let $S$ be a hooked Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x, \infty_1), (t, t + 3x + 2, \infty_2), (t, t + 3x + 4, \infty_3)$$

for each $t \in \mathbb{Z}_{6x+4}$. Then the union of $\mathcal{M}_\infty$, $T$ and the following set of triples is a Mendelsohn triple system $\mathcal{M}$ of order $n$ with underlying set $\mathbb{Z}_{6x+4} \cup \{\infty_1, \infty_2, \infty_3\}$:

$$\{(t, t + i, t + x + b_i), (t, t - i, t - x - b_i) : i = 1, 2, \ldots, x, (a_i, b_i) \in S, t \in \mathbb{Z}_{6x+4}\}.$$ 

It follows that if we define $a * a = a$ for all $a \in \mathbb{Z}_{6x+4} \cup \{\infty_1, \infty_2, \infty_3\}$ and $a * b = c, b * c = a$ and $c * a = b$ for each triple $(a, b, c)$ in $\mathcal{M}$ then we obtain an idempotent, semi-symmetric 3-type quasigroup of order $n$. □

Lemma 5.5. For all $n \equiv 3(\text{mod } 6)$ with $n \geq 9$, there exists an idempotent, semi-symmetric 3-type quasigroup of order $n$.

Proof. Let $n = 6x + 9$ and let $\mathcal{M}_\infty$ be a Mendelsohn triple system of order 3. Let $R$ consist of the triples

$$(t, t + 2x + 2, t + 4x + 4), (t, t + 4x + 4, t + 2x + 2)$$

for each $t \in \mathbb{Z}_{6x+6}$. If $x \equiv 1, 2(\text{mod } 4)$ then let $S$ be an $(x + 2)$-extended Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x + 2, \infty_1), (t, t + 3x + 3, \infty_2), (t, t + 3x + 4, \infty_3)$$

for each $t \in \mathbb{Z}_{6x+6}$. If $x \equiv 0, 3(\text{mod } 4)$ then let $S$ be a hooked $(x + 2)$-extended Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x + 1, \infty_1), (t, t + 3x + 3, \infty_2), (t, t + 3x + 5, \infty_3)$$

for each $t \in \mathbb{Z}_{6x+6}$. Then the union of $R, \mathcal{M}_\infty, T$ and the following set of triples is a Mendelsohn triple system $\mathcal{M}$ of order $n$ with underlying set $\mathbb{Z}_{6x+6} \cup \{\infty_1, \infty_2, \infty_3\}$:

$$\{(t, t + i, t + x + b_i), (t, t - i, t - x - b_i) : i = 1, 2, \ldots, x, (a_i, b_i) \in S, t \in \mathbb{Z}_{6x+6}\}.$$ 

It follows that if we define $a * a = a$ for all $a \in \mathbb{Z}_{6x+6} \cup \{\infty_1, \infty_2, \infty_3\}$ and $a * b = c, b * c = a$ and $c * a = b$ for each triple $(a, b, c)$ in $\mathcal{M}$ then we obtain an idempotent, semi-symmetric 3-type quasigroup of order $n$. □

Lemma 5.6. For all $n \equiv 3(\text{mod } 6)$ with $n \geq 9$, there exists an idempotent, semi-symmetric 4-type quasigroup of order $n$. 

Proof. Let $n = 6x + 9$ and let $\mathcal{M}_\infty$ be a Mendelsohn triple system of order 4. If $x \equiv 0, 1(\text{mod } 4)$ then let $S$ be a Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x + 1, \infty_1), (t, t + 3x + 2, \infty_2), (t, t + 3x + 3, \infty_3), (t, t + 3x + 4, \infty_4)$$

for each $t \in \mathbb{Z}_{6x+5}$. If $x \equiv 2, 3(\text{mod } 4)$ then let $S$ be a hooked Skolem sequence of order $x$ and let $T$ consist of the triples

$$(t, t + 3x, \infty_1), (t, t + 3x + 2, \infty_2), (t, t + 3x + 3, \infty_3), (t, t + 3x + 5, \infty_4)$$

for each $t \in \mathbb{Z}_{6x+5}$. Then the union of $\mathcal{M}_\infty$, $T$ and the following set of triples is a Mendelsohn triple system $\mathcal{M}$ of order $n$ with underlying set $\mathbb{Z}_{6x+5} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$:

$$\{(t + i, t + x + bi), (t - i, t - x - bi) : i = 1, 2, \ldots, x, (a_i, b_i) \in S, t \in \mathbb{Z}_{6x+5}\}.$$ 

It follows that if we define $a * a = a$ for all $a \in \mathbb{Z}_{6x+5} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and $a * b = c, b * c = a$ and $c * a = b$ for each triple $(a, b, c)$ in $\mathcal{M}$ then we obtain an idempotent, semi-symmetric 4-type quasigroup of order $n$. □

Since the spectra for cyclic and 1-rotational Mendelsohn triple systems are known (see Theorems 25.11 and 25.14 of [8] respectively, or [7] and [5] for the original solutions), we have the spectra for idempotent, semi-symmetric 0-type and 1-type quasigroups described in Theorem 2.3. From Theorem 2.2, there are no idempotent, semi-symmetric quasigroups of orders equivalent to 2 (mod 3), and hence there are certainly no idempotent, semi-symmetric $f$-type quasigroups where $f \equiv 2(\text{mod } 3)$.

To establish the spectra for the case $f \equiv 0(\text{mod } 3)$, $f \geq 3$, we first determine the spectra for $f = 3$. Lemmas 5.4 and 5.5 give the existence of idempotent, semi-symmetric 3-type quasigroups of orders $n \equiv 1, 3(\text{mod } 6), n \geq 7$ respectively. From the spectra of 0-type idempotent, semi-symmetric quasigroups and Lemma 5.2, we have that there also exist idempotent, semi-symmetric 3-type quasigroups of orders $n \equiv 0, 4(\text{mod } 6), n \neq 12$. A computer search verified that there is indeed no idempotent, semi-symmetric 3-type quasigroup of order 12. The necessary conditions in Lemma 5.1 give that these constructions yield all the possible orders for $f = 3$.

To obtain the spectrum for the case $f \equiv 0(\text{mod } 3)$, $f > 3$, we apply Lemma 5.2 inductively on $f$. This constructs idempotent, semi-symmetric $f$-type quasigroups of order $n$ for all $n \equiv 0, 1(\text{mod } 3), n \geq 2f + 1$, where $f \equiv 0(\text{mod } 3), f \geq 9$. Note that Theorem 2.2 precludes the existence of idempotent 6-type quasigroups. By Lemma 5.1, this construction yields all possible orders for $f \equiv 0(\text{mod } 3)$.

To establish the spectrum for the case $f \equiv 1(\text{mod } 3)$, $f \geq 4$ we first determine the spectra for $f = 4$. Lemma 5.6 gives the existence of idempotent, semi-symmetric 4-type quasigroups of order $n \equiv 3(\text{mod } 6), n \geq 9$. From the spectrum of idempotent, semi-symmetric 1-type quasigroups and Lemma 5.2, we have that there also exist idempotent, semi-symmetric 4-type quasigroups of orders $n \equiv 0, 1, 4(\text{mod } 6), n \neq 13$. The following quasigroup is an idempotent, semi-symmetric 4-type quasigroup of order 13.

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The necessary conditions in Lemma 5.1 give that these constructions yield all the possible orders for $f = 4$. In a similar manner as above, for $f \equiv 1(\text{mod } 3)$, $f > 4$, Lemmas 5.1 and 5.2 and Theorem 2.2 give that there exist idempotent, semi-symmetric $f$-type quasigroups of order $n$ if and only if $n \equiv 0, 1(\text{mod } 3), n \geq 2f + 1$. 

This discussion establishes the spectra for idempotent, semi-symmetric $f$-type quasigroups claimed in Theorem 2.3. By Theorem 3.1 there exists an idempotent, semi-symmetric $f$-type quasigroup of order $n$ if and only if there exists a unipotent, semi-symmetric $(f + 1)$-type quasigroup of order $n + 1$. Combining this with the existence results for idempotent, semi-symmetric $f$-type quasigroups discussed above, we obtain the spectra for unipotent, semi-symmetric $f$-type quasigroups given in Theorem 2.3.

5.2. Arbitrary semi-symmetric $f$-type quasigroups

A semi-symmetric quasigroup that is not idempotent or unipotent does not correspond to a Mendelsohn triple system, but it does correspond to an extended Mendelsohn triple system. An extended Mendelsohn triple $(a, b, c)$ is a cyclically-ordered (or directed) 3-element multiset and an extended Mendelsohn triple system of order $n$ is a pair $(V, B)$ where $V$ is an $n$-set and $B$ is a collection of extended Mendelsohn triples of elements of $V$ such that each ordered pair of elements in $V$ occurs in exactly one extended Mendelsohn triple of $B$. Thus, in addition to triples with three distinct elements, extended Mendelsohn triple systems may also contain triples of the form $(a, a, b)$ or $(a, a, a)$. The correspondence is $a \circ b = c$ if and only if $(a, b, c)$ is an extended Mendelsohn triple (here $a, b, c$ are not necessarily distinct).

**The Case** $f \equiv 0(\text{mod } 3)$.

Since the spectrum for cyclic extended Mendelsohn triple systems has been found by Micale and Pennisi [16], we have the spectrum for arbitrary semi-symmetric 0-type quasigroups given in Theorem 2.3.

We next establish the spectrum for arbitrary, semi-symmetric 3-type quasigroups. The spectrum for idempotent, semi-symmetric 3-type quasigroups of order $n$ is all $n \equiv 0, 1(\text{mod } 3), n \geq 7$, with $n \neq 12$. Arbitrary, semi-symmetric 3-type quasigroups of orders 6 and 12 can be obtained by applying Lemma 5.2 to arbitrary, semi-symmetric 0-type quasigroups of orders 3 and 9 respectively. Hence, the necessary conditions in Lemma 5.1 give that there exists an arbitrary, semi-symmetric 3-type quasigroup if and only if $n \equiv 0, 1(\text{mod } 3), n \geq 6$.

For $f \equiv 0(\text{mod } 3), f \geq 6$, we may use Lemma 5.2 inductively on $f$ to establish the existence of arbitrary, semi-symmetric $f$-type quasigroups of order $n$ for all $n \equiv 0, 1(\text{mod } 3), n \geq 2f$. By Lemma 5.1, these are all possible orders. This discussion establishes the results given in Theorem 2.3 on arbitrary, semi-symmetric $f$-type quasigroups where $f \equiv 0(\text{mod } 3)$.

**The Case** $f \equiv 1(\text{mod } 3)$.

Since the spectrum for 1-rotational extended Mendelsohn triple systems has been found by Micale and Pennisi [16], we have the spectrum for arbitrary, semi-symmetric 1-type quasigroups given in Theorem 2.3. If $f \equiv 1(\text{mod } 3), f \geq 4$, then the spectra for semi-symmetric idempotent and unipotent $f$-type quasigroups establish the existence of arbitrary, semi-symmetric $f$-type quasigroups of order $n$ for all $n \geq 2f$, with the single exception of the arbitrary, semi-symmetric 7-type quasigroup of order 14. This specific quasigroup can be constructed by applying Lemma 5.2 twice to an arbitrary, semi-symmetric 1-type quasigroup of order 8.

**The Case** $f \equiv 2(\text{mod } 3)$.

We first establish the spectrum for arbitrary, semi-symmetric 2-type quasigroups. Note that the spectrum for unipotent, semi-symmetric 2-type quasigroups gives the existence of arbitrary, semi-symmetric 2-type quasigroups of order $n$ for all $n \equiv 2, 4, 5(\text{mod } 6), n \neq 11$, where $n \geq 2f$.

There exists an arbitrary, semi-symmetric 2-type quasigroup of order 11, as follows.

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To complete the spectrum for \( f = 2 \), we require the following lemma.

**Lemma 5.7.** For all \( n \equiv 1(\text{mod } 6) \) with \( n \geq 7 \), there exists a semi-symmetric 2-type quasigroup of order \( n \).

**Proof.** Let \( n = 6x + 7 \) and let \( M_\infty \) be an extended Mendelsohn triple system of order 2. The construction given in the proof of Lemma 3.5 of [4] shows that there exists an edge disjoint decomposition of the complete graph with vertex set \( \mathbb{Z}_{6x+5} \) into \( x(6x + 5) \) triangles and two Hamilton cycles. This decomposition has the property that if \([a, b, c]\) is a triangle in the decomposition then so is \([a + 1, b + 1, c + 1]\). Furthermore, we may take the two Hamilton cycles to be \( H_1 \) consisting of the edges \([i, i + 1]\) for \( i \in \mathbb{Z}_{6x+5} \) and \( H_2 \) consisting of the edges \([i, i + 3x + 2]\) for \( i \in \mathbb{Z}_{6x+5} \) or the edges \([i, i + 3x + 1]\) for \( i \in \mathbb{Z}_{6x} \). (Note that \( gcd(3x + 5, 6x + 5) = gcd(3x + 2, 6x + 5) = 1 \).) These Hamilton cycles have the property that they remain unchanged under the permutation \( a \rightarrow a + 1 \).

Let \( \Delta \) be the set of triangles in such a decomposition, and denote the two Hamilton cycles by \((v_0, v_1, \ldots, v_{6x+4})\) and \((w_0, w_1, \ldots, w_{6x+4})\). Let \( T \) be the set of directed triples

\[\{(a, b, c), (b, a, c) : [a, b, c] \in \Delta\}.\]

It follows then that the union of \( M_\infty, T \) and the following set of directed triples is an extended Mendelsohn triple system \( M \) of order \( n \) with underlying set \( \mathbb{Z}_{6x+5} \cup \{\infty_1, \infty_2\} \):

\[\{(v_1, v_{i+1}, \infty_1), (v_{i+1}, v_1, \infty_2), (w_i, w_{i+1}, w_{i+1}) : 1 \leq i \leq 6x+5\}.\]

Hence if we define \( a \ast b = c, b \ast c = a \) and \( c \ast a = b \) for each triple \((a, b, c)\) in \( M \) then we obtain a semi-symmetric 2-type quasigroup of order \( n \).

The necessary conditions in Lemma 5.1 give that these constructions yield all the possible orders for \( f = 2 \). For \( f \equiv 2(\text{mod } 3), f \geq 5 \), the spectrum for unipotent, semi-symmetric \( f \)-type quasigroups and Lemma 5.1 give that there exists an arbitrary, semi-symmetric \( f \)-type quasigroup of order \( n \) if and only if \( n \equiv 1, 2(\text{mod } 3), n \geq 2f \).

6. Totally symmetric quasigroups

In this section we prove the results in Theorem 2.3 pertaining to totally symmetric \( f \)-type quasigroups. There is a well-known equivalence between Steiner triple systems and idempotent, totally symmetric quasigroups, see for example [8], p.24 or [9], p.75. Indeed, such quasigroups are often called Steiner quasigroups. For distinct \( a \) and \( b \), \( a \circ b = c \) if and only if \([a, b, c]\) is a triple in the Steiner triple system \((a \circ a = a \text{ for all } a \in Q)\). Since the spectra for cyclic and for 1-rotational Steiner triple systems are known (see Theorem 7.3 and Lemma 7.37 of [8] respectively, or [19] and [20] for the original solutions), we have the spectra for idempotent, totally symmetric 0-type and 1-type quasigroups. Since there is no idempotent quasigroup of order 2, there are no idempotent 2-type quasigroups, and since there are no commutative \( f \)-type quasigroups for \( f \geq 3 \) (by Theorem 4.1), there are certainly no totally symmetric \( f \)-type quasigroups for \( f \geq 3 \).

By Theorem 3.1 there exists an idempotent, totally symmetric \( f \)-type quasigroup of order \( n \) if and only if there exists a unipotent, totally symmetric \((f + 1)\)-type quasigroup of order \( n + 1 \). Combining this with the existence results for idempotent, totally symmetric \( f \)-type quasigroups discussed above, we obtain the spectra for unipotent, totally symmetric \( f \)-type quasigroups stated in Theorem 2.3.

We can treat the arbitrary, totally symmetric case just as we did for the semi-symmetric case. That is, by moving to extended triple systems, we find the triple system analogue of a totally symmetric quasigroup which need not be idempotent. The original motivation for Johnson and Mendelsohn [13] to introduce extended triple systems was their correspondence to totally symmetric quasigroups.

Since the spectra for cyclic and for 1-rotational extended triple systems are known (see [6]), we have the spectra for arbitrary, totally symmetric 0-type and 1-type quasigroups. Also, by Theorem 4.1, any totally symmetric 2-type quasigroup has even order \( 2k + 2 \) and satisfies \( a \circ a = \infty_1 \) and \( a \circ (a + k) = \infty_2 \), or \( a \circ a = \infty_2 \) and \( a \circ (a + k) = \infty_1 \) for all \( a \in \mathbb{Z}_{2k} \). Without loss of generality we can assume that \( a \circ a = \infty_2 \) for all \( a \in \mathbb{Z}_{2k} \). Hence any such quasigroup can be made unipotent by redefining, if necessary, \( \infty_1 \circ \infty_1 = \infty_2 \circ \infty_2 = \infty_2 \) and \( \infty_1 \circ \infty_2 = \infty_2 \circ \infty_1 = \infty_1 \). Note that this quasigroup satisfies \( \infty_2 \circ a = a \) for all \( a \in Q \). Hence by Theorem 3.1, there exists an arbitrary (in fact, unipotent), totally symmetric 2-type quasigroup of order \( n + 1 \) if and only if there exists an idempotent, totally symmetric 1-type quasigroup of order \( n \). Thus, we have the spectrum for arbitrary, totally symmetric 2-type
quasigroups given in Theorem 2.3. Finally, by Theorem 4.1, there are no totally symmetric \(f\)-type quasigroups for \(f \geq 3\).

7. Quasigroups with no specified conjugate symmetry

In this section we establish the spectrum for \(f\)-type quasigroups which have no prescribed symmetry between conjugates. Theorem 6 of [22], which we state next, settles the existence of arbitrary \(f\)-type quasigroups for all values of \(f\).

**Theorem 7.1.** A 0-type quasigroup of order \(n\) exists if and only if \(n\) is odd. For \(f \geq 1\), there exists an \(f\)-type quasigroup of order \(n\) if and only if \(n \geq 2f\).

To deal with the idempotent and unipotent cases, we need two simple results which use the notion of isotopism. Two quasigroups \((Q,\circ)\) and \((Q,\cdot)\) on the same set are said to be isotopic if there exist three permutations \(\alpha, \beta, \gamma\) of \(Q\) such that \(a \circ b = c\) if and only if \(\alpha(a) \cdot \beta(b) = \gamma(c)\) for all \(a, b, c \in Q\).

**Lemma 7.2.** Let \(f \geq 1\). Every \(f\)-type quasigroup of order \(m + f\) is isotopic to an \(f\)-type quasigroup containing the triples \((i, i, \infty_i)\) for each \(i \in \mathbb{Z}_m\). Also, if \(m > f\), every \(f\)-type quasigroup of order \(m + f\) is isotopic to an \(f\)-type quasigroup containing the triples \((i, i, i)\) for each \(i \in \mathbb{Z}_m\).

**Proof.** Let \(Q = \mathbb{Z}_m \cup \{\infty_1, \infty_2, \ldots, \infty_f\}\) and suppose that \((Q,\circ)\) is an \(f\)-type quasigroup. If \(m = 1\) then \(f = 1\), and the result is trivial since every 1-type quasigroup of order 2 is unipotent.

If \(m > 1\), let \(x \in Q\) satisfy \(0 \circ x = \infty_1\). Note that the definition of \(f\)-type implies that \(x \in \mathbb{Z}_m\). Define a new quasigroup \((Q,\cdot)\) by \(a \cdot b = a \circ (x + b)\) for all \(a, b \in Q\). It is routine to check from the definitions that \((Q,\cdot)\) is an \(f\)-type quasigroup isotopic to \((Q,\circ)\). Also, \(0 \cdot 0 = 0 \circ (x + 0) = 0 \circ x = \infty_1\), from which it follows that \((Q,\cdot)\) contains the triples \((i, i, \infty_i)\) for each \(i \in \mathbb{Z}_m\).

Now suppose that \(m > f\). Then there exists \(y, z \in \mathbb{Z}_m\) such that \(y \circ z = 0\). Define a third quasigroup \((Q,\ast)\) by \(a \ast b = (y + a) \circ (z + b)\) for all \(a, b \in Q\). Again it is routine to check that \((Q,\ast)\) is an \(f\)-type quasigroup isotopic to \((Q,\circ)\). But this time \(0 \ast 0 = (y + 0) \circ (z + 0) = y \circ z = 0\) so that \((Q,\ast)\) contains the triples \((i, i, i)\) for each \(i \in \mathbb{Z}_m\).

**Lemma 7.3.** Let \(f \geq 1\) and let \((Q,\circ)\) be an \(f\)-type quasigroup of order \(m + f\). Then there exists an unipotent \(f\)-type quasigroup of order \(m + f\). Also, if \(f \neq 2\) and \(m > f\), there exists an idempotent \(f\)-type quasigroup of order \(m + f\).

**Proof.** By Lemma 7.2, \((Q,\circ)\) is isotopic to a quasigroup \((Q,\ast)\) containing the triples \((i, i, \infty_i)\) for each \(i \in \mathbb{Z}_m\). Let \((\infty_1, \infty_2, \ldots, \infty_f, \ast)\) be an unipotent quasigroup satisfying \(a \ast a = \infty_1\) for all \(a \in \{\infty_1, \infty_2, \ldots, \infty_f\}\). Define the binary operation \(\cdot\) on \(Q\) by \(p \cdot q = p \ast q\) if \(p, q \in \{\infty_1, \infty_2, \ldots, \infty_f\}\), and \(p \cdot q = p \ast q\) otherwise. Then \((Q,\cdot)\) is a unipotent \(f\)-type quasigroup of order \(m + f\).

Now suppose that \(m > f\) and \(f \neq 2\). By Lemma 7.2, \((Q,\circ)\) is isotopic to a quasigroup \((Q,\ast)\) containing the triples \((i, i, i)\) for each \(i \in \mathbb{Z}_m\). Let \((\infty_1, \infty_2, \ldots, \infty_f, \ast)\) be an idempotent quasigroup. Such a quasigroup exists since \(f 
eq 2\). As before, define the binary operation \(\cdot\) by \(p \cdot q = p \ast q\) if \(p, q \in \{\infty_1, \infty_2, \ldots, \infty_f\}\), and \(p \cdot q = p \ast q\) otherwise. Then \((Q,\cdot)\) is an idempotent \(f\)-type quasigroup of order \(m + f\).

We can now establish the spectra for \(f\)-type quasigroups with no prescribed conjugate symmetry. Theorem 7.1 above settles the spectra for arbitrary \(f\)-type quasigroups for all values of \(f\). When \(f = 0\), Theorem 2.1 states that all 0-type quasigroups are necessarily of odd order. The existence of idempotent, commutative 0-type quasigroups for all odd orders thus establishes the spectrum for idempotent 0-type quasigroups with no prescribed conjugate symmetry. Note also that in the case \(f = 0\), any unipotent \(f\)-type quasigroup is necessarily of order 1.

Finally, for \(f \geq 1\), combining the results in Theorem 7.1, Theorem 2.2 and Lemma 7.3 gives the spectra for idempotent and unipotent \(f\)-type quasigroups as stated in Theorem 2.3.

References