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Kostant homology formulas for oscillator modules of Lie superalgebras

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Abstract

We provide a systematic approach to obtain formulas for characters and Kostant $\mathfrak{u}$-homology groups of the oscillator modules of the finite-dimensional general linear and ortho-symplectic superalgebras, via Howe dualities for infinite-dimensional Lie algebras. Specializing these Lie superalgebras to Lie algebras, we recover, in a new way, formulas for Kostant homology groups of unitarizable highest weight representations of Hermitian symmetric pairs. In addition, two new reductive dual pairs related to the above-mentioned $\mathfrak{u}$-homology computation are worked out.

Keywords: Lie superalgebras; Oscillator representations; Howe duality; Homology formulas

Contents

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1. Introduction

1.1. Inspired by the Borel–Weil–Bott theorem, in a classical work [25] Kostant computed the $\mathfrak{u}$-(co)homology groups for finite-dimensional simple modules of a semisimple Lie algebra, recovering the celebrated Weyl character formula in a purely algebraic way. Since then Kostant's calculation has been generalized in nontrivial ways to various setups, e.g. to integrable modules of Kac–Moody algebras [16], to unitarizable highest weight modules of Hermitian symmetric pairs [14], to finite-dimensional modules of general linear superalgebras [29] (and in a different way to polynomial representations [10]), and more recently to modules of infinite-dimensional Lie superalgebras [4].

On the other hand, Howe's theory of reductive dual pairs [17,18] has played important roles in the representation theory of real and p-adic Lie groups, and there have been generalizations in different directions. Dual pairs $(\mathfrak{g}, G)$ have been formulated systematically by the third author [31] between *infinite rank* affine Kac–Moody algebras $\mathfrak{g}$ [12,22] and finite-dimensional reductive Lie groups G. More recently, dual pairs between *finite-dimensional* Lie superalgebras $\bar{\mathfrak{g}}$ and Lie groups G have been studied in depth with application to irreducible characters of $\bar{\mathfrak{g}}$ in a number of papers [7,11,6]. A priori, no direct link between different dual pairs $(\bar{\mathfrak{g}}, G)$ and $(\mathfrak{g}, G)$ was expected to exist, except that the $\mathfrak{g}$-modules $\{L(\mathfrak{g}, \Lambda(\lambda))\}$ and $\bar{\mathfrak{g}}$-modules $\{L(\bar{\mathfrak{g}}, \widehat{\Lambda}_f(\lambda))\}$ appearing in these Howe dualities can both be parameterized by (part of) the same index set that parameterizes the finite-dimensional simple G-modules $\{V_G^\lambda\}$. Below in Table 1 is a list of the reductive dual pairs used in this paper which share the same G.

1.2. The main goal of this paper is to develop a conceptual approach to computing Kostant $\mathfrak{u}$-homology groups with coefficients in the so-called oscillator $\bar{\mathfrak{g}}$-modules, i.e. those appearing in the Howe duality $(\bar{\mathfrak{g}}, G)$ (see column I in Table 1). Remarkably, our results show that the $\mathfrak{u}$-homology groups of these oscillator $\bar{\mathfrak{g}}$-modules are dictated by those of the corresponding integrable $\mathfrak{g}$-modules, where $\mathfrak{g}$ is the infinite-dimensional classical "counterpart" (see column II) of $\bar{\mathfrak{g}}$. The Howe dualities $(\mathfrak{spo}(2m|2n+1), \mathrm{Pin}(d))$ and $(\mathfrak{b}^{\circ}_{\infty}, \mathrm{Pin}(d))$ appear to be new and are worked out in Appendix A.

Table 1
Reductive dual pairs.

Dual pairs I $(\bar{\mathfrak{g}}, G)$	Dual pairs II $(\mathfrak{g}, G)$	Dual pairs III $(\mathfrak{g}^*, G)$
$(\mathfrak{gl}(p+m\vert q+n), \mathrm{GL}(d))$	$(\mathfrak{a}_\infty, \mathrm{GL}(d))$	$(\mathfrak{a}_\infty, \mathrm{GL}(d))$
$(\mathfrak{spo}(2m\vert 2n+1), \mathrm{Pin}(d))$	$(\mathfrak{b}_\infty, \mathrm{Pin}(d))$	$(\mathfrak{b}^{\circ}_\infty, \mathrm{Pin}(d))$
$(\mathfrak{osp}(2m\vert 2n), \mathrm{Sp}(d))$	$(\mathfrak{c}_\infty, \mathrm{Sp}(d))$	$(\mathfrak{d}_\infty, \mathrm{Sp}(d))$
$(\mathfrak{spo}(2m\vert 2n), \mathrm{O}(d))$	$(\mathfrak{d}_\infty, \mathrm{O}(d))$	$(\mathfrak{c}_\infty, \mathrm{O}(d))$

The results in this paper provided the first supporting evidence for a super duality between categories $\overline{\mathcal{O}}$ of $\bar{\mathfrak{g}}$-modules and certain categories $\mathcal{O}_f$ of $\mathfrak{g}_f$-modules [5] (generalizing the super duality [9] for type A), where $\mathfrak{g}_f$ denotes certain finite-dimensional reductive Lie algebras corresponding to $\bar{\mathfrak{g}}$. As classical Kazhdan–Lusztig polynomials allow an interpretation in terms of $\mathfrak{u}$-homology [32], the main results here may be reformulated as an equality of certain Kazhdan–Lusztig polynomials for categories $\overline{\mathcal{O}}$ and $\mathcal{O}_f$.

In spite of the infinite-dimensionality, the $\mathfrak{g}$-modules above are integrable and hence there is a standard approach to compute their Kostant homology groups (cf. [16,20,27]). On the other hand, the structures of the infinite-dimensional oscillator modules of the finite-dimensional Lie superalgebras $\bar{\mathfrak{g}}$ are not so well understood. In the approach of this paper, we use essentially only the Howe dualities $(\bar{\mathfrak{g}}, G)$ and $(\mathfrak{g}, G)$ for the same G (see columns I and II in Table 1) together with some simple combinatorial manipulations with the characters of the integrable $\mathfrak{g}$-modules to derive first a character formula of the oscillator $\bar{\mathfrak{g}}$-modules in a suitable form. Then from a comparison of the Casimir eigenvalues of the corresponding $\mathfrak{g}$-module and $\bar{\mathfrak{g}}$-module, we obtain formulas of the corresponding $\mathfrak{u}$-homology groups with coefficients in $L(\bar{\mathfrak{g}}, \widehat{\Lambda}_f(\lambda))$, which are expressed in terms of the infinite Weyl group of $\mathfrak{g}$, from the corresponding formulas for $\mathrm{H}_*(\mathfrak{u}, L(\mathfrak{g}, \Lambda(\lambda)))$.

1.3. In the paper [4], the Howe duality $(\mathfrak{g}, G)$ was used together with another Howe duality $(\mathfrak{g}^s, G)$ (due to [8] for types A, B and [26] for types C, D) to derive a $\mathfrak{u}$-homology formula for modules over infinite-dimensional Lie superalgebras $\mathfrak{g}^s$ from a corresponding $\mathfrak{u}$-homology formula for modules of $\mathfrak{g}$. However, the connection between integrable modules over infinite rank affine Kac–Moody algebras and oscillator representations of finite-dimensional Lie (super)algebras were not suspected back then.

On the other hand, character formulas for $\bar{\mathfrak{g}}$-modules above were obtained in different ways and in different forms in earlier works [11,6]. In [11] a key role was played by a difficult theorem of Enright on $\mathfrak{u}$-homology for unitarizable highest weight modules of real reductive Lie algebras established by intricate arguments involving equivalences of categories and nontrivial combinatorics on Weyl groups [14]. Our present approach bypasses Enright's theorem. Indeed, in the special case when the Lie superalgebras specialize to Lie algebras (i.e. $q = n = 0$ in column I and rows 1, 3, and 4 of Table 1), we recover Kostant homology formulas for unitarizable highest weight modules of three Hermitian symmetric pairs of classical types (which can be shown by some combinatorial argument to be equivalent to Enright's formula).

1.4. The paper is organized as follows. In Section 2, we review and set up notations for various Howe dualities involving the infinite-dimensional Lie algebras $\mathfrak{g}$ and finite-dimensional Lie superalgebras $\bar{\mathfrak{g}}$. In Section 3, we compute the character formulas of the oscillator $\bar{\mathfrak{g}}$-modules from Howe dualities. In Section 4, the Casimir eigenvalues of modules of $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are computed and compared, and they are used in Section 5 to obtain formulas for Kostant homology groups for the oscillator $\bar{\mathfrak{g}}$-modules.

Generalizing the type A case in [4], we compute in Section 6 the character formulas and Kostant homology formulas for non-integrable $\mathfrak{g}^*$-modules at negative integral levels appearing in the Howe dualities $(\mathfrak{g}^*, G)$ listed in column III of Table 1. The dualities $(\mathfrak{g}^*, G)$ with $G = \mathrm{GL}(d)$, $\mathrm{Sp}(d)$ and $\mathrm{O}(d)$ were treated in [23] and [31]. The new case involving $\mathrm{Pin}(d)$ is worked out in Appendix A, where one can also find an additional Howe duality $(\mathfrak{spo}(2m|2n+1), \mathrm{Pin}(d))$.

1.5. Notations

Denote by $\mathcal{P}^+$ the set of partitions. For $\lambda \in \mathcal{P}^+$ we denote by $\ell(\lambda)$ the length of λ, and by $|\lambda|$ the size of λ. Given $d \in \mathbb{N}$, a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ of d integers will be called a *generalized partition of depth d*. For a generalized partition λ of depth d, we define

$$\lambda^+ := \big(\langle\lambda_1\rangle, \dots, \langle\lambda_d\rangle\big), \qquad \lambda^- := \big(\langle-\lambda_1\rangle, \dots, \langle-\lambda_d\rangle\big),$$

where here and further we set $\langle a\rangle = \max\{a, 0\}$ for $a \in \mathbb{Z}$. Then λ^+ is a partition and λ^- is a non-decreasing sequence of non-negative integers.

For a sequence of non-negative integers $\mu = (\mu_1, \mu_2, \dots)$ we let $\mu' = (\mu'_1, \mu'_2, \dots)$ be the partition with $\mu'_j := |\{i \mid \mu_i \geqslant j\}|$. Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}_+$ stand for the set of all, positive, and non-negative integers, respectively.

All vector spaces, algebras, etc. are over the complex field $\mathbb{C}$.

2. Howe dualities for Lie algebras $\mathfrak{g}$ and Lie superalgebras $\bar{\mathfrak{g}}$

In this section we review various Howe dualities involving infinite-dimensional Lie algebras $\mathfrak{g}$ and finite-dimensional Lie superalgebras $\bar{\mathfrak{g}}$.

2.1. Infinite rank affine Lie algebras

The infinite-dimensional Lie algebra $\widehat{\mathfrak{gl}}_\infty$ and its subalgebras $\mathfrak{b}_\infty$, $\mathfrak{c}_\infty$, $\mathfrak{d}_\infty$ of types B, C, D [12,22] are well known. Our notations regarding these Lie algebras in this paper will be the same as those in [4, Section 2.2], and we refer to [4] for explicit case-by-case description of the following standard terminologies based on matrix elements E_{ij} for $\widehat{\mathfrak{gl}}_\infty$, $\widetilde{E}_n$ for other types, and weights ϵ_i, etc.

- $\mathfrak{g}$: $\widehat{\mathfrak{gl}}_\infty \equiv \mathfrak{a}_\infty$, $\mathfrak{b}_\infty$, $\mathfrak{c}_\infty$ or $\mathfrak{d}_\infty$, with triangular decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$,
- $\mathfrak{h}$: a Cartan subalgebra of $\mathfrak{g}$,
- I: an index set for simple roots for $\mathfrak{g}$,
- $\mathfrak{l}$: the Levi subalgebra of $\mathfrak{g}$ with simple roots indexed by $S = I \setminus \{0\}$,
- $\mathfrak{u}_\pm$: the nilradicals with $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{l} \oplus \mathfrak{u}_-$, $\mathfrak{u}_- := \mathfrak{u}$,
- $\Pi = \{\alpha_i \mid i \in I\}$: a set of simple roots for $\mathfrak{g}$,
- $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$: a set of simple coroots for $\mathfrak{g}$,
- Δ^+ (resp. Δ^-): a set of positive (resp. negative) roots for $\mathfrak{g}$,
- Δ_S^+ (resp. Δ_S^-): a set of positive (resp. negative) roots for $\mathfrak{l}$,
- $\Delta^\pm(S)$: the subset of roots for $\mathfrak{u}_\pm$,
- $\Lambda_i^{\mathfrak{x}}$: the fundamental weights for $\mathfrak{x}_\infty$ with $i \in I$ and $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$,
- ρ_c: "half sum" of the positive roots in Δ^+,
- W (resp. W_0): Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{l}$),
- $W_k^0 \equiv W_k^0(\mathfrak{x})$: the set of the minimal length representatives of the right coset space $W_0 \setminus W$ of length k for $\mathfrak{x}_\infty$ with $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$,
- $L(\mathfrak{g}, \Lambda)$: the irreducible highest weight $\mathfrak{g}$-module of highest weight $\Lambda \in \mathfrak{h}^*$.

2.2. Reductive dual pairs on Fock spaces

2.2.1. Fermionic Fock spaces

We fix a positive integer $\ell \geqslant 1$ and consider ℓ pairs of free fermions $\psi^{\pm,i}(z)$ with $i = 1, \dots, \ell$. That is, we have

$$\psi^{+,i}(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \psi_r^{+,i} z^{-r-\frac{1}{2}}, \qquad \psi^{-,i}(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \psi_r^{-,i} z^{-r-\frac{1}{2}},$$

with nontrivial anti-commutation relations $[\psi_r^{+,i}, \psi_s^{-,j}] = \delta_{ij}\delta_{r+s,0}$. Let $\mathfrak{F}^\ell$ denote the corresponding Fock space generated by the vacuum vector $|0\rangle$, which is annihilated by $\psi_r^{+,i}, \psi_s^{-,i}$ for $r, s > 0$.

We introduce a neutral fermionic field $\phi(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \phi_r z^{-r-\frac{1}{2}}$ with nontrivial anti-commutation relations $[\phi_r, \phi_s] = \delta_{r+s,0}$. Denote by $\mathfrak{F}^{\frac{1}{2}}$ the Fock space of $\phi(z)$ generated by a vacuum vector that is annihilated by ϕ_r for $r > 0$. We denote by $\mathfrak{F}^{\ell+\frac{1}{2}}$ the tensor product of $\mathfrak{F}^\ell$ and $\mathfrak{F}^{\frac{1}{2}}$.

2.2.2. The $(\widehat{\mathfrak{gl}}_\infty, \mathrm{GL}(d))$-duality

We denote by e_{ij} $(1 \leqslant i, j \leqslant d)$ the elementary $d \times d$ matrix with 1 in the ith row and jth column and 0 elsewhere. Then $H = \sum_i \mathbb{C}e_{ii}$ is a Cartan subalgebra of $\mathfrak{gl}(d)$, while $\sum_{i\leqslant j} \mathbb{C}e_{ij}$ is a Borel subalgebra of $\mathfrak{gl}(d)$ containing H. An irreducible rational representation of $\mathfrak{gl}(d)$ (or $\mathrm{GL}(d)$) is determined by its highest weight $\lambda \in H^*$ with $\langle \lambda, e_{ii}\rangle = \lambda_i \in \mathbb{Z}$ $(1 \leqslant i \leqslant d)$ and $\lambda_1 \geqslant \dots \geqslant \lambda_d$. Identifying λ with $(\lambda_1, \dots, \lambda_d)$ as usual, we denote by $V^\lambda_{\mathrm{GL}(d)}$ the irreducible $\mathrm{GL}(d)$-module of highest weight λ. These irreducible modules are parameterized by the set of generalized partitions of depth d:

$$\mathcal{P}\big(\mathrm{GL}(d)\big) := \big\{\lambda = (\lambda_1, \dots, \lambda_d) \bigm| \lambda_i \in \mathbb{Z},\ \lambda_1 \geqslant \dots \geqslant \lambda_d\big\}.$$

Proposition 2.1. *(See [15], [31, Theorem 3.1].) There exists a commuting action of $\widehat{\mathfrak{gl}}_\infty$ and $\mathrm{GL}(d)$ on $\mathfrak{F}^d$. Furthermore, under this joint action, we have*

$$\mathfrak{F}^d \cong \bigoplus_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} L\big(\widehat{\mathfrak{gl}}_\infty, \Lambda^{\mathfrak{a}}(\lambda)\big) \otimes V^\lambda_{\mathrm{GL}(d)}, \tag{2.1}$$

where $\Lambda^{\mathfrak{a}}(\lambda) := d\Lambda_0^{\mathfrak{a}} + \sum_{j\geqslant 1}(\lambda^+)'_j \epsilon_j - \sum_{j\geqslant 0}(\lambda^-)'_{j+1}\epsilon_{-j} = \sum_{i=1}^d \Lambda^{\mathfrak{a}}_{\lambda_i}$.

We assume below that x_n $(n \in \mathbb{Z})$ and z_i $(i \in \mathbb{N})$ are formal indeterminates. Computing the trace of the operator $\prod_{n\in\mathbb{Z}} x_n^{E_{nn}} \prod_{i=1}^d z_i^{e_{ii}}$ on both sides of (2.1), we obtain the following identity:

$$\prod_{i=1}^d \prod_{n\in\mathbb{N}} (1 + x_n z_i)\big(1 + x_{1-n}^{-1} z_i^{-1}\big) = \sum_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} \operatorname{ch} L\big(\widehat{\mathfrak{gl}}_\infty, \Lambda^{\mathfrak{a}}(\lambda)\big) \operatorname{ch} V^\lambda_{\mathrm{GL}(d)}. \tag{2.2}$$

2.2.3. The $(\mathfrak{c}_\infty, \mathrm{Sp}(d))$-duality

Let d be even and $\mathrm{Sp}(d)$ denote the symplectic group, which may be viewed as the subgroup of $\mathrm{GL}(d)$ preserving the non-degenerate skew-symmetric bilinear form on $\mathbb{C}^d$ given by

$$\begin{pmatrix} 0 & J_{\frac{d}{2}} \\ -J_{\frac{d}{2}} & 0 \end{pmatrix}.$$

Here J_k is the following $k \times k$ matrix:

$$J_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Let $\mathfrak{sp}(d)$ be the Lie algebra of $\mathrm{Sp}(d)$. We take as a Borel subalgebra of $\mathfrak{sp}(d)$ the subalgebra of upper triangular matrices, and as a Cartan subalgebra H the subalgebra spanned by $e_i = e_{ii} - e_{d+1-i,d+1-i}$ $(1 \leqslant i \leqslant \frac{d}{2})$. A finite-dimensional irreducible representation of $\mathfrak{sp}(d)$ is determined by its highest weight $\lambda \in H^*$ with $\langle \lambda, e_i \rangle = \lambda_i \in \mathbb{Z}_+$ $(1 \leqslant i \leqslant \frac{d}{2})$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_{\frac{d}{2}}$. Furthermore each such a representation lifts to an irreducible representation of $\mathrm{Sp}(d)$, which is denoted by $V^\lambda_{\mathrm{Sp}(d)}$. Put

$$\mathcal{P}\big(\mathrm{Sp}(d)\big) := \big\{\lambda = (\lambda_1, \ldots, \lambda_{\frac{d}{2}}) \bigm| \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geqslant \cdots \geqslant \lambda_{\frac{d}{2}}\big\},$$

which is the set of partitions of length no more than $\frac{d}{2}$.

Proposition 2.2. *(See [31, Theorem 3.4].) There exists a commuting action of $\mathfrak{c}_\infty$ and $\mathrm{Sp}(d)$ on $\mathfrak{F}^{\frac{d}{2}}$. Furthermore, under this joint action, we have*

$$\mathfrak{F}^{\frac{d}{2}} \cong \bigoplus_{\lambda \in \mathcal{P}(\mathrm{Sp}(d))} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}(\lambda)\big) \otimes V^\lambda_{\mathrm{Sp}(d)}, \tag{2.4}$$

where $\Lambda^{\mathfrak{c}}(\lambda) := \frac{d}{2}\Lambda^{\mathfrak{c}}_0 + \sum_{k\geqslant 1} \lambda'_k \epsilon_k = \sum_{k=1}^{\frac{d}{2}} \Lambda^{\mathfrak{c}}_{\lambda_k}$.

Computing the trace of $\prod_{n\in\mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\frac{d}{2}} z_i^{e_i}$ on both sides of (2.4), we have

$$\prod_{i=1}^{\frac{d}{2}} \prod_{n\in\mathbb{N}} (1 + x_n z_i)\big(1 + x_n z_i^{-1}\big) = \sum_{\lambda \in \mathcal{P}(\mathrm{Sp}(d))} \mathrm{ch}\, L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}(\lambda)\big)\, \mathrm{ch}\, V^\lambda_{\mathrm{Sp}(d)}. \tag{2.5}$$

2.2.4. The $(\mathfrak{d}_\infty, \mathrm{O}(d))$-duality

Write $d \in \mathbb{N}$ as $d = 2\ell$ or $d = 2\ell + 1$ with $\ell \in \mathbb{N}$. Let $\mathrm{O}(d)$ denote the orthogonal group which is the subgroup of $\mathrm{GL}(d)$ preserving the non-degenerate symmetric bilinear form on $\mathbb{C}^d$ determined by J_d of (2.3). Let $\mathfrak{so}(d)$ be the Lie algebra of $\mathrm{O}(d)$. We take as a Cartan subalgebra H of $\mathfrak{so}(d)$ the subalgebra spanned by $e_i := e_{ii} - e_{d+1-i,d+1-i}$ $(1 \leqslant i \leqslant \ell)$, while we take as the Borel subalgebra the subalgebra of upper triangular matrices.

For $\lambda \in H^*$ let $\lambda_i = \langle \lambda, e_i \rangle$, for $1 \leqslant i \leqslant \ell$. Then a finite-dimensional irreducible module of $\mathfrak{so}(2\ell)$ is determined by its highest weight λ satisfying the condition $\lambda_1 \geqslant \cdots \geqslant \lambda_{\ell-1} \geqslant |\lambda_\ell|$ with either $\lambda_i \in \mathbb{Z}$ or else $\lambda_i \in \frac{1}{2} + \mathbb{Z}$, for all $1 \leqslant i \leqslant \ell$. Furthermore it lifts to a module of $\mathrm{SO}(2\ell)$ if and only if $\lambda_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant \ell$. Also a finite-dimensional irreducible module of $\mathfrak{so}(2\ell+1)$ is determined by its highest weight λ satisfying the conditions $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell$ with either $\lambda_i \in \mathbb{Z}_+$ or else $\lambda_i \in \frac{1}{2} + \mathbb{Z}_+$, for all $1 \leqslant i \leqslant \ell$. Furthermore it lifts to a module of $\mathrm{SO}(2\ell+1)$ if and only if $\lambda_i \in \mathbb{Z}_+$, for $1 \leqslant i \leqslant \ell$. Put

$$\mathcal{P}\big(\mathrm{O}(d)\big) := \big\{\lambda = (\lambda_1, \ldots, \lambda_d) \,\big|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geqslant \cdots \geqslant \lambda_d,\ \lambda'_1 + \lambda'_2 \leqslant d\big\}.$$

For $\lambda \in \mathcal{P}(\mathrm{O}(d))$, let $\tilde{\lambda}$ be the partition obtained from λ by replacing its first column with $d - \lambda'_1$.

Suppose that $d = 2\ell$ and let $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0) \in \mathcal{P}(\mathrm{O}(2\ell))$. For $\lambda_\ell > 0$, the irreducible $\mathrm{O}(2\ell)$-module $V^\lambda_{\mathrm{O}(2\ell)}$, viewed as an $\mathfrak{so}(2\ell)$-module, is isomorphic to the direct sum of irreducible modules of highest weights $(\lambda_1, \ldots, \lambda_\ell)$ and $(\lambda_1, \ldots, -\lambda_\ell)$. If $\lambda_\ell = 0$, the $\mathrm{O}(2\ell)$-module $V^\lambda_{\mathrm{O}(2\ell)}$, viewed as an $\mathfrak{so}(2\ell)$-module, is isomorphic to the irreducible module of highest weight $(\lambda_1, \ldots, \lambda_{\ell-1}, 0)$, on which the element $\tau = \sum_{i \neq \ell, \ell+1} e_{ii} + e_{\ell,\ell+1} + e_{\ell+1,\ell} \in \mathrm{O}(2\ell) \setminus \mathrm{SO}(2\ell)$ transforms trivially on highest weight vectors. Set $V^{\tilde{\lambda}}_{\mathrm{O}(2\ell)} := V^\lambda_{\mathrm{O}(2\ell)} \otimes \det$, where det is the one-dimensional nontrivial module of $\mathrm{O}(2\ell)$.

Suppose that $d = 2\ell + 1$ and let $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0) \in \mathcal{P}(\mathrm{O}(2\ell+1))$. Let $V^\lambda_{\mathrm{O}(2\ell+1)}$ be the irreducible $\mathrm{O}(2\ell+1)$-module isomorphic to the irreducible module of highest weight $(\lambda_1, \ldots, \lambda_\ell)$ as an $\mathrm{SO}(2\ell+1)$-module, on which $-I_d$ acts trivially. Here I_d is the $d \times d$ identity matrix. Also, we let $V^{\tilde{\lambda}}_{\mathrm{O}(2\ell+1)} := V^\lambda_{\mathrm{O}(2\ell+1)} \otimes \det$ (cf. e.g. [3,17]).

Proposition 2.3. *(See [31, Theorems 3.2 and 4.1].) There exists a commuting action of $\mathfrak{d}_\infty$ and $\mathrm{O}(d)$ on $\mathfrak{F}^{\frac{d}{2}}$. Furthermore, under this joint action, we have*

$$\mathfrak{F}^{\frac{d}{2}} \cong \bigoplus_{\lambda \in \mathcal{P}(\mathrm{O}(d))} L\big(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\lambda)\big) \otimes V^\lambda_{\mathrm{O}(d)}, \tag{2.6}$$

where $\Lambda^{\mathfrak{d}}(\lambda) := d\Lambda^{\mathfrak{d}}_0 + \sum_{k \geqslant 1} \lambda'_k \epsilon_k$.

Suppose that $d = 2\ell$. Computing the trace of $\prod_{n \in \mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\ell} z_i^{e_i}$ on (2.6) gives us

$$\prod_{i=1}^{\ell} \prod_{n \in \mathbb{N}} (1 + x_n z_i)\big(1 + x_n z_i^{-1}\big) = \sum_{\lambda \in \mathcal{P}(\mathrm{O}(2\ell))} \operatorname{ch} L\big(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\lambda)\big) \operatorname{ch} V^\lambda_{\mathrm{O}(2\ell)}. \tag{2.7}$$

Suppose that $d = 2\ell + 1$. Let ϵ be the eigenvalue of $-I_d$ on $\mathrm{O}(2\ell+1)$-modules satisfying $\epsilon^2 = 1$. From the computation of the trace of $\prod_{n \in \mathbb{N}} x_n^{E_n} \prod_{i=1}^{\ell} z_i^{e_i}(-I_d)$ on both sides of (2.6), we obtain

$$\prod_{i=1}^{\ell}\prod_{n\in\mathbb{N}}(1+\epsilon x_n z_i)\big(1+\epsilon x_n z_i^{-1}\big)(1+\epsilon x_n) = \sum_{\lambda\in\mathcal{P}(\mathrm{O}(2\ell+1))} \operatorname{ch} L\big(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell+1)}. \tag{2.8}$$

Note that $\operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell)}$ is a Laurent polynomial in $z_1,\dots,z_\ell$ and $\operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell)} = \operatorname{ch} V^{\tilde{\lambda}}_{\mathrm{O}(2\ell)}$, while $\operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell+1)}$ is the Laurent polynomial in $z_1,\dots,z_\ell,\epsilon$ and $\operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell+1)} = \epsilon \operatorname{ch} V^{\tilde{\lambda}}_{\mathrm{O}(2\ell+1)}$.

2.2.5. The $(\mathfrak{b}_\infty, \mathrm{Pin}(d))$*-duality*

Let d be even. The Lie group $\mathrm{Pin}(d)$ is a double cover of $\mathrm{O}(d)$, with $\mathrm{Spin}(d)$ as the inverse image of $\mathrm{SO}(d)$ under the covering map (see e.g. [3]). An irreducible representation of $\mathrm{Spin}(d)$ that does not factor through $\mathrm{SO}(d)$ is an irreducible representation of $\mathfrak{so}(d)$ of highest weight of the form

$$\left(\lambda_1+\frac{1}{2},\dots,\lambda_{\frac{d}{2}-1}+\frac{1}{2},\lambda_{\frac{d}{2}}+\frac{1}{2}\right), \quad \text{or} \quad \left(\lambda_1+\frac{1}{2},\dots,\lambda_{\frac{d}{2}-1}+\frac{1}{2},-\lambda_{\frac{d}{2}}-\frac{1}{2}\right), \tag{2.9}$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_{\frac{d}{2}}$ with $\lambda_i \in \mathbb{Z}_+$ for $1 \leqslant i \leqslant \frac{d}{2}$. We put

$$\mathcal{P}\big(\mathrm{Pin}(d)\big) := \big\{\lambda = \big(\lambda_1,\dots,\lambda_{\frac{d}{2}}\big) \bigm| \lambda_i \in \mathbb{Z}_+,\ \lambda_1 \geqslant \cdots \geqslant \lambda_{\frac{d}{2}}\big\}.$$

For $\lambda \in \mathcal{P}(\mathrm{Pin}(d))$, let us denote by $V^{\lambda}_{\mathrm{Pin}(d)}$ the irreducible representation of $\mathrm{Pin}(d)$ induced from the irreducible representation of $\mathrm{Spin}(d)$ whose highest weight is given by either of the two weights in (2.9). When restricted to $\mathrm{Spin}(d)$, $V^{\lambda}_{\mathrm{Pin}(d)}$ decomposes into a direct sum of two irreducible representations of highest weights given by those in (2.9). The modules in $\{V^{\lambda}_{\mathrm{Pin}(d)} \mid \lambda \in \mathcal{P}(\mathrm{Pin}(d))\}$ are precisely those finite-dimensional irreducible representations of $\mathrm{Pin}(d)$ that do not factor through $\mathrm{O}(d)$.

Proposition 2.4. *(See [31, Theorem 3.3].) There exists a commuting action of* $\mathfrak{b}_\infty$ *and* $\mathrm{Pin}(d)$ *on* $\mathfrak{F}^{\frac{d}{2}}$*. Furthermore, under this joint action, we have*

$$\mathfrak{F}^{\frac{d}{2}} \cong \bigoplus_{\lambda\in\mathcal{P}(\mathrm{Pin}(d))} L\big(\mathfrak{b}_\infty, \Lambda^{\mathfrak{b}}(\lambda)\big) \otimes V^{\lambda}_{\mathrm{Pin}(d)}, \tag{2.10}$$

where $\Lambda^{\mathfrak{b}}(\lambda) := d\Lambda^{\mathfrak{b}}_0 + \sum_{k\geqslant 1}\lambda'_k\epsilon_k$.

Computing the trace of the operator $\prod_{n\in\mathbb{Z}} x_n^{\widetilde{E}_n}\prod_{i=1}^{\frac{d}{2}} z_i^{e_i}$ on both sides of (2.10) gives

$$\prod_{i=1}^{\frac{d}{2}}\prod_{n\in\mathbb{N}}\big(z_i^{\frac{1}{2}}+z_i^{-\frac{1}{2}}\big)(1+x_n z_i)\big(1+x_n z_i^{-1}\big) = \sum_{\lambda\in\mathcal{P}(\mathrm{Pin}(d))} \operatorname{ch} L\big(\mathfrak{b}_\infty, \Lambda^{\mathfrak{b}}(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{Pin}(d)}. \tag{2.11}$$

2.3. *Formulas for $\mathfrak{u}_-$-homology groups of $\mathfrak{g}$-modules*

Recall that the Weyl group W can be written as $W = W_0 W^0$ with $W^0 = \bigsqcup_{k\geqslant 0} W^0_k$. For $\mu \in \mathfrak{h}^*$ and $w \in W$ we set $w \circ \mu := w(\mu + \rho_c) - \rho_c$.

Since we have $\langle w \circ \Lambda^{\mathfrak{x}}(\lambda), \alpha_j^\vee\rangle \in \mathbb{Z}_+$, for $w \in W^0$ and $j \in S$, it follows that we may find partitions $\lambda_w^\pm = ((\lambda_w^\pm)_1, (\lambda_w^\pm)_2, \ldots)$ and $\lambda_w = ((\lambda_w)_1, (\lambda_w)_2, \ldots)$ such that $w \circ \Lambda^{\mathfrak{x}}(\lambda)$ can be written as

$$w \circ \Lambda^{\mathfrak{x}}(\lambda) = \begin{cases} d\Lambda_0^{\mathfrak{x}} + \sum_{j>0}(\lambda_w^+)_j\epsilon_j - \sum_{j\geqslant 0}(\lambda_w^-)_{j+1}\epsilon_{-j}, & \text{if } \mathfrak{x} = \mathfrak{a}, \\ d\Lambda_0^{\mathfrak{x}} + \sum_{j>0}(\lambda_w)_j\epsilon_j, & \text{if } \mathfrak{x} = \mathfrak{b}, \mathfrak{d}, \\ \frac{d}{2}\Lambda_0^{\mathfrak{x}} + \sum_{j>0}(\lambda_w)_j\epsilon_j, & \text{if } \mathfrak{x} = \mathfrak{c}. \end{cases}$$

The following is obtained by applying the Kostant homology formula for integrable (= standard) modules over Kac–Moody algebra $\mathfrak{g}$ (cf. e.g. [20, Theorem 3.13]) and the Euler–Poincaré principle to the complex for the corresponding Lie algebra homology (see [4, Section 2.4, (2.18)]).

Proposition 2.5. *We have the following character formula:*

$$\operatorname{ch} L\big(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)\big) = \frac{1}{D^{\mathfrak{x}}} \sum_{k=0}^{\infty} (-1)^k \operatorname{ch} H_k\big(\mathfrak{u}_-; L\big(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)\big)\big),$$

where $\operatorname{ch} \mathrm{H}_k(\mathfrak{u}_-; L(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)))$ *is given by*

$$\begin{cases} \sum_{w\in W_k^0(\mathfrak{x})} s_{\lambda_w^+}(x_1, x_2, \ldots) s_{\lambda_w^-}(x_0^{-1}, x_{-1}^{-1}, \ldots), & \textit{if } \mathfrak{x} = \mathfrak{a}, \\ \sum_{w\in W_k^0(\mathfrak{x})} s_{\lambda_w}(x_1, x_2, \ldots), & \textit{if } \mathfrak{x} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \end{cases} \tag{2.12}$$

and

$$D^{\mathfrak{x}} := \begin{cases} \prod_{i,j}(1 - x_{-i+1}^{-1}x_j), & \textit{if } \mathfrak{x} = \mathfrak{a}, \\ \prod_i (1 - x_i)\prod_{i<j}(1 - x_i x_j), & \textit{if } \mathfrak{x} = \mathfrak{b}, \\ \prod_{i\leqslant j}(1 - x_i x_j), & \textit{if } \mathfrak{x} = \mathfrak{c}, \\ \prod_{i<j}(1 - x_i x_j), & \textit{if } \mathfrak{x} = \mathfrak{d}, \end{cases} \quad i, j \in \mathbb{N}.$$

Here, for a partition γ, s_γ denotes the corresponding Schur function. Observe that $hs_\lambda(y, \xi) = hs_{\lambda'}(\xi, y)$.

2.4. *The general linear and ortho-symplectic superalgebras*

Let $p, q, m, n \in \mathbb{Z}_+$. We briefly recall the general linear superalgebra $\mathfrak{gl}(p + m|q + n)$ and ortho-symplectic superalgebras $\mathfrak{spo}(2m|2n + 1)$, $\mathfrak{osp}(2m|2n)$, $\mathfrak{spo}(2m|2n)$ (see e.g. [21]), which will be called of type $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{d}$, respectively, for reasons of Howe dualities appearing later on. The following notations associated with these Lie superalgebras will be assumed throughout the paper.

- $\bar{\mathfrak{g}}$: $\mathfrak{gl}(p + m|q + n)$, $\mathfrak{spo}(2m|2n + 1)$, $\mathfrak{osp}(2m|2n)$, $\mathfrak{spo}(2m|2n)$, or their central extensions in Section 3.1,

- $\bar{\mathfrak{h}}$: a Cartan subalgebra of $\bar{\mathfrak{g}}$,
- $\bar{I}$: an index set for simple roots for $\bar{\mathfrak{g}}$,
- $\overline{\Pi} = \{\beta_i \mid i \in \bar{I}\}$: a set of simple roots for $\bar{\mathfrak{g}}$,
- $\overline{\Pi}^\vee = \{\beta_i^\vee \mid i \in \bar{I}\}$: a set of simple coroots for $\bar{\mathfrak{g}}$,
- $\overline{\Delta}^+$: a set of positive roots for $\bar{\mathfrak{g}}$,
- $L(\bar{\mathfrak{g}}, \lambda)$: the irreducible highest weight $\bar{\mathfrak{g}}$-module of highest weight $\lambda \in \bar{\mathfrak{h}}^*$.

2.4.1. Denote by $\mathbb{C}^{p+m|q+n}$ the complex superspace of dimension $(p+m|q+n)$ with basis $\{v_{-p}, \dots, v_{-1}, w_{-q}, \dots, w_{-1}, v_1, \dots, v_m, w_1, \dots, w_n\}$, where $\deg v_i := \bar{0}$ and $\deg w_j := \bar{1}$. With respect to this basis the general linear superalgebra $\mathfrak{gl}(p+m|q+n)$ may be regarded as the Lie superalgebra of complex matrices (a_{ij}), with $i, j \in I_{p+m|q+n}$, where $I_{p+m|q+n} := \{-p, \dots, -1, 1, \dots, m\} \cup \{-\bar{q}, \dots, -\bar{1}, \bar{1}, \dots, \bar{n}\}$. For notational convenience we declare a linear ordering of $I_{p+m|q+n}$ by setting

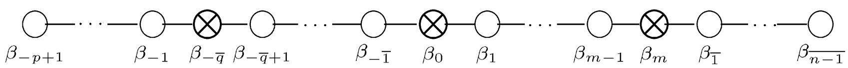
$$-p < \cdots < -1 < -\bar{q} < \cdots < -\bar{1} < 1 < \cdots < m < \bar{1} < \cdots < \bar{n}.$$

For $i \in I_{p+m|q+n} \setminus \{\bar{n}\}$, $i+1$ means the minimum of those $j \in I_{p+m|q+n}$ with $j > i$. In a similar way one defines $i-1$, for $i \in I_{p+m|q+n} \setminus \{-p\}$.

Denote by E_{ij}, $i, j \in I_{p+m|q+n}$, the elementary matrix with 1 at the ith row and the jth column, and zero elsewhere. Then $\bar{\mathfrak{h}}$ is spanned by E_{kk}, $k \in I_{p+m|q+n}$. Let ε_i and δ_j, with $i \in \{-p, \dots, -1, 1, \dots, m\}$ and $j \in \{-q, \dots, -1, 1, \dots, n\}$, form the basis of $\bar{\mathfrak{h}}^*$ dual to E_{ii} and $E_{\bar{j},\bar{j}}$, respectively. We choose our Borel subalgebra to be the subalgebra spanned by E_{ij} with $i \leqslant j$. With respect to this Borel subalgebra, we have

$$\begin{aligned}
\overline{\Pi}^\vee = \big\{&\beta_i^\vee = E_{i-1,i-1} - E_{ii} \ \big(i \in \{-p+1, \dots, -\bar{1}\} \setminus \{-\bar{q}\}\big), \\
&\beta_i^\vee = E_{ii} - E_{i+1,i+1} \ \big(i \in \{1, \dots, \overline{n-1}\} \setminus \{m\}\big), \\
&\beta_{-\bar{q}}^\vee = E_{-1,-1} + E_{-\bar{q},-\bar{q}},\ \beta_0^\vee = E_{-\bar{1},-\bar{1}} + E_{11},\ \beta_m^\vee = E_{mm} + E_{\bar{1},\bar{1}}\big\}, \\
\overline{\Pi} = \big\{&\beta_i = \varepsilon_{i-1} - \varepsilon_i \ (i = -p+1, \dots, -1),\ \beta_i = \varepsilon_i - \varepsilon_{i+1} \ (i = 1, \dots, m-1), \\
&\beta_{\bar{j}} = \delta_{j-1} - \delta_j \ (j = -q+1, \dots, -1),\ \beta_{\bar{j}} = \delta_j - \delta_{j+1} \ (j = 1, \dots, n-1), \\
&\beta_{-\bar{q}} = \varepsilon_{-1} - \delta_{-q},\ \beta_0 = \delta_{-1} - \varepsilon_1,\ \beta_m = \varepsilon_m - \delta_1\big\}, \\
\overline{\Delta}^+ = \big\{&\varepsilon_i - \varepsilon_j,\ \varepsilon_i - \delta_j,\ \delta_i - \varepsilon_j,\ \delta_i - \delta_j \ (i < j)\big\}.
\end{aligned}$$

The associated Dynkin diagram is as follows ($\bigotimes$ denotes an odd isotropic root):



2.4.2. The ortho-symplectic Lie superalgebra $\mathfrak{osp}(2m|2n)$ is a subalgebra of $\mathfrak{gl}(2m|2n)$ consisting of the linear transformations that preserve a non-degenerate even supersymmetric bilinear form $(\cdot|\cdot)$, namely, $\mathfrak{osp}(2m|2n) = \mathfrak{osp}(2m|2n)_{\bar{0}} \oplus \mathfrak{osp}(2m|2n)_{\bar{1}}$ with $\mathfrak{osp}(2m|2n)_\kappa$ equal to

$$\big\{A \in \mathfrak{gl}(2m|2n)_\kappa \ \big|\ (Av|w) = (-1)^{1+\kappa \deg v}(v|Aw) \text{ for homogeneous } v, w \in \mathbb{C}^{2m|2n}\big\}.$$

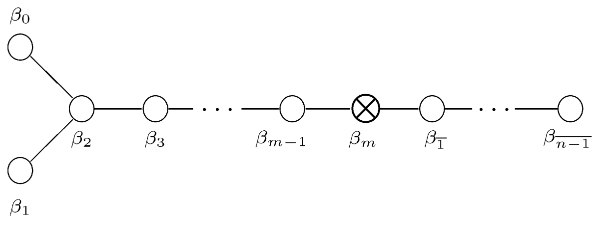
Below let $(\cdot|\cdot)$ be the supersymmetric bilinear form corresponding to the matrix (see (2.3) for J_k)

$$\begin{pmatrix} J_{2m} & 0 & 0 \\ 0 & 0 & J_n \\ 0 & -J_n & 0 \end{pmatrix}.$$

The Cartan subalgebra $\bar{\mathfrak{h}}$ of $\mathfrak{osp}(2m|2n)$ is spanned by $E_i := E_{ii} - E_{2m+1-i,2m+1-i}$ and $E_{\bar{j}} := E_{\bar{j},\bar{j}} - E_{\overline{2n-j+1},\overline{2n-j+1}}$, $i = 1, \dots, m$, and $j = 1, \dots, n$. We use the same notation for the restrictions of ε_i and δ_j to the Cartan subalgebra of $\mathfrak{osp}(2m|2n)$. Let

$$\begin{aligned}
\overline{\Pi}^\vee &= \bigl\{\beta_0^\vee = -E_1 - E_2,\ \beta_i^\vee = E_i - E_{i+1}\ \bigl(i \in I_{m|n} \setminus \{m, \bar{n}\}\bigr),\ \beta_m^\vee = E_m + E_{\bar{1}}\bigr\},\\
\overline{\Pi} &= \bigl\{\beta_0 = -\varepsilon_1 - \varepsilon_2,\ \beta_i = \varepsilon_i - \varepsilon_{i+1}\ (i = 1, \dots, m-1),\\
&\qquad \beta_m = \varepsilon_m - \delta_1,\ \beta_{\bar{j}} = \delta_j - \delta_{j+1}\ (j = 1, \dots, n-1)\bigr\},\\
\overline{\Delta}^+ &= \bigl\{\pm\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i - \delta_j,\ -2\delta_i,\ \pm\delta_i - \delta_j\ (i < j)\bigr\}.
\end{aligned}$$

The associated Dynkin diagram is as follows:



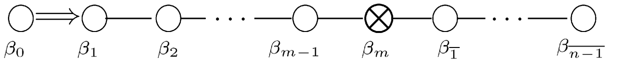
2.4.3. Taking the skew-supersymmetric bilinear form corresponding to the matrix

$$\begin{pmatrix} 0 & J_m & 0 \\ -J_m & 0 & 0 \\ 0 & 0 & J_{2n} \end{pmatrix},$$

we can construct similarly the Lie superalgebra $\mathfrak{spo}(2m|2n)$, which is isomorphic to $\mathfrak{osp}(2n|2m)$. Similarly we define the Cartan subalgebra $\bar{\mathfrak{h}}$, ε_i, and δ_j. Let

$$\begin{aligned}
\overline{\Pi}^\vee &= \bigl\{\beta_0^\vee = -E_1,\ \beta_i^\vee = E_i - E_{i+1}\ \bigl(i \in I_{m|n} \setminus \{m, \bar{n}\}\bigr),\ \beta_m^\vee = E_m + E_{\bar{1}}\bigr\},\\
\overline{\Pi} &= \bigl\{\beta_0 = -2\varepsilon_1,\ \beta_i = \varepsilon_i - \varepsilon_{i+1}\ (i = 1, \dots, m-1),\\
&\qquad \beta_m = \varepsilon_m - \delta_1,\ \beta_{\bar{j}} = \delta_j - \delta_{j+1}\ (j = 1, \dots, n-1)\bigr\},\\
\overline{\Delta}^+ &= \bigl\{-2\varepsilon_i,\ \pm\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i - \delta_j,\ \pm\delta_i - \delta_j\ (i < j)\bigr\}.
\end{aligned}$$

The associated Dynkin diagram is as follows:



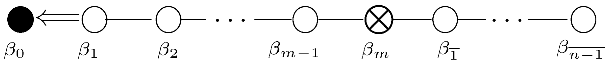
2.4.4. The skew-supersymmetric bilinear form corresponding to the matrix

$$\begin{pmatrix} 0 & J_m & 0 \\ -J_m & 0 & 0 \\ 0 & 0 & J_{2n+1} \end{pmatrix}$$

gives rise to $\mathfrak{spo}(2m|2n+1)$. The Cartan subalgebra $\bar{\mathfrak{h}}$ of $\mathfrak{spo}(2m|2n+1)$ is spanned by $E_i := E_{ii} - E_{2m+1-i,2m+1-i}$ and $E_{\bar{j}} := E_{\bar{j},\bar{j}} - E_{\overline{2n-j+2},\overline{2n-j+2}}$, $i = 1, \ldots, m$, and $j = 1, \ldots, n$. We define ε_i and δ_j analogously. Let

$$\begin{aligned} \overline{\Pi}^\vee &= \big\{\beta_0^\vee = E_1,\ \beta_i^\vee = E_i - E_{i+1}\ \big(i \in I_{m|n} \setminus \{m, \bar{n}\}\big),\ \beta_m^\vee = E_m + E_{\bar{1}}\big\}, \\ \overline{\Pi} &= \big\{\beta_0 = -\varepsilon_1,\ \beta_i = \varepsilon_i - \varepsilon_{i+1}\ (i = 1, \ldots, m-1), \\ &\qquad \beta_m = \varepsilon_m - \delta_1,\ \beta_{\bar{j}} = \delta_j - \delta_{j+1}\ (j = 1, \ldots, n-1)\big\}, \\ \overline{\Delta}^+ &= \big\{-\varepsilon_i,\ -2\varepsilon_i,\ \pm\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i - \delta_j,\ -\delta_j,\ \pm\delta_i - \delta_j\ (i < j)\big\}. \end{aligned}$$

The associated Dynkin diagram is as follows (● denotes an odd non-isotropic simple root)



2.5. *Reductive dual pairs* $(\bar{\mathfrak{g}}, G)$

Below we recall Howe dualities involving the Lie superalgebras $\mathfrak{gl}(p+m|q+n)$, $\mathfrak{spo}(2m|2n+1)$, $\mathfrak{osp}(2m|2n)$ and $\mathfrak{spo}(2m|2n)$.

Let $d \in \mathbb{N}$. Suppose that λ is a generalized partition of depth d with $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$. Then $\lambda^+_{m+1} \leqslant n$, and $\lambda^-_{d-p} \leqslant q$ (recall $\lambda^\pm$ from Section 1.5). Define $\Lambda^{\mathfrak{a}}_f(\lambda) \in \bar{\mathfrak{h}}^*$ to be

$$\begin{aligned} \Lambda^{\mathfrak{a}}_f(\lambda) := &- \sum_{i=-p}^{-1} \big(\langle\lambda^-_{d+i+1} - q\rangle + d\big)\varepsilon_i \\ &- \sum_{j=-q}^{-1} \big((\lambda^-)'_{-j} - d\big)\delta_j + \sum_{i=1}^{m} \lambda^+_i \varepsilon_i + \sum_{j=1}^{n} \langle(\lambda^+)'_j - m\rangle \delta_j. \end{aligned}$$

The following Howe duality was built on the special case when $p = q = 0$ obtained in [7, Theorem 3.2] (also cf. [30]). According to [6] $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*} \oplus \mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ carries a natural commuting action of $\mathfrak{gl}(p+m|q+n)$ and $\mathrm{GL}(d)$, which form a reductive dual pair in the sense of [17].

Proposition 2.6. *(See [6, Theorem 3.3].) As a* $\mathfrak{gl}(p+m|q+n) \times \mathrm{GL}(d)$*-module we have*

$$S\big(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*} \oplus \mathbb{C}^{m|n} \otimes \mathbb{C}^d\big) \cong \bigoplus_{\lambda} L\big(\mathfrak{gl}(p+m|q+n), \Lambda^{\mathfrak{a}}_f(\lambda)\big) \otimes V^{\lambda}_{\mathrm{GL}(d)},$$

where the summation is over all $\lambda \in \mathcal{P}(\mathrm{GL}(d))$*, subject to the conditions* $\lambda_{m+1} \leqslant n$ *and* $\lambda_{d-p} \geqslant -q$.

Let d be even. On the superspace $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ we have natural actions of $\mathfrak{osp}(2m|2n)$ and $\mathrm{Sp}(d)$, which form a reductive dual pair [17, Section 3].

Proposition 2.7. *(See [11, Theorem 5.2].) As an* $\mathfrak{osp}(2m|2n) \times \mathrm{Sp}(d)$*-module we have*

$$S\big(\mathbb{C}^{m|n} \otimes \mathbb{C}^d\big) \cong \bigoplus_{\lambda} L\big(\mathfrak{osp}(2m|2n), \Lambda^{\mathfrak{c}}_f(\lambda)\big) \otimes V^{\lambda}_{\mathrm{Sp}(d)},$$

where the summation is over all $\lambda \in \mathcal{P}(\mathrm{Sp}(d))$ *with* $\lambda_{m+1} \leqslant n$*, and*

$$\Lambda^{\mathfrak{c}}_f(\lambda) := \sum_{i=1}^{m} \left(\lambda_i + \frac{1}{2}d\right)\varepsilon_i + \sum_{j=1}^{n} \left(\langle \lambda'_j - m\rangle - \frac{1}{2}d\right)\delta_j.$$

Let d be even or odd. On the superspace $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ we have a natural action of $\mathfrak{spo}(2m|2n)$ and $\mathrm{O}(d)$, which form a reductive dual pair [17, Section 3].

Proposition 2.8. *(See [11, Theorem 5.1].) As an* $\mathfrak{spo}(2m|2n) \times \mathrm{O}(d)$*-module we have*

$$S\big(\mathbb{C}^{m|n} \otimes \mathbb{C}^d\big) \cong \bigoplus_{\lambda} L\big(\mathfrak{spo}(2m|2n), \Lambda^{\mathfrak{d}}_f(\lambda)\big) \otimes V^{\lambda}_{\mathrm{O}(d)},$$

where the sum is over all $\lambda \in \mathcal{P}(\mathrm{O}(d))$ *with* $\lambda_{m+1} \leqslant n$*, and*

$$\Lambda^{\mathfrak{d}}_f(\lambda) := \sum_{i=1}^{m} \left(\lambda_i + \frac{1}{2}d\right)\varepsilon_i + \sum_{j=1}^{n} \left(\langle \lambda'_j - m\rangle - \frac{1}{2}d\right)\delta_j.$$

The following new Howe duality is worked out in Appendix A.1.

Proposition 2.9. *(See Theorem A.1.) Let d be even. As an* $\mathfrak{spo}(2m|2n+1) \times \mathrm{Pin}(d)$*-module we have*

$$S\big(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}}\big) \cong \bigoplus_{\lambda} L\big(\mathfrak{spo}(2m|2n+1), \Lambda^{\mathfrak{b}}_f(\lambda)\big) \otimes V^{\lambda}_{\mathrm{Pin}(d)},$$

where the summation is over all $\lambda \in \mathcal{P}(\mathrm{Pin}(d))$ *with* $\lambda_{m+1} \leqslant n$*, and*

$$\Lambda^{\mathfrak{b}}_f(\lambda) := \sum_{i=1}^{m} \left(\lambda_i + \frac{1}{2}d\right)\varepsilon_i + \sum_{j=1}^{n} \left(\langle \lambda'_j - m\rangle - \frac{1}{2}d\right)\delta_j.$$

The simple $\bar{\mathfrak{g}}$-modules constructed in this section will be referred to as *oscillator modules* of $\bar{\mathfrak{g}}$.

From now on, we mean by $(\mathfrak{g}, G)$ and $(\bar{\mathfrak{g}}, G)$ one of the dual pairs of type $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ in Sections 2.2 and 2.5, respectively. We let $\mathcal{P}^{\bar{\mathfrak{g}}}(G)$ be the subset of (generalized) partitions in $\mathcal{P}(G)$ that appear in the $(\bar{\mathfrak{g}}, G)$-duality decomposition.

3. Character formulas of oscillator modules of Lie superalgebras

In this section we derive character formulas for the oscillator modules of $\bar{\mathfrak{g}}$, essentially only using the character formulas of the integrable modules of infinite-dimensional Lie algebra $\mathfrak{g}$. To that end, we introduce (trivial) central extensions of $\bar{\mathfrak{g}}$.

3.1. Central extensions

Given $\mathfrak{J} \in \mathfrak{gl}(p+m|q+n)$, we define

$$\beta_{\mathfrak{J}}(A, B) := \mathrm{Str}\big([\mathfrak{J}, A]B\big), \quad A, B \in \mathfrak{gl}(p+m|q+n),$$

where $\mathrm{Str}(c_{ij}) = \sum_{i=-p}^{-1} c_{ii} - \sum_{j=-\bar{q}}^{-\bar{1}} c_{jj} + \sum_{i=1}^{m} c_{ii} - \sum_{j=\bar{1}}^{\bar{n}} c_{jj}$. It follows that $\beta_{\mathfrak{J}}$ is a 2-cocycle which defines a central extension of $\mathfrak{gl}(p+m|q+n)$. We note that this cocycle is a coboundary, since we can construct this extension on the Lie superalgebra $\mathfrak{gl}(p+m|q+n) \oplus \mathbb{C}K$, with a central element K, as follows. Define for $A \in \mathfrak{gl}(p+m|q+n)$

$$\widehat{A} := A - \mathrm{Str}(\mathfrak{J}A)K.$$

Then $[\widehat{A}, \widehat{B}] = \widehat{[A, B]} + \beta_{\mathfrak{J}}(A, B)K$.

From now on we let

$$\mathfrak{J} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

We denote by $\widehat{\mathfrak{gl}}(p+m|q+n)$ the resulting central extension of $\mathfrak{gl}(p+m|q+n)$ by the one-dimensional center $\mathbb{C}K$. The simple roots of the derived algebra of $\widehat{\mathfrak{gl}}(p+m|q+n)$ are the same as those of the derived algebra of $\mathfrak{gl}(p+m|q+n)$. The simple coroots are also the same except that $\beta_0^\vee = E_{-\bar{1},-\bar{1}} + E_{11}$ is replaced by $\widehat{E}_{-\bar{1},-\bar{1}} + \widehat{E}_{11} + K$.

Consider the $(2m+2n) \times (2m+2n)$ matrix of the form

$$\mathfrak{J}' = \frac{1}{2} \begin{pmatrix} -I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}. \tag{3.2}$$

The cocycle $\beta_{\mathfrak{J}'}$ gives rise to a central extension of $\mathfrak{gl}(2m|2n)$ by a one-dimensional center $\mathbb{C}K$. The induced central extensions of $\mathfrak{osp}(2m|2n)$ and $\mathfrak{spo}(2m|2n)$ by a one-dimensional center $\mathbb{C}K$ are denoted by $\widehat{\mathfrak{osp}}(2m|2n)$ and $\widehat{\mathfrak{spo}}(2m|2n)$, respectively.

Similarly we let $\widehat{\mathfrak{spo}}(2m|2n+1)$ stand for the central extension of $\mathfrak{spo}(2m|2n+1)$ by the one-dimensional center $\mathbb{C}K$, induced from the central extension of $\mathfrak{gl}(2m|2n+1)$ associated with $\beta_{\mathfrak{J}''}$, where $\mathfrak{J}''$ is the following $(2m+2n+1) \times (2m+2n+1)$ matrix

$$\mathfrak{J}'' = \frac{1}{2}\begin{pmatrix} -I_m & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & -I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n \end{pmatrix}. \tag{3.3}$$

3.2. $\widehat{\mathfrak{gl}}(p+m|q+n)$

For later convenience of comparing with infinite-dimensional Lie algebras, we introduce an additional element K, which commutes with $\mathfrak{gl}(p+m|q+n)$ and $\mathrm{GL}(d)$, and regard $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*} \oplus \mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ as a module over $(\mathfrak{gl}(p+m|q+n) \oplus \mathbb{C}K) \times \mathrm{GL}(d)$, where we declare that K acts as the scalar d. For $\mathfrak{J}$ as in (3.1) set

$$\widehat{E}_{ij} := E_{ij} - \mathrm{Str}(\mathfrak{J}E_{ij})K. \tag{3.4}$$

Then $\sum_{i,j} \mathbb{C}\widehat{E}_{ij} \oplus \mathbb{C}K$ gives rise to the central extension $\widehat{\mathfrak{gl}}(p+m|q+n)$. Now $\widehat{\mathfrak{gl}}(p+m|q+n)$ and $\mathrm{GL}(d)$ form a reductive dual pair on $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*} \oplus \mathbb{C}^{m|n} \otimes \mathbb{C}^d)$.

Define $\overline{\Lambda}^{\mathfrak{a}}_0 \in \bar{\mathfrak{h}}^*$ by $\langle \overline{\Lambda}^{\mathfrak{a}}_0, K\rangle = 1$ and $\langle \overline{\Lambda}^{\mathfrak{a}}_0, E_{kk}\rangle = 0$ for all $k \in I_{p+m|q+n}$. For $\lambda \in \mathcal{P}(\mathrm{GL}(d))$ with $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$, we define

$$\begin{aligned}\widehat{\Lambda}^{\mathfrak{a}}_f(\lambda) := d\widetilde{\Lambda}^{\mathfrak{a}}_0 - \sum_{i=-p}^{-1} \langle \lambda^-_{d+i+1} - q\rangle \varepsilon_i \\ - \sum_{j=-q}^{-1} \big(\lambda^-\big)'_{-j}\delta_j + \sum_{i=1}^{m} \lambda^+_i \varepsilon_i + \sum_{j=1}^{n} \big\langle \big(\lambda^+\big)'_j - m\big\rangle \delta_j,\end{aligned}$$

where $\widetilde{\Lambda}^{\mathfrak{a}}_0 = \overline{\Lambda}^{\mathfrak{a}}_0 - \sum_{i=-p}^{-1} \varepsilon_i + \sum_{j=-q}^{-1} \delta_j$.

Recall that a partition $\lambda \in \mathcal{P}^+$ is called an $(m|n)$-*hook partition*, if $\lambda_{m+1} \leqslant n$ [2]. For an $(m|n)$-hook partition λ, we let $\lambda^\natural := \sum_{i=1}^m \lambda_i \varepsilon_i + \sum_{j=1}^n \langle \lambda'_j - m\rangle \delta_j \in \mathfrak{h}^*_{m|n}$.

Let $\xi := \{\xi_1, \ldots, \xi_n\}$, $y := \{y_1, \ldots, y_m\}$, $\eta^{-1} := \{\eta^{-1}_{-1}, \ldots, \eta^{-1}_{-q}\}$, and $x^{-1} := \{x^{-1}_{-1}, \ldots, x^{-1}_{-p}\}$ be indeterminates. The character of $L(\mathfrak{gl}(m|n), \lambda^\natural)$, i.e. the trace of the operator $\prod_{i=1}^m y_i^{E_{ii}} \times \prod_{j=1}^n \xi_j^{E_{\bar{j}\bar{j}}}$, is given by the *hook Schur polynomial* [2,29]

$$hs_\lambda(y, \xi) := \sum_{\mu \subseteq \lambda} s_\mu(y_1, \ldots, y_m) s_{\lambda'/\mu'}(\xi_1, \ldots, \xi_n). \tag{3.5}$$

Theorem 3.1. *For $\lambda \in \mathcal{P}(\mathrm{GL}(d))$ with $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$, we have*

$$\begin{aligned}&\mathrm{ch}\, L\big(\widehat{\mathfrak{gl}}(p+m|q+n), \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda)\big) \\ &\quad = \frac{\sum_{k=0}^\infty (-1)^k \sum_{w \in W^0_k(\mathfrak{a})} hs_{\lambda^+_w}(\xi, y)\, hs_{\lambda^-_w}(\eta^{-1}, x^{-1})}{\prod_{i,j,s,t} (1 - x_i^{-1} y_s)(1 + x_i^{-1}\xi_t)^{-1}(1 + \eta_i^{-1} y_s)^{-1}(1 - \eta_i^{-1}\xi_t)}.\end{aligned}$$

Proof. Computing the trace of the operator $\prod_{i,j,s,t} x_i^{\widehat{E}_{ii}} \eta_j^{\widehat{E}_{\bar{j},\bar{j}}} y_s^{\widehat{E}_{ss}} \xi_t^{\widehat{E}_{\bar{t},\bar{t}}} \prod_{k=1}^d z_k^{e_{kk}}$ on both sides of the isomorphism in Proposition 2.6, where $-p \leqslant i \leqslant -1$, $-q \leqslant j \leqslant -1$, $1 \leqslant s \leqslant m$, and $1 \leqslant t \leqslant n$, we obtain

$$\prod_{k=1}^d \prod_{i,j,s,t} \frac{(1+\eta_j^{-1} z_k^{-1})(1+\xi_t z_k)}{(1-x_i^{-1} z_k^{-1})(1-y_s z_k)} = \sum_\lambda \operatorname{ch} L\big(\widehat{\mathfrak{gl}}(p+m|q+n), \widehat{\Lambda}_f^{\mathfrak{a}}(\lambda)\big) \operatorname{ch} V_{\mathrm{GL}(d)}^\lambda, \tag{3.6}$$

where the summation is over all $\lambda \in \mathcal{P}(\mathrm{GL}(d))$ with $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$.

Recall the identity (2.2) which results from the $(\widehat{\mathfrak{gl}}_\infty, \mathrm{GL}(d))$-duality. Now (2.2) and Proposition 2.5 imply that

$$\begin{aligned}
&\prod_{k=1}^d \prod_{n\in\mathbb{N}} (1+x_n z_k)\big(1+x_{1-n}^{-1} z_k^{-1}\big) \\
&\quad = \sum_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} \operatorname{ch} L\big(\widehat{\mathfrak{gl}}_\infty, \Lambda^{\mathfrak{a}}(\lambda)\big) \operatorname{ch} V_{\mathrm{GL}(d)}^\lambda \\
&\quad = \sum_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} \frac{\sum_{k=0}^\infty (-1)^k \sum_{w\in W_k^0} s_{\lambda_w^+}(x_1, x_2, \ldots) s_{\lambda_w^-}(x_0^{-1}, x_{-1}^{-1}, \ldots)}{\prod_{i,j} (1 - x_{-i+1}^{-1} x_j)} \times \operatorname{ch} V_{\mathrm{GL}(d)}^\lambda.
\end{aligned}$$

We set $\xi_t := x_t$ $(t = 1, \ldots, n)$, $y_s := x_{n+s}$ $(s \in \mathbb{N})$, $\eta_{j-1} := x_j$ $(j = 0, \ldots, -q+1)$, and $x_i := x_{-q+1+i}$ $(i \in -\mathbb{N})$, and rewrite the above as

$$\begin{aligned}
&\prod_{k=1}^d \prod_{i,j,s,t} \big(1+x_i^{-1} z_k^{-1}\big)\big(1+\eta_j^{-1} z_k^{-1}\big)(1+y_s z_k)(1+\xi_t z_k) = \sum_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} \operatorname{ch} V_{\mathrm{GL}(d)}^\lambda \\
&\quad \times \frac{\sum_{k=0}^\infty (-1)^k \sum_{w\in W_k^0} s_{\lambda_w^+}(\xi_1, \ldots, \xi_n, y_1, \ldots) s_{\lambda_w^-}(\eta_{-1}^{-1}, \ldots, \eta_{-q}^{-1}, x_{-1}^{-1}, \ldots)}{\prod_{i,j,s,t} (1 - x_i^{-1} y_s)(1 - x_i^{-1} \xi_t)(1 - \eta_i^{-1} y_s)(1 - \eta_i^{-1} \xi_t)}.
\end{aligned}$$

Let ω be the standard involution of symmetric functions that interchanges elementary symmetric functions and complete symmetric functions (e.g. [28, (2.7)]). Note that ω sends s_μ to $s_{\mu'}$ and recall (3.5). Applying ω twice to the above identity, once on the variables $x_{-1}^{-1}, x_{-2}^{-1}, \ldots$ and another on the variables $y_1, y_2, \ldots$, we obtain

$$\begin{aligned}
&\prod_{k=1}^d \prod_{i,j,s,t} \frac{(1+\xi_t z_k)(1+\eta_j^{-1} z_k^{-1})}{(1-y_s z_k)(1-x_i^{-1} z_k^{-1})} = \sum_{\lambda\in\mathcal{P}(\mathrm{GL}(d))} \operatorname{ch} V_{\mathrm{GL}(d)}^\lambda \\
&\quad \times \frac{\sum_{k=0}^\infty (-1)^k \sum_{w\in W_k^0} hs_{\lambda_w^+}(\xi_1, \ldots, \xi_n, y_1, \ldots) hs_{\lambda_w^-}(\eta_{-1}^{-1}, \ldots, \eta_{-q}^{-1}, x_{-1}^{-1}, \ldots)}{\prod_{i,j,s,t} (1 - x_i^{-1} y_s)(1 + x_i^{-1} \xi_t)^{-1}(1 + \eta_i^{-1} y_s)^{-1}(1 - \eta_i^{-1} \xi_t)}.
\end{aligned}$$

Finally setting $x_{-p-i}^{-1} = y_{m+i} = 0$ for $i \in \mathbb{N}$, we get an identity which shares the same left-hand side as (3.6). The theorem now follows from comparison of the right-hand sides and the linear independence of the characters $\{\operatorname{ch} V_{\mathrm{GL}(d)}^\lambda \mid \lambda \in \mathcal{P}(\mathrm{GL}(d))\}$. $\square$

Remark 3.2. The character here, which is the trace of the corresponding operators with hats from $\widehat{\mathfrak{gl}}(p+m|q+n)$, differs from the usual one from $\mathfrak{gl}(p+m|q+n)$ by exactly a factor of $(\frac{\eta_{-q}\cdots\eta_{-1}}{x_{-p}\cdots x_{-1}})^d$. Similar remarks apply below to $\mathfrak{spo}/\mathfrak{osp}$-characters versus $\widehat{\mathfrak{spo}}/\widehat{\mathfrak{osp}}$-characters, where the power d here is replaced by $\frac{d}{2}$.

Our formula here differs from the character formula obtained in [6, Theorem 5.3].

Remark 3.3. Denote by $\omega^{\mathfrak{a}}$ a linear extension of the composition of maps in the proof of Theorem 3.1 that sends $\operatorname{ch} L(\widehat{\mathfrak{gl}}_\infty, \Lambda^{\mathfrak{a}}(\lambda))$ to $\operatorname{ch} L(\widehat{\mathfrak{gl}}(p+m|q+n), \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda))$. Since each map in the composite is either an involution ω or an evaluation of some variables at zero, $\omega^{\mathfrak{a}}$ respects the multiplication of characters.

3.3. $\widehat{\mathfrak{osp}}(2m|2n)$

Suppose that d is even. Recalling that $\widehat{\mathfrak{osp}}(2m|2n)$ is the central extension induced from $\widehat{\mathfrak{gl}}(2m|2n)$ with respect to $\mathfrak{J}'$ in (3.2), we first introduce an element K that acts on $S(\mathbb{C}^{m|n}\otimes\mathbb{C}^d)$ as the scalar $\frac{d}{2}$, and that commutes with the actions of $\mathfrak{osp}(2m|2n)$ and $\mathrm{Sp}(d)$. Setting

$$\widehat{A} := A - \operatorname{Str}\big(\mathfrak{J}'A\big)K \in \mathfrak{osp}(2m|2n)\oplus\mathbb{C}K, \quad A\in\mathfrak{osp}(2m|2n),$$

gives rise to an action of $\widehat{\mathfrak{osp}}(2m|2n)$ on $S(\mathbb{C}^{m|n}\otimes\mathbb{C}^d)$. Clearly $\widehat{\mathfrak{osp}}(2m|2n)$ and $\mathrm{Sp}(d)$ also form a reductive dual pair. Define an element $\overline{\Lambda}^{\mathfrak{c}}_0\in\bar{\mathfrak{h}}^*$ by $\langle\overline{\Lambda}^{\mathfrak{c}}_0, K\rangle = 1$ and $\langle\overline{\Lambda}^{\mathfrak{c}}_0, E_i\rangle = \langle\overline{\Lambda}^{\mathfrak{c}}_0, E_{\bar{j}}\rangle = 0$, for all $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$. Further for $\lambda\in\mathcal{P}(\mathrm{Sp}(d))$ with $\lambda_{m+1}\leqslant n$, we define

$$\widehat{\Lambda}^{\mathfrak{c}}_f(\lambda) := \frac{d}{2}\widetilde{\Lambda}^{\mathfrak{c}}_0 + \sum_{i=1}^m \lambda_i\varepsilon_i + \sum_{j=1}^n\langle\lambda'_j - m\rangle\delta_j,$$

where $\widetilde{\Lambda}^{\mathfrak{c}}_0 = \overline{\Lambda}^{\mathfrak{c}}_0 + \sum_{i=1}^m\varepsilon_i - \sum_{j=1}^n\delta_j$. Then $\{\varepsilon_i, \delta_j, \widetilde{\Lambda}^{\mathfrak{c}}_0\}$ is the basis dual to $\{\widehat{E}_i, \widehat{E}_{\bar{j}}, K\}$.

Theorem 3.4. *For $\lambda\in\mathcal{P}(\mathrm{Sp}(d))$ with $\lambda_{m+1}\leqslant n$, we have*

$$\operatorname{ch} L\big(\widehat{\mathfrak{osp}}(2m|2n), \widehat{\Lambda}^{\mathfrak{c}}_f(\lambda)\big) = \frac{\sum_{k=0}^{\infty}(-1)^k\sum_{w\in W^0_k(\mathfrak{c})} hs_{\lambda_w}(\eta, x)}{\prod_{1\leqslant i<j\leqslant m}\prod_{1\leqslant s\leqslant t\leqslant n}(1-\eta_i\eta_j)(1-x_sx_t)(1+\eta_ix_s)^{-1}},$$

where $\eta := \{\eta_1,\dots,\eta_n\}$, $x := \{x_1,\dots,x_m\}$.

Proof. Computing the trace of the operator $\prod_{i,j}\eta_i^{\widehat{E}_i}x_j^{\widehat{E}_{\bar{j}}}\prod_{k=1}^{\frac{d}{2}}z_k^{e_k}$ on both sides of the isomorphism in Proposition 2.7, where $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$, we obtain

$$\prod_{k=1}^{\frac{d}{2}}\prod_{i,j}\frac{(1+\eta_jz_k^{-1})(1+\eta_jz_k)}{(1-x_iz_k^{-1})(1-x_iz_k)} = \sum_{\substack{\lambda\in\mathcal{P}(\mathrm{Sp}(d))\\ \lambda_{m+1}\leqslant n}} \operatorname{ch} L\big(\widehat{\mathfrak{osp}}(2m|2n), \widehat{\Lambda}^{\mathfrak{c}}_f(\lambda)\big)\operatorname{ch} V^{\lambda}_{\mathrm{Sp}(d)}. \quad (3.7)$$

Replacing $\operatorname{ch} L(\mathfrak{g}, \Lambda^{\mathfrak{c}}(\lambda))$ in (2.5) by the expression in Proposition 2.5, we obtain an identity of symmetric functions in variables $x_1, x_2, \dots$. We apply to this identity the involution on the ring of symmetric functions in $x_1, x_2, \ldots$. Next, we replace x_i by η_i, $i = 1, \ldots, n$, and then x_{i+n} by x_i, $i \in \mathbb{N}$. Finally, we put $x_i = 0$ for $i \geqslant n+1$. Under the composition of those maps, we obtain a new identity which shares the same left-hand side as (3.7). Comparing the right-hand sides of this new identity and of (3.7), we obtain the result thanks to the linear independence of $\{\mathrm{ch}\, V^{\lambda}_{\mathrm{Sp}(d)} \mid \lambda \in \mathcal{P}(\mathrm{Sp}(d))\}$. □

We denote by $\omega^{\mathfrak{c}}$ the map which sends $\mathrm{ch}\, L(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}(\lambda))$ to $\mathrm{ch}\, L(\widehat{\mathfrak{osp}}(2m|2n), \widehat{\Lambda}^{\mathfrak{c}}_f(\lambda))$ and extend $\omega^{\mathfrak{c}}$ by linearity (compare (2.5) and (3.7)). Note that $\omega^{\mathfrak{c}}$ respects the multiplication of characters.

3.4. $\widehat{\mathfrak{spo}}(2m|2n)$

Recall the central extension $\widehat{\mathfrak{spo}}(2m|2n)$ induced from $\widehat{\mathfrak{gl}}(2m|2n)$ determined by $\mathfrak{J}'$ in (3.2). Introduce an element K that acts on $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ as the scalar $\frac{d}{2}$ and commutes with the actions of $\mathfrak{spo}(2m|2n)$ and $\mathrm{O}(d)$. Putting $\widehat{A} := A - \mathrm{Str}(\mathfrak{J}'A)K \in \mathfrak{spo}(2m|2n) \oplus \mathbb{C}K$, for $A \in \mathfrak{spo}(2m|2n)$, defines an action of $\widehat{\mathfrak{spo}}(2m|2n)$ on $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$. Then $\widehat{\mathfrak{spo}}(2m|2n)$ and $\mathrm{O}(d)$ form a reductive dual pair. Define an element $\overline{\Lambda}^{\mathfrak{d}}_0 \in \overline{\mathfrak{h}}^*$ by $\langle \overline{\Lambda}^{\mathfrak{d}}_0, K\rangle = 1$ and $\langle \overline{\Lambda}^{\mathfrak{d}}_0, E_i\rangle = \langle \overline{\Lambda}^{\mathfrak{d}}_0, E_{\bar{j}}\rangle = 0$, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

When $d = 2\ell$, calculating the trace of the operator $\prod_{i,j} \eta_i^{\widehat{E}_i} x_j^{\widehat{E}_{\bar{j}}} \prod_{k=1}^{\ell} z_k^{e_k}$ on both sides of the isomorphism in Proposition 2.8, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, gives

$$\prod_{k=1}^{\ell} \prod_{i,j} \frac{(1+\eta_j z_k^{-1})(1+\eta_j z_k)}{(1-x_i z_k^{-1})(1-x_i z_k)} = \sum_{\substack{\lambda \in \mathcal{P}(\mathrm{O}(2\ell)) \\ \lambda_{m+1} \leqslant n}} \mathrm{ch}\, L\big(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda)\big)\, \mathrm{ch}\, V^{\lambda}_{\mathrm{O}(2\ell)}, \tag{3.8}$$

and when $d = 2\ell+1$, the trace of the operator $\prod_{i,j} \eta_i^{\widehat{E}_i} x_j^{\widehat{E}_{\bar{j}}} \prod_{k=1}^{\ell} z_k^{e_k}(-I_d)$ gives

$$\prod_{k=1}^{\ell} \prod_{i,j} \frac{(1+\epsilon\eta_j z_k^{-1})(1+\epsilon\eta_j z_k)(1+\epsilon\eta_j)}{(1-\epsilon x_i z_k^{-1})(1-\epsilon x_i z_k)(1-\epsilon x_i)}$$
$$= \sum_{\substack{\lambda \in \mathcal{P}(\mathrm{O}(2\ell+1)) \\ \lambda_{m+1} \leqslant n}} \mathrm{ch}\, L\big(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda)\big)\, \mathrm{ch}\, V^{\lambda}_{\mathrm{O}(2\ell+1)}. \tag{3.9}$$

Here for $\lambda \in \mathcal{P}(\mathrm{O}(d))$ with $\lambda_{m+1} \leqslant n$,

$$\widehat{\Lambda}^{\mathfrak{d}}_f(\lambda) := \frac{d}{2}\widetilde{\Lambda}^{\mathfrak{d}}_0 + \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \langle \lambda'_j - m\rangle \delta_j,$$

with $\widetilde{\Lambda}^{\mathfrak{d}}_0 = \overline{\Lambda}^{\mathfrak{d}}_0 + \sum_{i=1}^{m} \varepsilon_i - \sum_{j=1}^{n} \delta_j$.

Now the next theorem follows by similar arguments as for Theorem 3.4 using (3.8), (3.9), (2.7) and (2.8) and Proposition 2.5.

Theorem 3.5. *Put $\eta := \{\eta_1, \ldots, \eta_m\}$ and $x := \{x_1, \ldots, x_n\}$. Let $\lambda \in \mathcal{P}(\mathrm{O}(d))$ with $\lambda_{m+1} \leqslant n$, be given.*

(1) *If $d = 2\ell + 1$, then we have*

$$\operatorname{ch} L\big(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda)\big) = \frac{\sum_{k=0}^{\infty}(-1)^k \sum_{w\in W^0_k(\mathfrak{d})} hs_{\lambda_w}(\eta, x)}{\prod_{1\leqslant i\leqslant j\leqslant m}\prod_{1\leqslant s<t\leqslant n}(1-\eta_i\eta_j)(1-x_s x_t)(1+\eta_i x_s)^{-1}}.$$

(2) *If $d = 2\ell$, then we have*

$$\operatorname{ch} L\big(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda)\big) + \operatorname{ch} L\big(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\tilde{\lambda})\big)$$
$$= \frac{\sum_{k=0}^{\infty}(-1)^k \sum_{w\in W^0_k(\mathfrak{d})}[hs_{\lambda_w}(\eta, x) + hs_{\tilde{\lambda}_w}(\eta, x)]}{\prod_{1\leqslant i\leqslant j\leqslant m}\prod_{1\leqslant s<t\leqslant n}(1-\eta_i\eta_j)(1-x_s x_t)(1+\eta_i x_s)^{-1}}.$$

Remark 3.6. Theorem 3.5(2) gives the character of the sum of two irreducible modules. However, in the case when $\lambda'_1 = \ell$, we have $\tilde{\lambda} = \lambda$, and hence $\lambda_w = \tilde{\lambda}_w$. Thus in this case it actually gives the character of one irreducible module, i.e. we have an identity as in (1). Similar remark applies in the sequel as well when dealing with $G = \mathrm{O}(2\ell)$ and $\lambda = \tilde{\lambda}$. In particular it applies to the character formulas of Kostant homology groups in Theorems 5.7 and 6.5, and also to (5.1).

Character formulas in forms different from Theorems 3.4 and 3.5 were also obtained in [11, Theorems 6.2, 6.3] using [14, Theorem 2.2]. Our approach uses Howe dualities involving infinite-dimensional Lie algebras, thus bypassing [14].

Introduce a map $\omega^{\mathfrak{d}}$ such that $\omega^{\mathfrak{d}}(\operatorname{ch} L(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\lambda))) = \operatorname{ch} L(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda))$ if $d = 2\ell + 1$, and $\omega^{\mathfrak{d}}[\operatorname{ch} L(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\lambda)) + \operatorname{ch} L(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}(\tilde{\lambda}))] = \operatorname{ch} L(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\lambda)) + \operatorname{ch} L(\widehat{\mathfrak{spo}}(2m|2n), \widehat{\Lambda}^{\mathfrak{d}}_f(\tilde{\lambda}))$ if $d = 2\ell$. Extended by linearity, $\omega^{\mathfrak{d}}$ sends either side of (2.7) and (2.8) to the corresponding side of (3.8) and (3.9), respectively.

3.5. $\widehat{\mathfrak{spo}}(2m|2n+1)$

Recall that $\widehat{\mathfrak{spo}}(2m|2n+1)$ is the central extension induced from $\widehat{\mathfrak{gl}}(2m|2n+1)$ determined by $\mathfrak{J}''$ in (3.3). Let K act on $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$ as the scalar $\frac{d}{2}$ and commute with the actions of $\mathfrak{spo}(2m|2n+1)$ and $\mathrm{Pin}(d)$. Putting $\widehat{A} := A - \mathrm{Str}(\mathfrak{J}''A)K \in \mathfrak{spo}(2m|2n+1) \oplus \mathbb{C}K$, for $A \in \mathfrak{spo}(2m|2n+1)$, defines an action of $\widehat{\mathfrak{spo}}(2m|2n+1)$ on $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$. Then $\widehat{\mathfrak{spo}}(2m|2n+1)$ and $\mathrm{Pin}(d)$ form a reductive dual pair. Define an element $\overline{\Lambda}^{\mathfrak{b}}_0 \in \bar{\mathfrak{h}}^*$ by $\langle \overline{\Lambda}^{\mathfrak{b}}_0, K\rangle = 1$ and $\langle \overline{\Lambda}^{\mathfrak{b}}_0, E_i\rangle = \langle \overline{\Lambda}^{\mathfrak{b}}_0, E_{\bar{j}}\rangle = 0$, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Computing the trace of the operator $\prod_{i,j} \eta_i^{\widehat{E}_i} x_j^{\widehat{E}_{\bar{j}}} \prod_{k=1}^{\ell} z_k^{e_k}$ on both sides of the isomorphism in Proposition 2.9 gives

$$\prod_{k=1}^{\ell}\prod_{i,j}\big(z_k^{\frac{1}{2}} + z_k^{-\frac{1}{2}}\big)\frac{(1+\eta_j z_k^{-1})(1+\eta_j z_k)}{(1-x_i z_k^{-1})(1-x_i z_k)}$$
$$= \sum_{\substack{\lambda\in\mathcal{P}(\mathrm{Pin}(2\ell))\\ \lambda_{m+1}\leqslant n}} \operatorname{ch} L\big(\widehat{\mathfrak{spo}}(2m|2n+1), \widehat{\Lambda}^{\mathfrak{b}}_f(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{Pin}(2\ell)}, \tag{3.10}$$

where

$$\widehat{\Lambda}^{\mathfrak{b}}_f(\lambda) := \frac{d}{2}\widetilde{\Lambda}^{\mathfrak{b}}_0 + \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \langle \lambda'_j - m \rangle \delta_j,$$

with $\widetilde{\Lambda}^{\mathfrak{b}}_0 = \overline{\Lambda}^{\mathfrak{b}}_0 + \sum_{i=1}^{m} \varepsilon_i - \sum_{j=1}^{n} \delta_j$.

Now the next theorem follows from similar arguments as for Theorem 3.4 using (3.10), (2.11), and Proposition 2.5.

Theorem 3.7. *Put $\eta := \{\eta_1, \ldots, \eta_m\}$ and $x := \{x_1, \ldots, x_n\}$. For $\lambda \in \mathcal{P}(\mathrm{Pin}(d))$ with $\lambda_{m+1} \leqslant n$, we have*

$$\begin{aligned}&\operatorname{ch} L\big(\widehat{\mathfrak{spo}}(2m|2n+1), \widehat{\Lambda}^{\mathfrak{b}}_f(\lambda)\big) \\ &\quad = \frac{\sum_{k=0}^{\infty} (-1)^k \sum_{w \in W^0_k(\mathfrak{b})} hs_{\lambda_w}(\eta, x)}{\prod_{1 \leqslant i \leqslant j \leqslant m} \prod_{1 \leqslant s < t \leqslant n} (1 - \eta_i \eta_j)(1 - x_s x_t)(1 - \eta_j)(1 + \eta_i x_s)^{-1}(1 + x_s)^{-1}}.\end{aligned}$$

We denote by $\omega^{\mathfrak{b}}$ the map which sends $\operatorname{ch} L(\mathfrak{b}_\infty, \Lambda^{\mathfrak{b}}(\lambda))$ to $\operatorname{ch} L(\widehat{\mathfrak{osp}}(2m|2n+1), \widehat{\Lambda}^{\mathfrak{b}}_f(\lambda))$ and we then extend $\omega^{\mathfrak{b}}$ by linearity.

4. The bilinear forms and Casimir operators for Lie (super)algebras

4.1. The bilinear form $(\cdot|\cdot)_c$ and the Casimir operator Ω of $\mathfrak{g}$

Recall the symmetric bilinear form $(\cdot|\cdot)_c$ defined on $\mathfrak{h}^*$ in [4, Section 4.1], which induces a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, also denoted by $(\cdot|\cdot)_c$. Using this one defines a Casimir operator Ω which commutes with the action of $\mathfrak{g}$ on a highest weight $\mathfrak{g}$-module V of highest weight λ, and which acts as the scalar $(\lambda + 2\rho_c|\lambda)_c$ on V. The details here are completely analogous to Section 4.2 below, and we refer to [4, Section 4.1] for more detail.

4.2. The bilinear form $(\cdot|\cdot)_s$ and the Casimir operator $\overline{\Omega}$ of $\bar{\mathfrak{g}}$

Suppose first that $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n)$. Note that $\{\varepsilon_i, \delta_j\} \cup \{\widetilde{\Lambda}^{\mathfrak{a}}_0\}$ is the basis of $\bar{\mathfrak{h}}^*$ dual to the basis $\{\widehat{E}_{kk}\} \cup \{K\}$ of $\bar{\mathfrak{h}}$. Set

$$\rho_s := \sum_{i=-p}^{-1} (-i - q)\varepsilon_i + \sum_{i=1}^{m} (1 - i)\varepsilon_i + \sum_{j=-q}^{-1} (-j - 1)\delta_j + \sum_{j=1}^{n} (m - j)\delta_j. \tag{4.1}$$

We choose a symmetric bilinear form $(\cdot|\cdot)_s$ on $\bar{\mathfrak{h}}^*$ satisfying

$$\begin{aligned}(\lambda|\varepsilon_i)_s &= -\left\langle \lambda, E_{ii} + \frac{K}{2} \right\rangle, \quad i \in \{-p, \ldots, -1, 1, \ldots, m\}, \\ (\lambda|\delta_j)_s &= \left\langle \lambda, E_{\bar{j}\bar{j}} - \frac{K}{2} \right\rangle, \quad j \in \{-q, \ldots, -1, 1, \ldots, n\}.\end{aligned}$$

We check easily that

$$(\varepsilon_i|\varepsilon_j)_s = -\delta_{ij}, \qquad (\varepsilon_i|\delta_j)_s = 0, \qquad (\delta_i|\delta_j)_s = \delta_{ij}.$$

Also we have $(\widetilde{\Lambda}^{\mathfrak{a}}_0|\varepsilon_{-i})_s = (\widetilde{\Lambda}^{\mathfrak{a}}_0|\delta_{-j})_s = -(\widetilde{\Lambda}^{\mathfrak{a}}_0|\varepsilon_i)_s = -(\widetilde{\Lambda}^{\mathfrak{a}}_0|\delta_j)_s = \frac{1}{2}$, for $i, j > 0$. Furthermore, recalling the simple roots β_k for $\bar{\mathfrak{g}}$, we have

$$(\rho_s|\beta_k)_s = \frac{1}{2}(\beta_k|\beta_k)_s, \quad k \in \overline{I}.$$

Next, suppose that $\bar{\mathfrak{g}}$ is $\widehat{\mathfrak{spo}}(2m|2n+1)$, $\widehat{\mathfrak{osp}}(2m|2n)$ or $\widehat{\mathfrak{spo}}(2m|2n)$. Note that $\{\varepsilon_i, \delta_j\} \cup \{\widetilde{\Lambda}^{\mathfrak{x}}_0\}$ is the basis of $\bar{\mathfrak{h}}^*$ dual to $\{\widehat{E}_i, \widehat{E}_{\bar{j}}\} \cup \{K\}$, $\mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$. We set

$$\rho_s := \begin{cases} \sum_{i=1}^m (-i + \frac{1}{2})\varepsilon_i + \sum_{j=1}^n (m - j + \frac{1}{2})\delta_j, & \text{for } \mathfrak{x} = \mathfrak{b}, \\ \sum_{i=1}^m (1 - i)\varepsilon_i + \sum_{j=1}^n (m - j)\delta_j, & \text{for } \mathfrak{x} = \mathfrak{c}, \\ \sum_{i=1}^m -i\varepsilon_i + \sum_{j=1}^n (m - j + 1)\delta_j, & \text{for } \mathfrak{x} = \mathfrak{d}. \end{cases} \tag{4.2}$$

We choose a symmetric bilinear form $(\cdot|\cdot)_s$ on $\bar{\mathfrak{h}}^*$ satisfying for $\lambda \in \bar{\mathfrak{h}}^*$

$$(\lambda|\varepsilon_i)_s = \langle \lambda, E_i \rangle, \quad i \in \{1, \dots, m\},$$
$$(\lambda|\delta_j)_s = -\langle \lambda, E_{\bar{j}} \rangle, \quad j \in \{1, \dots, n\}.$$

Similarly, we can check that $(\varepsilon_i|\varepsilon_j)_s = \delta_{ij}$, $(\varepsilon_i|\delta_j)_s = 0$, $(\delta_i|\delta_j)_s = -\delta_{ij}$. Also we have $(\widetilde{\Lambda}^{\mathfrak{x}}_0|\varepsilon_i)_s = (\widetilde{\Lambda}^{\mathfrak{x}}_0|\delta_j)_s = 1$, and $(\rho_s|\beta_k)_s = \frac{1}{2}(\beta_k|\beta_k)_s$ for $k \in \overline{I}$.

For $i \in \overline{I}$ define

$$\bar{s}^{\mathfrak{a}}_i := \begin{cases} -1, & \text{if } i \in \{-p+1, \dots, -1, -\bar{q}, 1, \dots, m\}, \\ 1, & \text{if } i \in \{-\overline{q-1}, \dots, -\bar{1}, 0, \bar{1}, \dots, \overline{n-1}\}. \end{cases}$$

$$\bar{s}^{\mathfrak{x}}_i := \begin{cases} -1, & \text{if } i = 0 \text{ and } \mathfrak{x} = \mathfrak{b}, \\ 1, & \text{if } i = 0 \text{ and } \mathfrak{x} = \mathfrak{c}, \\ 2, & \text{if } i = 0 \text{ and } \mathfrak{x} = \mathfrak{d}, \\ 1, & \text{if } i \in \{1, \dots, m\} \text{ and } \mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}, \\ -1, & \text{if } i \in \{\bar{1}, \dots, \overline{n-1}\} \text{ and } \mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}. \end{cases}$$

Then we have

$$(\lambda|\beta_i)_s = \bar{s}^{\mathfrak{x}}_i \langle \lambda, \beta_i^\vee \rangle, \quad i \in \overline{I},\ \lambda \in \bar{\mathfrak{h}}^*.$$

By defining $(\beta_i^\vee|\beta_j^\vee)_s := (\bar{s}^{\mathfrak{x}}_i \bar{s}^{\mathfrak{x}}_j)^{-1}(\beta_i|\beta_j)_s$, we obtain a symmetric bilinear form on the Cartan subalgebra of $\bar{\mathfrak{g}}' = [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]$, which can be extended to a non-degenerate invariant supersymmetric bilinear form on $\bar{\mathfrak{g}}'$ such that

$$(\bar{e}_i|\overline{f}_j)_s = \delta_{ij}/\bar{s}^{\mathfrak{x}}_i,$$

where $\bar{e}_i$ and $\overline{f}_i$ denote the Chevalley generators of $\bar{\mathfrak{g}}'$ with $[\bar{e}_i, \overline{f}_i] = \beta_i^\vee$.

Let $\bar{\mathfrak{g}}_\beta$ be the root space of $\beta \in \overline{\Delta}^{\pm}$. Take $u_\beta \in \bar{\mathfrak{g}}_\beta$ and $u^\beta \in \bar{\mathfrak{g}}_{-\beta}$ for $\beta \in \overline{\Delta}^{+}$ such that $(u_\alpha|u^\beta)_s = \delta_{\alpha\beta}$. For any highest weight $\bar{\mathfrak{g}}$-module V, with weight space decomposition $V = \bigoplus_\mu V_\mu$, we define $\overline{\Gamma}_1 : V \to V$ to be the linear map that acts as the scalar $(\mu + 2\rho_s|\mu)_s$ on V_μ. Let $\overline{\Gamma}_2 := 2\sum_{\beta\in\overline{\Delta}^+} u^\beta u_\beta$. Define the Casimir operator to be

$$\overline{\Omega} := \overline{\Gamma}_1 + \overline{\Gamma}_2. \tag{4.3}$$

It is straightforward to check the following.

Proposition 4.1. *The operator $\overline{\Omega}$ commutes with the action of $\bar{\mathfrak{g}}$ on a highest weight module V with highest weight λ, and acts on V as the scalar $(\lambda + 2\rho_s|\lambda)_s$.*

4.3. The sets of weights $\mathcal{P}^+_{\mathfrak{l}}$ and $\mathcal{P}^{++}_{\mathfrak{l}}$ in $\mathfrak{h}^*$

Recall the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with simple roots indexed by S. Let $c \in \mathbb{C}$.

For $\mathfrak{g} = \mathfrak{a}_\infty$, let $\mathcal{P}^+_{\mathfrak{l},c}$ consist of $\mu \in \mathfrak{h}^*$ of the following form:

$$\mu = c\Lambda_0^{\mathfrak{a}} + \sum_{i\geqslant 1} \eta_i \epsilon_i - \sum_{j\geqslant 0} \zeta_j \epsilon_{-j}, \tag{4.4}$$

where $\eta = (\eta_1, \eta_2, \ldots)$ and $\zeta = (\zeta_0, \zeta_1, \ldots) \in \mathcal{P}^+$. (Note the shift of index for ζ.) We denote by $\mathcal{P}^{++}_{\mathfrak{l},c}$ the subset of $\mathcal{P}^+_{\mathfrak{l},c}$ which consists of μ above with η and ζ being $(n|m)$- and $(q|p)$-hook partitions, respectively.

For $\mathfrak{g} = \mathfrak{b}_\infty, \mathfrak{c}_\infty, \mathfrak{d}_\infty$, let $\mathcal{P}^+_{\mathfrak{l},c}$ consist of $\mu \in \mathfrak{h}^*$ of the following form:

$$\mu = c\Lambda_0^{\mathfrak{x}} + \sum_{i\geqslant 1} \mu_i \epsilon_i, \tag{4.5}$$

where $(\mu_1, \mu_2, \ldots) \in \mathcal{P}^+$. Denote by $\mathcal{P}^{++}_{\mathfrak{l},c}$ the subset of $\mathcal{P}^+_{\mathfrak{l},c}$ of the form above with μ being an $(n|m)$-hook partition. We put

$$\mathcal{P}^+_{\mathfrak{l}} := \bigsqcup_{c\in\mathbb{C}} \mathcal{P}^+_{\mathfrak{l},c}, \qquad \mathcal{P}^{++}_{\mathfrak{l}} := \bigsqcup_{c\in\mathbb{C}} \mathcal{P}^{++}_{\mathfrak{l},c}.$$

4.4. The sets of weights $\mathcal{P}^{++}_{\bar{\mathfrak{l}}}$ in $\bar{\mathfrak{h}}^*$

Let $\overline{\Delta} := \overline{\Delta}^+ \cup \overline{\Delta}^-$ be the set of roots of $\bar{\mathfrak{g}}$, where $\overline{\Delta}^- = -\overline{\Delta}^+$. Put

$$S := \bar{I} \setminus \{0\}.$$

Let $\overline{\Delta}^{\pm}_S := \overline{\Delta}^{\pm} \cap (\sum_{r\in S} \mathbb{Z}\beta_r)$ and $\overline{\Delta}^{\pm}(S) := \overline{\Delta}^{\pm} \setminus \overline{\Delta}^{\pm}_S$. Let

$$\bar{\mathfrak{u}}_{\pm} := \sum_{\beta\in\overline{\Delta}^{\pm}(S)} \bar{\mathfrak{g}}_\beta, \qquad \bar{\mathfrak{l}} := \sum_{\beta\in\overline{\Delta}^+_S\cup\overline{\Delta}^-_S} \bar{\mathfrak{g}}_\beta \oplus \bar{\mathfrak{h}}$$

so that $\bar{\mathfrak{g}} = \bar{\mathfrak{u}}_+ \oplus \bar{\mathfrak{l}} \oplus \bar{\mathfrak{u}}_-$. If $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n)$, then $\bar{\mathfrak{l}} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(m|n) \oplus \mathbb{C}K$, and $\bar{\mathfrak{l}} = \mathfrak{gl}(m|n) \oplus \mathbb{C}K$, otherwise. The Lie superalgebras $\bar{\mathfrak{l}}$ and $\bar{\mathfrak{g}}$ share the same Cartan subalgebra $\bar{\mathfrak{h}}$.

Let $c \in \mathbb{C}$. For $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n)$, let $\mathcal{P}^{++}_{\bar{\mathfrak{l}},c}$ consist of $\overline{\mu} \in \bar{\mathfrak{h}}^*$ of the following form:

$$\overline{\mu} = c\widetilde{\Lambda}^{\mathfrak{a}}_0 + \tau_1\varepsilon_1 + \cdots + \tau_m\varepsilon_m + \langle \tau'_1 - m\rangle\delta_1 + \cdots + \langle \tau'_n - m\rangle\delta_n$$
$$- \big(\xi_1\delta_{-1} + \cdots + \xi_q\delta_{-q} + \langle \xi'_1 - q\rangle\varepsilon_{-1} + \cdots + \langle \xi'_p - q\rangle\varepsilon_{-p}\big),$$

where $\tau = (\tau_1, \tau_2, \ldots)$ and $\xi = (\xi_1, \xi_2, \ldots)$ are $(m|n)$- and $(q|p)$-hook partitions, respectively.

When $\bar{\mathfrak{g}}$ is $\widehat{\mathfrak{spo}}(2m|2n+1)$, $\widehat{\mathfrak{osp}}(2m|2n)$ or $\widehat{\mathfrak{spo}}(2m|2n)$, we let $\mathcal{P}^{++}_{\bar{\mathfrak{l}},c}$ consist of $\overline{\mu} \in \bar{\mathfrak{h}}^*$ of the following form:

$$\overline{\mu} = c\widetilde{\Lambda}^{\mathfrak{x}}_0 + \tau_1\varepsilon_1 + \cdots + \tau_m\varepsilon_m + \langle \tau'_1 - m\rangle\delta_1 + \cdots + \langle \tau'_n - m\rangle\delta_n,$$

where $\tau = (\tau_1, \tau_2, \ldots)$ runs over all $(m|n)$-hook partitions. We put

$$\mathcal{P}^{++}_{\bar{\mathfrak{l}}} := \bigsqcup_{c\in\mathbb{C}} \mathcal{P}^{++}_{\bar{\mathfrak{l}},c}.$$

4.5. *Casimir eigenvalues of $\widehat{\mathfrak{gl}}(p+m|q+n)$ versus those of $\widehat{\mathfrak{gl}}_\infty$*

Let $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n)$. Let $\mu \in \mathcal{P}^{++}_{\mathfrak{l},c}$ be as in (4.4). Set $\tau = \eta'$, $\nu = (\nu_1, \nu_2, \ldots) := (\tau_{m+1}, \tau_{m+2}, \ldots)$ and $\chi = (\chi_1, \chi_2, \ldots) := (\zeta_q, \zeta_{q+1}, \ldots)$. Define a bijection

$$\vartheta : \mathcal{P}^{++}_{\mathfrak{l},c} \to \mathcal{P}^{++}_{\bar{\mathfrak{l}},c}$$

by

$$\vartheta(\mu) := c\widetilde{\Lambda}^{\mathfrak{a}}_0 + \tau_1\varepsilon_1 + \cdots + \tau_m\varepsilon_m + \nu'_1\delta_1 + \cdots + \nu'_n\delta_n$$
$$- \big(\zeta_0\delta_{-1} + \cdots + \zeta_{q-1}\delta_{-q} + \chi'_1\varepsilon_{-1} + \cdots + \chi'_p\varepsilon_{-p}\big).$$

Note that

$$\vartheta\big(\Lambda^{\mathfrak{a}}(\lambda)\big) = \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda), \quad \text{for } \lambda \in \mathcal{P}\big(\mathrm{GL}(d)\big) \text{ with } \lambda_{m+1} \leqslant n \text{ and } \lambda_{d-p} \geqslant -q. \tag{4.6}$$

For η and ζ as above it is convenient to introduce the following symbols:

$$(\eta + 2\rho_1|\eta)_1 := \sum_{i\geqslant 1} \eta_i(\eta_i - 2i),$$
$$(\zeta + 2\rho_2|\zeta)_2 := \sum_{j\geqslant 0} \zeta_j(\zeta_j - 2j).$$

By [4, Lemma 7.1] or [28, (1.7)], for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}^+$, we have

$$\sum_{i\geqslant 1} \lambda_i(\lambda_i - 2i) = -\sum_{i\geqslant 1} \lambda'_i\big(\lambda'_i - 2(i-1)\big). \tag{4.7}$$

Lemma 4.2. *For $\mu \in \mathcal{P}^{++}_{\mathfrak{l},c}$, we have*

$$\big(\vartheta(\mu) + 2\rho_s \big| \vartheta(\mu)\big)_s = (\mu + 2\rho_c|\mu)_c + \overline{C}, \tag{4.8}$$

where

$$\overline{C} = c^2\big(\widetilde{\Lambda}^{\mathfrak{a}}_0 \big| \widetilde{\Lambda}^{\mathfrak{a}}_0\big)_s + 2c\big(\rho_s \big| \widetilde{\Lambda}^{\mathfrak{a}}_0\big)_s.$$

In particular, for $\lambda \in \mathcal{P}(\mathrm{GL}(d))$ with $\lambda_{m+1} \leqslant n$ and $\lambda_{d-p} \geqslant -q$, $(\Lambda^{\mathfrak{a}}(\lambda) + 2\rho_c|\Lambda^{\mathfrak{a}}(\lambda))_c = (\mu + 2\rho_c|\mu)_c$ if and only if $(\widehat{\Lambda}^{\mathfrak{a}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{a}}_f(\lambda))_s = (\vartheta(\mu) + 2\rho_s|\vartheta(\mu))_s$.

Proof. Let $\mu \in \mathcal{P}^{++}_{\mathfrak{l},c}$ be as in (4.4). Using (4.7), we have

$$\begin{aligned}(\mu + 2\rho_c|\mu)_c &= (\eta + 2\rho_1|\eta)_1 + (\zeta + 2\rho_2|\zeta)_2 - c\big(|\eta| + |\zeta|\big) \\ &= -(\tau + 2\rho_2|\tau)_2 + (\zeta + 2\rho_2|\zeta)_2 - c\big(|\eta| + |\zeta|\big).\end{aligned}$$

For convenience, put $\eta = \sum_{i \geqslant 1} \eta_i \epsilon_i$ and $\zeta = \sum_{j \geqslant 0} \zeta_j \epsilon_{-j}$. Now, we have

$$\begin{aligned}&\big(\vartheta(\mu) + 2\rho_s \big| \vartheta(\mu)\big)_s \\ &\quad = \big(c\widetilde{\Lambda}^{\mathfrak{a}}_0 + \vartheta(\eta) - \vartheta(\zeta) + 2\rho_s \big| c\widetilde{\Lambda}^{\mathfrak{a}}_0 + \vartheta(\eta) - \vartheta(\zeta)\big)_s \\ &\quad = c^2\big(\widetilde{\Lambda}^{\mathfrak{a}}_0 \big| \widetilde{\Lambda}^{\mathfrak{a}}_0\big)_s + 2c\big(\rho_s \big| \widetilde{\Lambda}^{\mathfrak{a}}_0\big)_s + \big(\vartheta(\eta) - \vartheta(\zeta) + 2\rho_s \big| \vartheta(\eta) - \vartheta(\zeta)\big)_s \\ &\qquad + 2c\big(\widetilde{\Lambda}^{\mathfrak{a}}_0 \big| \vartheta(\eta) - \vartheta(\zeta)\big)_s \\ &\quad = \overline{C} + \big(\vartheta(\eta) + 2\rho_s \big| \vartheta(\eta)\big)_s + \big(\vartheta(\zeta) - 2\rho_s \big| \vartheta(\zeta)\big)_s - c\big(|\eta| + |\zeta|\big).\end{aligned}$$

First, we have

$$\begin{aligned}&\big(\vartheta(\eta) + 2\rho_s \big| \vartheta(\eta)\big)_s \\ &\quad = -\big[\tau_1(\tau_1 - 2\cdot 0) + \cdots + \tau_m\big(\tau_m - 2(m-1)\big)\big] \\ &\qquad + \big[\nu'_1\big(\nu'_1 + 2(m-1)\big) + \cdots + \nu'_n\big(\nu'_n + 2(m-n)\big)\big] \\ &\quad = -\big[\tau_1(\tau_1 - 2\cdot 0) + \cdots + \tau_m\big(\tau_m - 2(m-1)\big)\big] \\ &\qquad + \big[\nu'_1\big(\nu'_1 - 2\cdot 1\big) + \cdots + \nu'_n\big(\nu'_n - 2n\big)\big] + 2m\sum_{k=1}^{n} \nu'_k \\ &\quad = -\big[\tau_1(\tau_1 - 2\cdot 0) + \cdots + \tau_m\big(\tau_m - 2(m-1)\big)\big] \\ &\qquad - \big[\nu_1(\nu_1 - 2\cdot 0) + \nu_2(\nu_2 - 2\cdot 1) + \cdots\big] + 2m\sum_{k \geqslant 1} \nu_k \\ &\quad = -\big[\tau_1(\tau_1 - 2\cdot 0) + \cdots + \tau_m\big(\tau_m - 2(m-1)\big)\big] \\ &\qquad - \big[\tau_{m+1}(\tau_{m+1} - 2\cdot 0) + \tau_{m+2}(\tau_{m+2} - 2\cdot 1) + \cdots\big] + 2m\sum_{k \geqslant 1} \tau_{m+k} \\ &\quad = -(\tau + 2\rho_2|\tau)_2.\end{aligned} \tag{4.9}$$

Similarly, we have

$$
\begin{aligned}
&\big(\vartheta(\zeta)-2\rho_s\big|\vartheta(\zeta)\big)_s\\
&\quad=\big[\zeta_0(\zeta_0-2\cdot 0)+\cdots+\zeta_{q-1}\big(\zeta_{q-1}-2(q-1)\big)\big]\\
&\qquad-\big[\chi_1'\big(\chi_1'+2(q-1)\big)+\cdots+\chi_p'\big(\chi_p'+2(q-p)\big)\big]\\
&\quad=\big[\zeta_0(\zeta_0-2\cdot 0)+\cdots+\zeta_{q-1}\big(\zeta_{q-1}-2(q-1)\big)\big]\\
&\qquad-\big[\chi_1'\big(\chi_1'-2\cdot 1\big)+\cdots+\chi_p'\big(\chi_p'-2\cdot p\big)\big]-2q\sum_{k=1}^{p}\chi_k'\\
&\quad=\big[\zeta_0(\zeta_0-2\cdot 0)+\cdots+\zeta_{q-1}\big(\zeta_{q-1}-2(q-1)\big)\big]\\
&\qquad+\big[\chi_1(\chi_1-2\cdot 0)+\chi_2(\chi_2-2\cdot 1)+\cdots\big]-2q\sum_{k\geqslant 1}\chi_k\\
&\quad=\big[\zeta_0(\zeta_0-2\cdot 0)+\cdots+\zeta_{q-1}\big(\zeta_{q-1}-2(q-1)\big)\big]\\
&\qquad+\big[\zeta_q(\zeta_q-2\cdot 0)+\zeta_{q+1}(\zeta_{q+1}-2\cdot 1)+\cdots\big]-2q\sum_{k\geqslant 0}\zeta_{q+k}\\
&\quad=(\zeta+2\rho_2|\zeta)_2.
\end{aligned}
$$

This completes the proof of (4.8). The remaining part of the lemma follows from the definition of the bijection ϑ, (4.6), and (4.8). □

4.6. Casimir eigenvalues of $\widehat{\mathfrak{osp}}$ and $\widehat{\mathfrak{spo}}$ versus those of $\mathfrak{x}_\infty$

Suppose that $(\bar{\mathfrak{g}}, G)$ is $(\widehat{\mathfrak{spo}}(2m|2n+1), \mathrm{Pin}(d))$, $(\widehat{\mathfrak{osp}}(2m|2n), \mathrm{Sp}(d))$ or $(\widehat{\mathfrak{spo}}(2m|2n), \mathrm{O}(d))$. Let $\mu \in \mathcal{P}^{++}_{\mathfrak{l},c}$ be as in (4.5) with $\mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ and $c \in \mathbb{C}$. Then $\nu := (\nu_1, \nu_2, \ldots) = (\mu_1, \mu_2, \ldots)'$ is of $(m|n)$-hook shape. Set $\tau := (\tau_1, \tau_2, \ldots) := (\nu_{m+1}, \nu_{m+2}, \ldots)$. We define a bijection $\vartheta : \mathcal{P}^{++}_{\mathfrak{l},c} \to \mathcal{P}^{++}_{\bar{\mathfrak{l}},\bar{c}}$ by letting

$$
\vartheta(\mu) := \bar{c}\widetilde{\Lambda}^{\mathfrak{x}}_0 + \nu_1\varepsilon_1 + \cdots + \nu_m\varepsilon_m + \tau_1'\delta_1 + \cdots + \tau_n'\delta_n,
$$

where $\bar{c} = c\langle \Lambda^{\mathfrak{x}}_0, K\rangle$. Note that

$$
\vartheta\big(\Lambda^{\mathfrak{x}}(\lambda)\big) = \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda), \quad \text{for } \lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G).
$$

Lemma 4.3. *For $\mu \in \mathcal{P}^{++}_{\mathfrak{l},c}$, we have*

$$
\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s = -(\mu+2\rho_c|\mu)_c + \overline{C},
$$

where $\overline{C} = \bar{c}^2(\widetilde{\Lambda}^{\mathfrak{x}}_0|\widetilde{\Lambda}^{\mathfrak{x}}_0)_s + 2\bar{c}(\rho|\widetilde{\Lambda}^{\mathfrak{x}}_0)_s$. In particular, $(\Lambda^{\mathfrak{x}}(\lambda)+2\rho_c|\Lambda^{\mathfrak{x}}(\lambda))_c = (\mu+2\rho_c|\mu)_c$ if and only if $(\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)+2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s = (\vartheta(\mu)+2\rho_s|\vartheta(\mu))_s$, for $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$.

Proof. The equivalence of the identities follows easily once we establish the first identity. Let $\mu = c\Lambda_0^{\mathfrak{x}} + \sum_{i\geqslant 1}\mu_i\epsilon_i \in \mathcal{P}^{++}_{\mathfrak{l},c}$, and so $\mu^\circ := (\mu_1, \mu_2, \ldots)$ is of $(n|m)$-hook shape.

Case 1. $\bar{\mathfrak{g}} = \widehat{\mathfrak{spo}}(2m|2n+1)$. Let $(\cdot|\cdot)_c$ stand for the bilinear form on the dual Cartan subalgebra of $\mathfrak{b}_\infty$. We compute

$$(\mu + 2\rho_c|\mu)_c = \Big(\sum_{i\geqslant 1}\mu_i\epsilon_i + 2\rho_c\Big|\sum_{i\geqslant 1}\mu_i\epsilon_i\Big)_c + 2c\Big(\sum_{i\geqslant 1}\mu_i\epsilon_i\Big|\Lambda_0^{\mathfrak{b}}\Big)_c$$
$$= \sum_{i\geqslant 1}\mu_i(\mu_i - 2i + 1) - c|\mu^\circ| = (\mu^\circ + 2\rho_1|\mu^\circ)_1 - (c-1)|\mu^\circ|.$$

On the other hand we have

$$\begin{aligned}
&\big(\vartheta(\mu) + 2\rho_s\big|\vartheta(\mu)\big)_s \\
&= \overline{C} + c\Big(\sum_{i=1}^m \nu_i\varepsilon_i + \sum_{j=1}^n \tau'_j\delta_j\Big|\widetilde{\Lambda}_0^{\mathfrak{b}}\Big)_s \\
&\quad + \Big(\sum_{i=1}^m \nu_i\varepsilon_i + \sum_{j=1}^n \tau'_j\delta_j + 2\rho_s\Big|\sum_{i=1}^m \nu_i\varepsilon_i + \sum_{j=1}^n \tau'_j\delta_j\Big)_s \\
&= \overline{C} + c|\mu^\circ| + \Big(\sum_{i=1}^m(\nu_i - 2i + 1)\varepsilon_i + \sum_{j=1}^n\big(\tau'_j + 2m - 2i + 1\big)\delta_j\Big|\sum_{i=1}^m \nu_i\varepsilon_i + \sum_{j=1}^n \tau'_j\delta_j\Big)_s \\
&= \overline{C} + c|\mu^\circ| + \sum_{i=1}^m \nu_i(\nu_i - 2i + 1) - \sum_{j=1}^n \tau'_j\big(\tau'_j + 2m - 2j + 1\big) \\
&= \overline{C} + (c-1)|\mu^\circ| + \sum_{i=1}^m \nu_i\big(\nu_i - 2(i-1)\big) + \sum_{j=1}^n \tau_j\big(\tau_j - 2(m+j-1)\big) \\
&= \overline{C} + (c-1)|\mu^\circ| + \sum_{i=1}^m \nu_i\big(\nu_i - 2(i-1)\big) + \sum_{j=1}^n \nu_{m+j}\big(\nu_{m+j} - 2(m+j-1)\big) \\
&= \overline{C} + (c-1)|\mu^\circ| - \sum_{i\geqslant 1}\mu_i(\mu_i - 2i) = \overline{C} + (c-1)|\mu^\circ| - \big(\mu^\circ + 2\rho_1|\mu^\circ\big)_1.
\end{aligned}$$

Above we have used (4.7) in the third to last identity.

Case 2. $\bar{\mathfrak{g}} = \widehat{\mathfrak{osp}}(2m|2n)$. By (4.7), we have

$$(\mu + 2\rho_c|\mu)_c = \sum_{i\geqslant 1}\mu_i(\mu_i - 2i) - 2c|\mu^\circ| = -(\nu + 2\rho_2|\nu)_2 - 2c|\mu^\circ|,$$

where $(\cdot|\cdot)_c$ is the bilinear form on the dual Cartan subalgebra of $\mathfrak{c}_\infty$.

On the other hand,

$$\begin{aligned}&\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s\\&\quad=c^2\big(\widetilde{\Lambda}^{\mathfrak{c}}_0\big|\widetilde{\Lambda}^{\mathfrak{c}}_0\big)_s+2c\big(\rho_s\big|\widetilde{\Lambda}^{\mathfrak{c}}_0\big)_s+\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s+2c\big(\widetilde{\Lambda}^{\mathfrak{c}}_0\big|\vartheta(\mu)\big)_s\\&\quad=\overline{C}+\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s+2c\big|\mu^\circ\big|.\end{aligned}$$

And by the same argument as in (4.9), we can show that

$$\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s=(\nu+2\rho_2|\nu)_2.$$

Case 3. $\bar{\mathfrak{g}}=\widehat{\mathfrak{spo}}(2m|2n)$. By (4.7), we have

$$(\mu+2\rho_c|\mu)_c=\sum_{i\geqslant 1}\mu_i\big(\mu_i-2(i-1)\big)-c\big|\mu^\circ\big|=-(\nu+2\rho_1|\nu)_1-c\big|\mu^\circ\big|,$$

where $(\cdot|\cdot)_c$ is the bilinear form on the dual Cartan subalgebra of $\mathfrak{d}_\infty$.

On the other hand,

$$\begin{aligned}&\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s\\&\quad=(c/2)^2\big(\widetilde{\Lambda}^{\mathfrak{d}}_0\big|\widetilde{\Lambda}^{\mathfrak{d}}_0\big)_s+c\big(\rho_s\big|\widetilde{\Lambda}^{\mathfrak{d}}_0\big)_s+\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s+c\big(\widetilde{\Lambda}^{\mathfrak{d}}_0\big|\vartheta(\mu)\big)_s\\&\quad=\overline{C}+\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s+c\big|\mu^\circ\big|.\end{aligned}$$

Similarly, arguing as in (4.9), we have

$$\big(\vartheta(\mu)+2\rho_s\big|\vartheta(\mu)\big)_s=(\nu+2\rho_1|\nu)_1.$$

This completes the proof of the first identity in the lemma. □

5. Kostant homology formulas for Lie superalgebras

5.1. The $\mathfrak{u}_-$-homology groups of $\mathfrak{g}$-modules

Recall the dual pair $(\mathfrak{g}, G)$ of type $\mathfrak{x}\in\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}\}$ in Section 2.2, the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, and $\mathcal{P}^+_{\mathfrak{l}}$, the set of dominant weights for $\mathfrak{l}$. We denote by $L(\mathfrak{l},\mu)$ the irreducible highest weight $\mathfrak{l}$-module with highest weight $\mu\in\mathfrak{h}^*$. The following results for integrable modules of Kac–Moody algebras apply to our setting.

Proposition 5.1. *(Cf. [16], [20, Proposition 3.1], [27, Proposition 18 and Lemma 20].) Let $\lambda\in\mathcal{P}(G)$.*

1. *If a weight $\eta\in\mathcal{P}^+_{\mathfrak{l}}$ with $(\eta+2\rho_c|\eta)_c=(\Lambda^{\mathfrak{x}}(\lambda)+2\rho_c|\Lambda^{\mathfrak{x}}(\lambda))_c$ appears in $\Lambda^k\mathfrak{u}_-\otimes L(\mathfrak{g},\Lambda^{\mathfrak{x}}(\lambda))$, then $\eta=w\circ\Lambda^{\mathfrak{x}}(\lambda)$ for some $w\in W^0_k(\mathfrak{x})$ and η appears with multiplicity one.*
2. *The $\mathfrak{l}$-module $\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{g},\Lambda^{\mathfrak{x}}(\lambda)))$ is completely reducible. Moreover, if $L(\mathfrak{l},\eta)$ is a summand of $\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{g},\Lambda^{\mathfrak{x}}(\lambda)))$, then $(\eta+2\rho_c|\eta)_c=(\Lambda^{\mathfrak{x}}(\lambda)+2\rho_c|\Lambda^{\mathfrak{x}}(\lambda))_c$.*

For $\lambda \in \mathcal{P}(G)$ and $w \in W^0$, we put

$$\mathcal{L}\big(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)\big) := \begin{cases} L(\mathfrak{g}, \Lambda^{\mathfrak{d}}(\lambda)) \oplus L(\mathfrak{g}, \Lambda^{\mathfrak{d}}(\tilde{\lambda})), & \text{if } G = \mathrm{O}(2\ell), \\ L(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)), & \text{otherwise}, \end{cases}$$

$$\mathcal{L}\big(\mathfrak{l}, w \circ \Lambda^{\mathfrak{x}}(\lambda)\big) := \begin{cases} L(\mathfrak{l}, w \circ \Lambda^{\mathfrak{d}}(\lambda)) \oplus L(\mathfrak{l}, w \circ \Lambda^{\mathfrak{d}}(\tilde{\lambda})), & \text{if } G = \mathrm{O}(2\ell), \\ L(\mathfrak{l}, w \circ \Lambda^{\mathfrak{x}}(\lambda)), & \text{otherwise}. \end{cases} \tag{5.1}$$

5.2. The $\bar{\mathfrak{u}}_-$-homology groups of $\bar{\mathfrak{g}}$-modules

Recall the dual pair $(\bar{\mathfrak{g}}, G)$ of type $\mathfrak{x} \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$ given in Section 2.5. For $\mu \in \bar{\mathfrak{h}}^*$ we denote by $L(\bar{\mathfrak{l}}, \mu)$ the irreducible highest weight $\bar{\mathfrak{l}}$-module with highest weight μ.

Lemma 5.2. *Let $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$. The $\bar{\mathfrak{l}}$-module $L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ is completely reducible.*

Proof. Suppose that $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n)$. By [7, Theorem 3.2] (also cf. [30]), $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*})$ and $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ are completely reducible over $\mathfrak{gl}(p|q)$ and $\mathfrak{gl}(m|n)$, respectively. Indeed $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ lies in the semisimple tensor category $\mathcal{O}^{++}_{m|n}$ of finite-dimensional $\mathfrak{gl}(m|n)$-modules with composition factors of the form $L(\mathfrak{gl}(m|n), \lambda^{\natural})$, for some $\lambda \in \mathcal{P}^+$ with $\lambda_{m+1} \leqslant n$ [4, Theorem 3.1]. Also, $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*})$ lies in a similar semisimple tensor category $\mathcal{O}^{++*}_{p|q}$. Thus $S(\mathbb{C}^{p|q*} \otimes \mathbb{C}^{d*} \oplus \mathbb{C}^{m|n} \otimes \mathbb{C}^d)$, as a $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p|q)$-module, lies in "$\mathcal{O}^{++}_{m|n} \otimes \mathcal{O}^{++*}_{p|q}$", and hence is completely reducible over $\bar{\mathfrak{l}} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p|q) \oplus \mathbb{C}K$. Therefore $L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda))$ is completely reducible over $\bar{\mathfrak{l}}$.

If $\bar{\mathfrak{g}}$ is of type $\mathfrak{b}, \mathfrak{c}, \mathfrak{d}$, then $\bar{\mathfrak{l}} = \mathfrak{gl}(m|n) \oplus \mathbb{C}K$. Thus $L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ is completely reducible over $\bar{\mathfrak{l}}$, since it is a submodule of $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d)$ or $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^d) \otimes \Lambda(\mathbb{C}^{\frac{d}{2}})$. □

Now consider the homology groups of Lie superalgebras $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))$, which are defined analogously (see e.g. [24]).

Lemma 5.3. *Let $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$. The $\bar{\mathfrak{l}}$-module $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))$ is completely reducible.*

Proof. Suppose that $\bar{\mathfrak{g}} = \widehat{\mathfrak{gl}}(p+m|q+n) \supseteq \bar{\mathfrak{l}} = \mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p|q) \oplus \mathbb{C}K$. It follows from [7, Theorem 3.3] that $\Lambda^k(\bar{\mathfrak{u}}_-)$ lies in "$\mathcal{O}^{++}_{m|n} \otimes \mathcal{O}^{++*}_{p|q}$". By Lemma 5.2 $L(\mathfrak{g}, \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda))$, as an $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p|q)$-module, lies in "$\mathcal{O}^{++}_{m|n} \otimes \mathcal{O}^{++*}_{p|q}$", and hence $\Lambda^k(\bar{\mathfrak{u}}_-) \otimes L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{a}}_f(\lambda))$ is a completely reducible $\bar{\mathfrak{l}}$-module. Since any subquotient of a completely reducible module is also completely reducible, the result follows.

The ortho-symplectic cases are easier and hence omitted. □

The proof of Lemma 5.3 implies that if $L(\bar{\mathfrak{l}}, \gamma)$ is an $\bar{\mathfrak{l}}$-submodule of $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))$, then $\gamma \in \mathcal{P}^{++}_{\bar{\mathfrak{l}}}$. We have the following super-analogue of the action of the Casimir operator on homology groups (see Proposition 5.1).

Proposition 5.4. *Let $\gamma \in \mathcal{P}^{++}_{\bar{\mathfrak{l}}}$. If $L(\bar{\mathfrak{l}}, \gamma)$ is a summand of $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))$, then $(\gamma + 2\rho_s|\gamma)_s = (\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s$.*

Proof. The proof follows the same type of arguments as for [27, Proposition 18] and thus will be omitted. We only remark that in the process we use the same bilinear form $(\cdot|\cdot)_s$ and the same ρ_s to define the corresponding Casimir operator for $\bar{\mathfrak{l}}$ as in (4.3). □

5.3. Formulas for the $\bar{\mathfrak{u}}_-$-homology groups of $\bar{\mathfrak{g}}$-modules

Recalling $\vartheta(\Lambda^{\mathfrak{x}}(\lambda)) = \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)$ for $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$, we define $\mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ and $\mathcal{L}(\bar{\mathfrak{l}}, \vartheta(w \circ \Lambda^{\mathfrak{x}}(\lambda)))$ similarly as in (5.1).

Lemma 5.5. *Let $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$ and $\bar{\mu} \in \mathcal{P}^{++}_{\bar{\mathfrak{l}}}$. If $L(\bar{\mathfrak{l}}, \bar{\mu})$ appears in the decomposition of $\Lambda^k \bar{\mathfrak{u}}_- \otimes \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ with multiplicity $m_{\overline{\mu}}$, then there exists a unique $\mu \in \mathcal{P}^{++}_{\mathfrak{l}}$ with $\vartheta(\mu) = \bar{\mu}$, and $L(\mathfrak{l}, \mu)$ appears in the decomposition of $\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda))$ with the same multiplicity $m_{\overline{\mu}}$.*

Proof. Since $\omega^{\mathfrak{x}}(\operatorname{ch} \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda))) = \operatorname{ch} \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$, for $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$, and $\omega^{\mathfrak{x}}(\operatorname{ch} \Lambda^k(\mathfrak{u}_-)) = \operatorname{ch} \Lambda^k(\bar{\mathfrak{u}}_-)$, we conclude that

$$\omega^{\mathfrak{x}}\big(\operatorname{ch}\big[\Lambda^k(\mathfrak{u}_-) \otimes \mathcal{L}\big(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)\big)\big]\big) = \operatorname{ch}\big[\Lambda^k(\bar{\mathfrak{u}}_-) \otimes \mathcal{L}\big(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)\big)\big].$$

Since ϑ is a bijection on $\mathcal{P}^{++}_{\mathfrak{l}}$, there exists a unique $\mu \in \mathcal{P}^{++}_{\mathfrak{l}}$ such that $\vartheta(\mu) = \overline{\mu}$. Therefore $L(\mathfrak{l}, \mu)$ is a composition factor of $\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{a}}(\lambda))$, if $L(\bar{\mathfrak{l}}, \vartheta(\mu))$ is a composition factor of $\Lambda^k \bar{\mathfrak{u}}_- \otimes \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}(\lambda))$. Furthermore they have the same multiplicity. □

Lemma 5.6. *Let $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$ and $\eta \in \mathcal{P}^{++}_{\bar{\mathfrak{l}}}$ such that $L(\bar{\mathfrak{l}}, \eta)$ is a summand of $\Lambda^k(\bar{\mathfrak{u}}_-) \otimes \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$. Then $(\eta + 2\rho_s|\eta)_s = (\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s$ if and only if there exists $w \in W^0_k(\mathfrak{x})$ with $\eta = \vartheta(w \circ \Lambda^{\mathfrak{x}}(\lambda))$.*

Proof. Assume that there exists $w \in W^0_k(\mathfrak{x})$ with $w \circ \Lambda^{\mathfrak{x}}(\lambda) \in \mathcal{P}^{++}_{\mathfrak{l}}$. Set $\eta = \vartheta(w \circ \Lambda^{\mathfrak{x}}(\lambda))$. Then by Lemmas 4.2 and 4.3 and the W-invariance of $(\cdot|\cdot)_c$ we have

$$\begin{aligned}(\eta + 2\rho_s|\eta)_s &= \pm\big(w \circ \Lambda^{\mathfrak{x}}(\lambda) + 2\rho_c\big|w \circ \Lambda^{\mathfrak{x}}(\lambda)\big)_c + \overline{C} \\ &= \pm\big(\Lambda^{\mathfrak{x}}(\lambda) + 2\rho_c\big|\Lambda^{\mathfrak{x}}(\lambda)\big)_c + \overline{C} = \big(\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s\big|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)\big)_s.\end{aligned}$$

On the other hand suppose that $(\eta + 2\rho_s|\eta)_s = (\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s$. By Lemma 5.5 there exists a unique $\vartheta^{-1}(\eta) \in \mathcal{P}^{++}_{\mathfrak{l}}$ with $\vartheta(\vartheta^{-1}(\eta)) = \eta$ satisfying the conditions of Lemma 5.5. By Lemmas 4.2 and 4.3, we have

$$\big(\vartheta^{-1}(\eta) + 2\rho_c\big|\vartheta^{-1}(\eta)\big)_c = \big(\Lambda^{\mathfrak{x}}(\lambda) + 2\rho_c\big|\Lambda^{\mathfrak{x}}(\lambda)\big)_c.$$

By Proposition 5.1(1) there exists $w \in W^0_k(\mathfrak{x})$ such that $\vartheta^{-1}(\eta) = w \circ \Lambda^{\mathfrak{x}}(\lambda)$. □

Let $W^0_k(\mathfrak{x})'$ denote the subset of $W^0_k(\mathfrak{x})$ consisting of those w with $\vartheta(w \circ \Lambda^{\mathfrak{x}}(\lambda)) \in \mathcal{P}^{++}_{\bar{\mathfrak{l}}}$. The proof of the following theorem is inspired by [1].

Theorem 5.7. *Let* $\lambda \in \mathcal{P}^{\bar{\mathfrak{g}}}(G)$, $k \in \mathbb{Z}_+$, *and* ϑ *as before. As* $\bar{\mathfrak{l}}$*-modules we have*

$$\mathrm{H}_k\big(\bar{\mathfrak{u}}_-; \mathcal{L}\big(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)\big)\big) \cong \bigoplus_{w\in W^0_k(\mathfrak{x})'} \mathcal{L}\big(\bar{\mathfrak{l}}, \vartheta\big(w \circ \Lambda^{\mathfrak{x}}(\lambda)\big)\big).$$

In particular, $\mathrm{ch}[\mathrm{H}_k(\bar{\mathfrak{u}}_-; \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))] = \omega^{\mathfrak{x}}(\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-; \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)))])$.

Proof. Let $\mu \in \mathcal{P}^+_{\mathfrak{l}}$ be such that $L(\mathfrak{l}, \mu)$ is a summand of $\mathrm{H}_k(\mathfrak{u}_-; \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)))$. Then it is precisely a summand of $\Lambda^k\mathfrak{u}_- \otimes \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda))$ with $(\mu + 2\rho_c|\mu)_c = (\Lambda^{\mathfrak{x}}(\lambda) + 2\rho_c|\Lambda^{\mathfrak{x}}(\lambda))_c$ by Proposition 5.1. Furthermore each appears with multiplicity one or two (cf. [4, Remark 5.2]). By Lemma 5.6 the corresponding $\vartheta(\mu)$'s, for $\mu \in \mathcal{P}^{++}_{\mathfrak{l}}$, are precisely the weights in $\mathcal{P}^{++}_{\bar{\mathfrak{l}}}$ such that $L(\bar{\mathfrak{l}}, \vartheta(\mu))$ appear as summands of $\Lambda^k\bar{\mathfrak{u}}_- \otimes \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ with $(\vartheta(\mu) + 2\rho_s|\vartheta(\mu))_s = (\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s$; moreover it appears with the same multiplicity.

Theorems 3.1, 3.4, and 3.5 together with the Euler–Poincaré principle imply that

$$\sum_{k=0}^{\infty}(-1)^k\,\mathrm{ch}\big[\mathrm{H}_k\big(\bar{\mathfrak{u}}_-; \mathcal{L}\big(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)\big)\big)\big] = \sum_{k=0}^{\infty}(-1)^k \sum_{w\in W^0_k(\mathfrak{x})'} \mathcal{H}_{\lambda,w},$$

where

$$\begin{aligned}\mathcal{H}_{\lambda,w} &= \begin{cases} hs_{\lambda^+_w}(\xi, y)hs_{\lambda^-_w}(\eta^{-1}, x^{-1}), & \text{if } G = \mathrm{GL}(d),\\ hs_{\lambda_w}(\eta, x), & \text{if } G = \mathrm{Sp}(d), \mathrm{O}(2\ell+1), \mathrm{Pin}(d),\\ hs_{\lambda_w}(\eta, x) + hs_{\tilde{\lambda}_w}(\eta, x), & \text{if } G = \mathrm{O}(2\ell)\end{cases}\\ &= \mathrm{ch}\,\mathcal{L}\big(\bar{\mathfrak{l}}, \vartheta\big(w \circ \Lambda^{\mathfrak{x}}(\lambda)\big)\big).\end{aligned}$$

Since all the highest weights are distinct, we conclude from Proposition 5.4 that

$$\mathrm{ch}\big[\mathrm{H}_k\big(\bar{\mathfrak{u}}_-; \mathcal{L}\big(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)\big)\big)\big] = \sum_{w\in W^0_k(\mathfrak{x})'} \mathrm{ch}\big[\mathcal{L}\big(\bar{\mathfrak{l}}, \vartheta\big(w \circ \Lambda^{\mathfrak{x}}(\lambda)\big)\big)\big],$$

which is equal to $\omega^{\mathfrak{x}}(\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-; \mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}(\lambda)))])$ by (2.12). □

Corollary 5.8.

(1) *The character of* $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\mathfrak{gl}(p+m|q+n), \Lambda^{\mathfrak{a}}_f(\lambda)))$, *i.e. the trace of the operator* $\prod_{i,j,s,t} x_i^{E_{ii}}\eta_j^{E_{\bar{j},\bar{j}}} y_s^{E_{ss}}\xi_t^{E_{\bar{t},\bar{t}}}$, *is given by*

$$\sum_{w\in W^0_k(\mathfrak{a})'}\left(\frac{\eta_{-q}\cdots\eta_{-1}}{x_{-p}\cdots x_{-1}}\right)^d hs_{\lambda^+_w}(\xi, y)hs_{\lambda^-_w}\big(\eta^{-1}, x^{-1}\big).$$

(2) *The character of* $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\mathfrak{spo}(2m|2n+1), \Lambda^{\mathfrak{b}}_f(\lambda)))$, *i.e. the trace of* $\prod_{i,j} \eta_i^{E_i} x_j^{E_{\bar{j}}}$, *is*

$$\sum_{w\in W_k^0(\mathfrak{b})'} \left(\frac{\eta_1\cdots\eta_m}{x_1\cdots x_n}\right)^{\frac{d}{2}} hs_{\lambda_w}(\eta, x).$$

(3) *The character of* $\mathrm{H}_k(\bar{\mathfrak{u}}_-; L(\mathfrak{osp}(2m|2n), \Lambda^{\mathfrak{c}}_f(\lambda)))$, *i.e. the trace of* $\prod_{i,j} \eta_i^{E_i} x_j^{E_{\bar{j}}}$, *is*

$$\sum_{w\in W_k^0(\mathfrak{c})'} \left(\frac{\eta_1\cdots\eta_m}{x_1\cdots x_n}\right)^{\frac{d}{2}} hs_{\lambda_w}(\eta, x).$$

(4) *The character of* $\mathrm{H}_k(\bar{\mathfrak{u}}_-; \mathcal{L}(\mathfrak{spo}(2m|2n), \Lambda^{\mathfrak{d}}_f(\lambda)))$, *i.e. the trace of* $\prod_{i,j} \eta_i^{E_i} x_j^{E_{\bar{j}}}$, *is*

$$\begin{cases} \sum_{w\in W_k^0(\mathfrak{d})'} \left(\frac{\eta_1\cdots\eta_m}{x_1\cdots x_n}\right)^{\frac{d}{2}} hs_{\lambda_w}(\eta, x), & \text{if } G = \mathrm{O}(2\ell+1), \\ \sum_{w\in W_k^0(\mathfrak{d})'} \left(\frac{\eta_1\cdots\eta_m}{x_1\cdots x_n}\right)^{\frac{d}{2}} [hs_{\lambda_w}(\eta, x) + hs_{\tilde{\lambda}_w}(\eta, x)], & \text{if } G = \mathrm{O}(2\ell). \end{cases}$$

These character formulas of $\mathrm{H}_k(\bar{\mathfrak{u}}_-; \mathcal{L}(\bar{\mathfrak{g}}, \Lambda^{\mathfrak{x}}_f(\lambda)))$ fit well with those of $\bar{\mathfrak{g}}$-modules in Section 3 by the Euler–Poincaré principle.

Theorem 5.7 and Lemma 5.6 also imply the following super-analogue of Proposition 5.1(2), which is the converse of Proposition 5.4.

Corollary 5.9. *If* $L(\bar{\mathfrak{l}}, \eta)$ *with* $(\eta + 2\rho_s|\eta)_s = (\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda) + 2\rho_s|\widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))_s$ *appears in* $\Lambda^k(\bar{\mathfrak{u}}_-) \otimes \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda))$ *with multiplicity* m_η, *then* $L(\bar{\mathfrak{l}}, \eta)$ *is a summand of* $\mathrm{H}_k(\bar{\mathfrak{u}}_-; \mathcal{L}(\bar{\mathfrak{g}}, \widehat{\Lambda}^{\mathfrak{x}}_f(\lambda)))$ *with the same multiplicity* m_η.

Remark 5.10. Specializing to $q = n = 0$ in Corollary 5.8(1), (3) and (4) we obtain Kostant type homology formulas of the unitarizable highest weight modules of the Lie groups $\mathrm{SU}(p, m)$, $\mathrm{SO}^*(2m)$, and the double cover of $\mathrm{Sp}(2m, \mathbb{R})$. For unitarizable highest weight modules such homology groups were first computed by Enright [14, Theorem 2.2], and they involve a complicated subset of the corresponding finite Weyl group [13, Definition 3.6]. Corollary 5.8 provides an alternative description involving an infinite Weyl group. Ngau Lam has informed us that both forms of the formula are equivalent [19].

Remark 5.11. Although all the discussions above dealt with $\mathfrak{u}_-$-homology and $\bar{\mathfrak{u}}_-$-homology groups, our calculations also give formulas for the corresponding (restricted) $\mathfrak{u}_+$-cohomology and $\bar{\mathfrak{u}}_+$-cohomology groups (see [27, Lemma 9]).

6. Homology formulas for oscillator modules at negative levels

In this section, we shall compute the character formulas and the $\mathfrak{u}_-$-homology groups of various modules of $\mathfrak{b}^{\circ}_\infty$, $\mathfrak{c}_\infty$ and $\mathfrak{d}_\infty$ at negative levels. As the approach is parallel to that of the earlier sections, the presentation here will be rather sketchy.

6.1. *The character formulas*

We fix a positive integer $\ell \geqslant 1$. Consider ℓ pairs of free bosons

$$\gamma^{\pm,i}(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \gamma_r^{\pm,i} z^{-r-1/2}$$

with $i = 1, \ldots, \ell$. Let $\mathfrak{F}^{-\ell}$ denote the corresponding Fock space generated by the vacuum vector $|0\rangle$, which is annihilated by $\gamma_r^{\pm,i}$ for $r > 0$. We also denote by $\mathfrak{F}^{-\ell+\frac{1}{2}}$ the tensor product of $\mathfrak{F}^{-\ell}$ and $\mathfrak{F}^{\frac{1}{2}}$.

6.1.1. *The case of* $\mathfrak{d}_\infty$

Suppose that $\mathfrak{g} = \mathfrak{d}_\infty$ and d is even. By [31, Theorem 5.2] there exists a commuting action of $\mathfrak{d}_\infty$ and $\mathrm{Sp}(d)$ on $\mathfrak{F}^{-\frac{d}{2}}$. Furthermore, under this joint action, we have

$$\mathfrak{F}^{-\frac{d}{2}} \cong \bigoplus_{\lambda\in\mathcal{P}(\mathrm{Sp}(d))} L\big(\mathfrak{d}_\infty, \Lambda_-^{\mathfrak{d}}(\lambda)\big) \otimes V_{\mathrm{Sp}(d)}^{\lambda}, \tag{6.1}$$

where $\Lambda_-^{\mathfrak{d}}(\lambda) := -d\Lambda_0^{\mathfrak{d}} + \sum_{k=1}^{\frac{d}{2}} \lambda_k \epsilon_k$.

Computing the trace of the operator $\prod_{n\in\mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\frac{d}{2}} z_i^{e_i}$ on both sides of (6.1), we obtain the following identity:

$$\prod_{i=1}^{\frac{d}{2}} \prod_{n\in\mathbb{N}} \frac{1}{(1 - x_n z_i)(1 - x_n z_i^{-1})} = \sum_{\lambda\in\mathcal{P}(\mathrm{Sp}(d))} \mathrm{ch}\, L\big(\mathfrak{d}_\infty, \Lambda_-^{\mathfrak{d}}(\lambda)\big)\, \mathrm{ch}\, V_{\mathrm{Sp}(d)}^{\lambda}. \tag{6.2}$$

By (2.5) and similar arguments as in Theorems 3.1–3.5, we obtain the following.

Theorem 6.1. *For $\lambda \in \mathcal{P}(\mathrm{Sp}(d))$, we have*

$$\mathrm{ch}\, L\big(\mathfrak{d}_\infty, \Lambda_-^{\mathfrak{d}}(\lambda)\big) = \frac{\sum_{k=0}^{\infty} \sum_{w\in W_k^0(\mathfrak{c})} (-1)^k s_{(\lambda_w)'}(x_1, x_2, \ldots)}{\prod_{i<j}(1 - x_i x_j)}.$$

The involution ω on the ring of symmetric functions in $x_1, x_2, \ldots$ maps the left-hand side of (2.5) to that of (6.2). Thus it follows that $\omega(\mathrm{ch}\, L(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}(\lambda))) = \mathrm{ch}\, L(\mathfrak{d}_\infty, \Lambda_-^{\mathfrak{d}}(\lambda))$.

6.1.2. *The case of* $\mathfrak{c}_\infty$

By [31, Theorems 5.3 and 6.2] there exists a commuting action of $\mathfrak{c}_\infty$ and $\mathrm{O}(d)$ on $\mathfrak{F}^{-\frac{d}{2}}$. Furthermore, under this joint action, we have

$$\mathfrak{F}^{-\frac{d}{2}} \cong \bigoplus_{\lambda\in\mathcal{P}(\mathrm{O}(d))} L\big(\mathfrak{c}_\infty, \Lambda_-^{\mathfrak{c}}(\lambda)\big) \otimes V_{\mathrm{O}(d)}^{\lambda}, \tag{6.3}$$

where $\Lambda_-^{\mathfrak{c}}(\lambda) := -\frac{d}{2}\Lambda_0^{\mathfrak{c}} + \sum_{k=1}^{d} \lambda_k \epsilon_k$.

If $d = 2\ell$, then the calculation of the trace of the operator $\prod_{n\in\mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\ell} z_i^{e_i}$ on both sides of (6.3) gives the following identity

$$\prod_{i=1}^{\ell}\prod_{n\in\mathbb{N}} \frac{1}{(1-x_n z_i)(1-x_n z_i^{-1})} = \sum_{\lambda\in\mathcal{P}(\mathrm{O}(2\ell))} \operatorname{ch} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{-}(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell)}. \tag{6.4}$$

If $d = 2\ell + 1$, then the trace of the operator $\prod_{n\in\mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\ell} z_i^{e_i}(-I_d)$ gives

$$\begin{aligned}\prod_{i=1}^{\ell}\prod_{n\in\mathbb{N}} &\frac{1}{(1-\epsilon x_n z_i)(1-\epsilon x_n z_i^{-1})(1-\epsilon x_n)} \\ &= \sum_{\lambda\in\mathcal{P}(\mathrm{O}(2\ell+1))} \operatorname{ch} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{-}(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{O}(2\ell+1)}.\end{aligned} \tag{6.5}$$

Applying the same arguments as in Theorem 6.1 to (2.7) and (2.8), we obtain the following.

Theorem 6.2. *Let $\lambda \in \mathcal{P}(\mathrm{O}(d))$ be given.*

(1) *If $d = 2\ell + 1$, then we have*

$$\operatorname{ch} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{-}(\lambda)\big) = \frac{\sum_{k=0}^{\infty}\sum_{w\in W_k^0(\mathfrak{d})} (-1)^k s_{(\lambda_w)'}(x_1, x_2, \ldots)}{\prod_{i\leqslant j}(1-x_i x_j)}.$$

(2) *If $d = 2\ell$, then we have*

$$\begin{aligned}&\operatorname{ch} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{-}(\lambda)\big) + \operatorname{ch} L\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{-}(\tilde{\lambda})\big) \\ &\quad = \frac{\sum_{k=0}^{\infty}\sum_{w\in W_k^0(\mathfrak{d})} (-1)^k [s_{(\lambda_w)'}(x_1, x_2, \ldots) + s_{(\tilde{\lambda}_w)'}(x_1, x_2, \ldots)]}{\prod_{i\leqslant j}(1-x_i x_j)}.\end{aligned}$$

Note that ω maps (2.7) (resp. (2.8)) to (6.4) (resp. (6.5)).

6.1.3. The case of $\mathfrak{b}^{\circ}_\infty$

Let d be even. Recall the duality from Theorem A.3. Computing the trace of the operator $\prod_{n\in\mathbb{N}} x_n^{\widetilde{E}_n} \prod_{i=1}^{\frac{d}{2}} z_i^{e_i}$ on both sides of (A.9) gives the following identity

$$\prod_{i=1}^{\frac{d}{2}}\prod_{n\in\mathbb{N}} \frac{(z_i^{\frac{1}{2}} + z_i^{-\frac{1}{2}})}{(1-x_n z_i)(1-x_n z_i^{-1})} = \sum_{\lambda\in\mathcal{P}(\mathrm{Pin}(d))} \operatorname{ch} L\big(\mathfrak{b}^{\circ}_\infty, \Lambda^{\mathfrak{b}^{\circ}}_{-}(\lambda)\big) \operatorname{ch} V^{\lambda}_{\mathrm{Pin}(d)}. \tag{6.6}$$

By (2.11) and similar arguments as in the proof of Theorem 6.1 we obtain the following.

Theorem 6.3. *For $\lambda \in \mathcal{P}(\mathrm{Pin}(d))$, we have*

$$\operatorname{ch} L\big(\mathfrak{b}^{\circ}_{\infty}, \Lambda^{\mathfrak{b}^{\circ}}_{-}(\lambda)\big) = \frac{\sum_{k=0}^{\infty} \sum_{w \in W^0_k(\mathfrak{b})} (-1)^k s_{(\lambda_w)'}(x_1, x_2, \ldots)}{\prod_i (1+x_i)^{-1} \prod_{i \leqslant j}(1 - x_i x_j)}.$$

Now ω maps the left-hand side of (2.11) to that of (6.6). Thus it follows that $\omega(\operatorname{ch} L(\mathfrak{b}_\infty, \Lambda^{\mathfrak{b}}(\lambda))) = \operatorname{ch} L(\mathfrak{b}^{\circ}_{\infty}, \Lambda^{\mathfrak{b}^{\circ}}_{-}(\lambda))$.

6.2. Formulas for the $\mathfrak{u}_-$-homology groups

Recall $\mathcal{P}^{+}_{\mathfrak{l},c}$ and $\mathcal{P}^{+}_{\mathfrak{l}}$ from Section 4.3 and Appendix A.2, which we shall now denote by $\mathcal{P}^{+,\mathfrak{x}}_{\mathfrak{l},c}$ and $\mathcal{P}^{+,\mathfrak{x}}_{\mathfrak{l}}$ to keep track that $\mathfrak{l}$ is a subalgebra of $\mathfrak{x}_\infty$ for $\mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$.

Let $\mu = c\Lambda^{\mathfrak{c}}_0 + \sum_{i \geqslant 1} \mu_i \epsilon_i \in \mathcal{P}^{+,\mathfrak{c}}_{\mathfrak{l},c}$ be given. Define a bijective map $\vartheta : \mathcal{P}^{+,\mathfrak{c}}_{\mathfrak{l},c} \to \mathcal{P}^{+,\mathfrak{d}}_{\mathfrak{l},-2c}$

$$\vartheta(\mu) := -2c\Lambda^{\mathfrak{d}}_0 + \sum_{i \geqslant 1} \mu'_i \epsilon_i. \tag{6.7}$$

In particular, we have $\vartheta(\Lambda^{\mathfrak{c}}_{\pm}(\lambda)) = \Lambda^{\mathfrak{d}}_{\mp}(\lambda)$, for $\lambda \in \mathcal{P}(\mathrm{Sp}(d))$, where it is understood that $\Lambda^{\mathfrak{x}}_{+}(\lambda) = \Lambda^{\mathfrak{x}}(\lambda)$ for $\mathfrak{x} \in \{\mathfrak{c}, \mathfrak{d}\}$.

For $\mu = 2c\Lambda^{\mathfrak{b}}_0 + \sum_{i \geqslant 1} \mu_i \epsilon_i \in \mathcal{P}^{+,\mathfrak{b}}_{\mathfrak{l},2c}$, we define $\vartheta : \mathcal{P}^{+,\mathfrak{b}}_{\mathfrak{l},2c} \to \mathcal{P}^{+,\mathfrak{b}^{\circ}}_{\mathfrak{l},-c}$

$$\vartheta(\mu) := -c\Lambda^{\mathfrak{b}^{\circ}}_0 + \sum_{i \geqslant 1} \mu'_i \epsilon_i. \tag{6.8}$$

We have $\vartheta(\Lambda^{\mathfrak{b}}(\lambda)) = \Lambda^{\mathfrak{b}^{\circ}}_{-}(\lambda)$.

For $\lambda \in \mathcal{P}(G)$ and $w \in W^0$, we define $\mathcal{L}(\mathfrak{g}, \Lambda^{\mathfrak{x}}_{-}(\lambda))$ and $\mathcal{L}(\mathfrak{l}, \vartheta^{\pm 1}(w \circ \Lambda^{\mathfrak{x}}_{+}(\lambda)))$ similarly as in (5.1). Then it is easy to see that $\omega(\operatorname{ch} \mathcal{L}(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{\pm}(\lambda))) = \operatorname{ch} \mathcal{L}(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}_{\mp}(\lambda))$, and

$$\omega\big(\operatorname{ch}\big[\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}\big(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{\pm}(\lambda)\big)\big]\big) = \operatorname{ch}\big[\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}\big(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}_{\mp}(\lambda)\big)\big].$$

Using analogous arguments as in Lemma 5.5, we can check that for $\mu \in \mathcal{P}^{+,\mathfrak{c}}_{\mathfrak{l}}$, $L(\mathfrak{l}, \mu)$ is a component in $\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}(\mathfrak{c}_\infty, \Lambda^{\mathfrak{c}}_{\pm}(\lambda))$ if and only if $L(\mathfrak{l}, \vartheta(\mu))$ is a component in $\Lambda^k \mathfrak{u}_- \otimes \mathcal{L}(\mathfrak{d}_\infty, \Lambda^{\mathfrak{d}}_{\mp}(\lambda))$ with the same multiplicity, while $\omega(\operatorname{ch} L(\mathfrak{l}, \mu)) = \operatorname{ch} L(\mathfrak{l}, \vartheta(\mu))$. We have a similar correspondence between $\mathfrak{l}$-modules inside $\Lambda^k \mathfrak{u}_- \otimes L(\mathfrak{b}_\infty, \Lambda^{\mathfrak{b}}(\lambda))$ and $\Lambda^k \mathfrak{u}_- \otimes L(\mathfrak{b}^{\circ}_{\infty}, \Lambda^{\mathfrak{b}^{\circ}}_{-}(\lambda))$.

Lemma 6.4. *For $\mu \in \mathcal{P}^{+,\mathfrak{x}}_{\mathfrak{l}}$, we have*

$$(\mu + 2\rho_c|\mu)_c = -\big(\vartheta(\mu) + 2\rho_c\big|\vartheta(\mu)\big)_c, \quad \mathfrak{x} \in \{\mathfrak{b}, \mathfrak{c}\}.$$

In particular, we have

(1) $(\Lambda^{\mathfrak{c}}_{\pm}(\lambda) + 2\rho_c|\Lambda^{\mathfrak{c}}_{\pm}(\lambda))_c = (\mu + 2\rho_c|\mu)_c$ *if and only if* $(\Lambda^{\mathfrak{d}}_{\mp}(\lambda) + 2\rho_c|\Lambda^{\mathfrak{d}}_{\mp}(\lambda))_c = (\vartheta(\mu) + 2\rho_c|\vartheta(\mu))_c$.

(2) $(\Lambda^{\mathfrak{b}}(\lambda)+2\rho_c|\Lambda^{\mathfrak{b}}(\lambda))_c=(\mu+2\rho_c|\mu)_c$ *if and only if* $(\Lambda^{\mathfrak{b}^\circ}_-(\lambda)+2\rho_c|\Lambda^{\mathfrak{b}^\circ}_-(\lambda))_c=(\vartheta(\mu)+2\rho_c|\vartheta(\mu))_c$.

Proof. Let $\mu=c\Lambda_0^{\mathfrak{x}}+\sum_{i\geqslant 1}\mu_i\epsilon_i$. Consider first the case of $\mathfrak{x}=\mathfrak{c}$. We have

$$(\mu+2\rho_c|\mu)_c=\Big(\sum_{i\geqslant 1}\mu_i\epsilon_i+2\rho_c\Big|\sum_{i\geqslant 1}\mu_i\epsilon_i\Big)_c+2c\Big(\Lambda_0^{\mathfrak{c}}\Big|\sum_{i\geqslant 1}\mu_i\epsilon_i\Big)_c$$
$$=\sum_{i\geqslant 1}\mu_i(\mu_i-2i)-2c\big|\mu^\circ\big|=\big(\mu^\circ+\rho_1\big|\mu^\circ\big)_1-2c\big|\mu^\circ\big|,$$

where $\mu^\circ=(\mu_1,\mu_2,\ldots)$. On the other hand, we have

$$\big(\vartheta(\mu)+2\rho_c\big|\vartheta(\mu)\big)_c=\Big(\sum_{i\geqslant 1}\mu_i'\epsilon_i+2\rho_c\Big|\sum_{i\geqslant 1}\mu_i'\epsilon_i\Big)_c-4c\Big(\Lambda_0^{\mathfrak{d}}\Big|\sum_{i\geqslant 1}\mu_i'\epsilon_i\Big)_c$$
$$=\sum_{i\geqslant 1}\mu_i'\big(\mu_i'-2(i-1)\big)+2c\big|\mu^\circ\big|$$
$$=-\big(\mu^\circ+\rho_1\big|\mu^\circ\big)_1+2c\big|\mu^\circ\big|=-(\mu+2\rho_c|\mu)_c.$$

The identity (4.7) was used in the second last identity above.

Now let $\mathfrak{x}=\mathfrak{b}$. One shows that

$$(\mu+2\rho_c|\mu)_c+\big(\vartheta(\mu)+2\rho_c\big|\vartheta(\mu)\big)_c=\sum_{i\geqslant 1}\mu_i(\mu_i-2i+1)+\sum_{i\geqslant 1}\mu_i'\big(\mu_i'-2i+1\big).$$

Now (4.7) says that the right-hand side is zero. □

Theorem 6.5. *Let* $k\in\mathbb{Z}_+$, *and* ϑ *as in* (6.7) *or* (6.8). *We have the following isomorphisms of* $\mathfrak{l}$*-modules:*

$$\mathrm{H}_k\big(\mathfrak{u}_-;L\big(\mathfrak{d}_\infty,\Lambda^{\mathfrak{d}}_-(\lambda)\big)\big)\cong\bigoplus_{w\in W_k^0(\mathfrak{c})}L\big(\mathfrak{l},\vartheta\big(w\circ\Lambda^{\mathfrak{c}}_+(\lambda)\big)\big),\quad \lambda\in\mathcal{P}\big(\mathrm{Sp}(d)\big),$$
$$\mathrm{H}_k\big(\mathfrak{u}_-;\mathcal{L}\big(\mathfrak{c}_\infty,\Lambda^{\mathfrak{c}}_-(\lambda)\big)\big)\cong\bigoplus_{w\in W_k^0(\mathfrak{d})}\mathcal{L}\big(\mathfrak{l},\vartheta^{-1}\big(w\circ\Lambda^{\mathfrak{d}}_+(\lambda)\big)\big),\quad \lambda\in\mathcal{P}\big(\mathrm{O}(d)\big),$$
$$\mathrm{H}_k\big(\mathfrak{u}_-;L\big(\mathfrak{b}^\circ_\infty,\Lambda^{\mathfrak{b}^\circ}_-(\lambda)\big)\big)\cong\bigoplus_{w\in W_k^0(\mathfrak{b})}L\big(\mathfrak{l},\vartheta\big(w\circ\Lambda^{\mathfrak{b}}(\lambda)\big)\big),\quad \lambda\in\mathcal{P}\big(\mathrm{Pin}(d)\big).$$

In particular, we have $\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{d}_\infty,\Lambda^{\mathfrak{d}}_\mp(\lambda)))]=\omega(\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{c}_\infty,\Lambda^{\mathfrak{c}}_\pm(\lambda)))])$ *and* $\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{b}^\circ_\infty,\Lambda^{\mathfrak{b}^\circ}_-(\lambda)))]=\omega(\mathrm{ch}[\mathrm{H}_k(\mathfrak{u}_-;L(\mathfrak{b}_\infty,\Lambda^{\mathfrak{b}}(\lambda)))])$.

Proof. The result follows from the same type of argument as the one used in the proof of Theorem 5.7, now using Lemma 6.4. We leave the details to the reader. □

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Appendix A. Two new reductive dual pairs

A.1. The $(\mathfrak{spo}(2m|2n+1), \mathrm{Pin}(d))$-duality

Let $d = 2\ell$ be even.

There exists a commuting action of $\mathfrak{spo}(2m|2n+1)$ and $\mathrm{Pin}(d)$ on $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$ as follows. We have

$$S\big(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}}\big) \cong S\big(\mathbb{C}^{m|n} \otimes \mathbb{C}^{d}\big) \otimes S\big(\mathbb{C}^{0|1} \otimes \mathbb{C}^{\frac{d}{2}}\big).$$

On $S(\mathbb{C}^{m|n} \otimes \mathbb{C}^{d})$ we have an action of the Howe dual pair $(\mathfrak{spo}(2m|2n), \mathrm{O}(d))$ by Proposition 2.8. On the other hand on $S(\mathbb{C}^{0|1} \otimes \mathbb{C}^{\frac{d}{2}}) \cong \Lambda(\mathbb{C}^{\frac{d}{2}})$ the Lie algebra $\mathfrak{so}(d)$ acts by two irreducible spin representations, giving rise to an irreducible representation of $\mathrm{Pin}(d)$. Since representations of $\mathrm{O}(d)$ pull back to representations of $\mathrm{Pin}(d)$ we obtain a commuting action of $\mathfrak{spo}(2m|2n)$ and $\mathrm{Pin}(d)$ on $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$. This action of $\mathrm{Pin}(d)$ does not factor through $\mathrm{O}(d)$. Furthermore the commuting action of $\mathfrak{spo}(2m|2n)$ extends to a commuting action of $\mathfrak{spo}(2m|2n+1)$. It follows from Appendix A.1.1 and arguments similar to [11, Proposition 4.1] that $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$ is a unitarizable, and hence a completely reducible, $\mathfrak{spo}(2m|2n+1)$-module.

A.1.1. Formulas for $\mathfrak{so}(d)$- and $\mathfrak{spo}(2m|2n+1)$-action on $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$

We introduce even indeterminates $\mathbf{x} := \{x_i^k, \overline{x}_i^k\}$ and odd indeterminates $\xi := \{\xi_j^k, \overline{\xi}_j^k, \xi_0^k\}$, where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant \ell$. We identify $S(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}})$ with the polynomial superalgebra $\mathbb{C}[\mathbf{x}, \xi]$. Then the actions of $\mathfrak{spo}(2m|2n+1)$ and $\mathfrak{so}(d)$ may be realized as differential operators as follows.

- *Formulas for the $\mathfrak{so}(d)$-action.*

$$\frac{1}{2}\delta_{ij} - \xi_0^j \frac{\partial}{\partial \xi_0^i} + \sum_{t=1}^{n}\left(\xi_t^i \frac{\partial}{\partial \xi_t^j} - \overline{\xi}_t^j \frac{\partial}{\partial \overline{\xi}_t^i}\right) + \sum_{s=1}^{m}\left(x_s^i \frac{\partial}{\partial x_s^j} - \overline{x}_s^j \frac{\partial}{\partial \overline{x}_s^i}\right), \quad 1 \leqslant i, j \leqslant \ell, \tag{A.1}$$

$$\frac{\partial}{\partial \xi_0^i}\frac{\partial}{\partial \xi_0^j} + \sum_{t=1}^{n}\left(\xi_t^i \frac{\partial}{\partial \overline{\xi}_t^j} - \xi_t^j \frac{\partial}{\partial \overline{\xi}_t^i}\right) + \sum_{s=1}^{m}\left(x_s^i \frac{\partial}{\partial \overline{x}_s^j} - x_s^j \frac{\partial}{\partial \overline{x}_s^i}\right), \quad 1 \leqslant i, j \leqslant \ell;\ i \neq j,$$

$$\xi_0^i \xi_0^j + \sum_{t=1}^{n}\left(\overline{\xi}_t^i \frac{\partial}{\partial \xi_t^j} - \overline{\xi}_t^j \frac{\partial}{\partial \xi_t^i}\right) + \sum_{s=1}^{m}\left(\overline{x}_s^i \frac{\partial}{\partial x_s^j} - \overline{x}_s^j \frac{\partial}{\partial x_s^i}\right), \quad 1 \leqslant i, j \leqslant \ell;\ i \neq j. \tag{A.2}$$

- *Formulas for the $\mathfrak{spo}(2m|2n+1)$-action.*

$$\sum_{k=1}^{\ell}\left(x_i^k \frac{\partial}{\partial x_j^k} + \overline{x}_i^k \frac{\partial}{\partial \overline{x}_j^k}\right) + \ell\delta_{ij}, \quad 1 \leqslant i, j \leqslant m,$$

$$I_{x_ix_j} := \sum_{k=1}^{\ell}\big(x_i^k\bar{x}_j^k + \bar{x}_i^k x_j^k\big), \qquad \Delta_{x_ix_j} := \sum_{k=1}^{\ell}\left(\frac{\partial}{\partial x_i^k}\frac{\partial}{\partial \bar{x}_j^k} + \frac{\partial}{\partial \bar{x}_i^k}\frac{\partial}{\partial x_j^k}\right),$$
$$1 \leqslant i, j \leqslant m, \tag{A.3}$$
$$\sum_{k=1}^{\ell}\left(\xi_i^k\frac{\partial}{\partial \xi_j^k} + \bar{\xi}_i^k\frac{\partial}{\partial \bar{\xi}_j^k}\right) - \ell\delta_{ij}, \quad 1 \leqslant i, j \leqslant n,$$

$$I_{\xi_i\xi_j} := \sum_{k=1}^{\ell}\big(\xi_i^k\bar{\xi}_j^k + \bar{\xi}_i^k\xi_j^k\big), \qquad \Delta_{\xi_i\xi_j} := \sum_{k=1}^{\ell}\left(\frac{\partial}{\partial \xi_i^k}\frac{\partial}{\partial \bar{\xi}_j^k} + \frac{\partial}{\partial \bar{\xi}_i^k}\frac{\partial}{\partial \xi_j^k}\right), \quad 1 \leqslant i, j \leqslant n;\ i \neq j,$$

$$I_{\xi_0\xi_i} := \sum_{k=1}^{\ell}\left(\xi_i^k\xi_0^k + \bar{\xi}_i^k\frac{\partial}{\partial \xi_0^k}\right), \qquad \Delta_{\xi_0\xi_i} := \sum_{k=1}^{\ell}\left(\frac{\partial}{\partial \xi_0^k}\frac{\partial}{\partial \xi_i^k} + \xi_0^k\frac{\partial}{\partial \bar{\xi}_i^k}\right), \quad 1 \leqslant i \leqslant n,$$

$$\Delta_{\xi_0x_i} := \sum_{k=1}^{\ell}\left(\frac{\partial}{\partial \xi_0^k}\frac{\partial}{\partial x_i^k} + \xi_0^k\frac{\partial}{\partial \bar{x}_i^k}\right), \qquad I_{\xi_0x_i} := \sum_{k=1}^{\ell}\left(\xi_0^k x_i^k + \bar{x}_i^k\frac{\partial}{\partial \xi_0^k}\right), \quad 1 \leqslant i \leqslant m,$$

$$\Delta_{x_i\xi_j} := \sum_{k=1}^{\ell}\left(\frac{\partial}{\partial \xi_j^k}\frac{\partial}{\partial \bar{x}_i^k} + \frac{\partial}{\partial \bar{\xi}_j^k}\frac{\partial}{\partial x_i^k}\right), \qquad I_{x_i\xi_j} := \sum_{k=1}^{\ell}\big(x_i^k\bar{\xi}_j^k + \bar{x}_i^k\xi_j^k\big),$$
$$1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n, \tag{A.4}$$
$$\sum_{k=1}^{\ell}\left(x_i^k\frac{\partial}{\partial \xi_j^k} + \bar{x}_i^k\frac{\partial}{\partial \bar{\xi}_j^k}\right), \qquad \sum_{k=1}^{\ell}\left(\xi_j^k\frac{\partial}{\partial x_i^k} + \bar{\xi}_j^k\frac{\partial}{\partial \bar{x}_i^k}\right), \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n. \tag{A.5}$$

A.1.2. The module decomposition

It is evident from (A.1) that the action of $\mathrm{Pin}(d)$ on $\mathbb{C}[\mathbf{x}, \xi]$ does not factor through $\mathrm{O}(d)$. Since $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathbf{x}, \xi])^{\mathrm{Pin}(d)}$ is generated by $\mathfrak{spo}(2m|2n+1)$ in Appendix A.1.1, it follows from the double commutant theorem that with respect to the $\mathfrak{spo}(2m|2n+1) \times \mathrm{Pin}(d)$-action we have

$$\mathbb{C}[\mathbf{x}, \xi] \cong \bigoplus_{\lambda \in \Lambda} L\big(\mathfrak{spo}(2m|2n+1), \Lambda_f^{\mathfrak{b}}(\lambda)\big) \otimes V_{\mathrm{Pin}(d)}^{\lambda}, \tag{A.6}$$

where $\Lambda \subseteq \mathcal{P}(\mathrm{Pin}(d))$, and $\Lambda_f^{\mathfrak{b}} : \Lambda \to \bar{\mathfrak{h}}^*$ is an injection. It remains to determine the set Λ and the map $\Lambda_f^{\mathfrak{b}}$.

Theorem A.1. *As an $\mathfrak{spo}(2m|2n+1) \times \mathrm{Pin}(d)$-module we have*

$$S\big(\mathbb{C}^{2m|2n+1} \otimes \mathbb{C}^{\frac{d}{2}}\big) \cong \bigoplus_{\lambda} L\big(\mathfrak{spo}(2m|2n+1), \Lambda_f^{\mathfrak{b}}(\lambda)\big) \otimes V_{\mathrm{Pin}(d)}^{\lambda},$$

where the summation is over all $\lambda \in \mathcal{P}(\mathrm{Pin}(d))$ with $\lambda_{m+1} \leqslant n$, and $\Lambda_f^{\mathfrak{b}}(\lambda) = \lambda^{\natural} + \frac{d}{2}\mathbf{1}_{m|n}$.

Proof. Let $\bar{\mathfrak{u}}_+$ be the algebra generated by the Δ-operators, i.e. $\Delta_{x_ix_j}$, $\Delta_{\xi_i\xi_j}$, $\Delta_{\xi_0\xi_j}$, $\Delta_{\xi_0x_i}$, and $\Delta_{x_i\xi_j}$. Then $\bar{\mathfrak{u}}_+$ is invariant under the adjoint action of $\mathfrak{gl}(m|n)$.

An element $f \in \mathbb{C}[\mathbf{x}, \xi]$ is called *harmonic*, if f is annihilated by $\bar{\mathfrak{u}}^+$. The space of harmonics will be denoted by $\mathcal{H}$ and it evidently admits an action of $\mathfrak{gl}(m|n) \times \mathrm{Pin}(d)$. Furthermore, since $S(\mathbb{C}^{2m|2n+1}\otimes\mathbb{C}^{\frac{d}{2}})$ is a completely reducible $\mathfrak{gl}(m|n)$-module, $L(\mathfrak{spo}(2m|2n+1),\mu)^{\bar{\mathfrak{u}}^+}$ is also completely reducible over $\mathfrak{gl}(m|n)$, for any irreducible $\mathfrak{spo}(2m|2n+1)$-module $L(\mathfrak{spo}(2m|2n+1),\mu)$ that appears in $S(\mathbb{C}^{2m|2n+1}\otimes\mathbb{C}^{\frac{d}{2}})$. By irreducibility of $L(\mathfrak{spo}(2m|2n+1),\mu)$ we must have

$$L\big(\mathfrak{spo}(2m|2n+1),\mu\big)^{\bar{\mathfrak{u}}^+}\cong L\big(\mathfrak{gl}(m|n),\mu\big).$$

So, by (A.6) $(\mathrm{Pin}(d),\mathfrak{gl}(m|n))$ forms a Howe dual pair on $\mathcal{H}$. Thus, proving the theorem is equivalent to establishing the following decomposition of $\mathcal{H}$ as a $\mathfrak{gl}(m|n)\times\mathrm{Pin}(d)$-module:

$$\mathcal{H}\cong\bigoplus_{\lambda}L\big(\mathfrak{gl}(m|n),\Lambda_f^{\mathfrak{b}}(\lambda)\big)\otimes V_{\mathrm{Pin}(d)}^{\lambda},\tag{A.7}$$

where the summation is over all $\lambda\in\mathcal{P}(\mathrm{Pin}(d))$, i.e. $\ell(\lambda)\leqslant\ell$, with $\lambda_{m+1}\leqslant n$, and $\Lambda_f^{\mathfrak{b}}(\lambda)=\lambda^{\natural}+\ell\mathbf{1}_{m|n}$.

We first consider the limit case $n=\infty$ with the space of harmonics denoted by $\mathcal{H}^\infty$. Here the only restriction on λ is $\ell(\lambda)\leqslant\ell$, and we observe that the vector given in [7, Theorems 4.1 and 4.2] associated to such a partition λ is indeed annihilated by $\bar{\mathfrak{u}}^+$ and hence is a joint $\mathfrak{gl}(m|n)\times\mathrm{Pin}(d)$-highest weight vector of weight $(\lambda,\lambda^{\natural}+\ell\mathbf{1}_{m|n})$. Hence all the summands on the right-hand side of (A.7) occur in the space of harmonics, and in particular, all irreducible representations of $\mathrm{Pin}(d)$ occur. Therefore, we have established (A.7) in the case $n=\infty$.

Now consider the finite n case. We may regard $S(\mathbb{C}^{2m|1+2n}\otimes\mathbb{C}^{\frac{d}{2}})\subseteq S(\mathbb{C}^{2m|1+2\infty}\otimes\mathbb{C}^{\frac{d}{2}})$ with compatible actions $\mathfrak{spo}(2m|1+2n)\subseteq\mathfrak{spo}(2m|1+2\infty)$. From the formulas of the Δ-operators we see that $\mathcal{H}\subseteq\mathcal{H}^\infty$. Thus $\mathcal{H}$ is obtained from $\mathcal{H}^\infty$ by setting the variables $\xi_j^k=\bar{\xi}_j^k=0$, for $j>n$. However, it is clear, from the explicit formulas of the joint highest vectors in $\mathcal{H}^\infty$, that, when setting the variables $\xi_j^k=\bar{\xi}_j^k=0$ for $j>n$, precisely those vectors corresponding to λ with $\lambda_{m+1}\leqslant n$ will survive. □

Remark A.2. Theorem A.1 is a finite-dimensional analogue of [8, Theorem 8.2].

A.2. The Lie superalgebra $\mathfrak{b}_\infty^\circ$ and the $(\mathfrak{b}_\infty^\circ,\mathrm{Pin}(d))$-duality

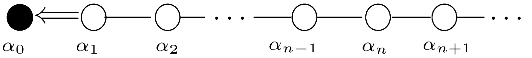
Consider the superspace with basis $\{v_{\frac12},v_k\mid k\in\mathbb{Z}\}$ with $\deg v_k=\bar{0}$, for $k\in\mathbb{Z}$, and $\deg v_{\frac12}=\bar{1}$. The Lie superalgebra $\overline{\mathfrak{b}_\infty^\circ}$ is the subalgebra of the general linear superalgebra preserving the even super-skewsymmetric bilinear form determined by $(v_i|v_j):=(-1)^i\delta_{i,1-j}$, $i,j\in\mathbb{Z}$, and $(v_{\frac12}|v_{\frac12}):=1$. Now consider the central extension of general linear superalgebra corresponding to the 2-cocycle $\gamma(A,B):=\mathrm{Str}((\sum_{j<1}E_{jj})[A,B])$, where $\mathrm{Str}(c_{ij}):=-c_{\frac12\frac12}+\sum_{j\in\mathbb{Z}}c_{jj}$. The Lie superalgebra $\mathfrak{b}_\infty^\circ$ is the central extension of $\overline{\mathfrak{b}_\infty^\circ}$ by a one-dimensional center $\mathbb{C}K$, obtained via restriction of the cocycle γ.

By construction one sees that $\mathfrak{c}_\infty\subseteq\mathfrak{b}_\infty^\circ$, and also $\mathfrak{c}_\infty$ and $\mathfrak{b}_\infty^\circ$ share the same Cartan subalgebra $\mathfrak{h}$. This allows us to identify their Cartan and dual Cartan subalgebras. For convenience of the reader we list below the simple roots and coroots for $\mathfrak{b}_\infty^\circ$.

$$\Pi^\vee=\big\{\alpha_0^\vee=\widetilde{E}_1+K,\ \alpha_1^\vee=\widetilde{E}_i-\widetilde{E}_{i+1}\ (i\in\mathbb{N})\big\},$$
$$\Pi=\big\{\alpha_0=-\epsilon_1,\ \alpha_i=\epsilon_i-\epsilon_{i+1}\ (i\in\mathbb{N})\big\},$$

$$\Delta_+ = \{\pm\epsilon_i - \epsilon_j,\ -\epsilon_i,\ -2\epsilon_i\ (i \in \mathbb{N},\ i < j)\}.$$

The associated Dynkin diagram is as follows:



Let $\rho_c, \Lambda_0^{\mathfrak{b}^\circ} \in \mathfrak{h}^*$ be determined by $\langle \rho_c, \widetilde{E}_j \rangle = -j + \frac{1}{2}$, $\langle \rho_c, K \rangle = \langle \Lambda_0^{\mathfrak{b}^\circ}, \widetilde{E}_j \rangle = 0$, and $\langle \Lambda_0^{\mathfrak{b}^\circ}, K \rangle = 1$, $j \in \mathbb{N}$. For $c \in \mathbb{C}$ let $\mathcal{P}^{+,\mathfrak{b}^\circ}_{\mathfrak{l},c}$ consist of elements in $\mathfrak{h}^*$ of the form $c\Lambda_0^{\mathfrak{b}^\circ} + \sum_{i \geqslant 1} \mu_i \epsilon_i$, where $(\mu_1, \mu_2, \ldots) \in \mathcal{P}^+$.

In Section 6 the following bilinear form $(\cdot|\cdot)_c$ on $\mathfrak{b}^\circ_\infty$ is used. We first choose a bilinear form $(\cdot|\cdot)_c$ on $\mathfrak{h}^*$ satisfying

$$(\lambda|\epsilon_i)_c = \langle \lambda, \widetilde{E}_i - K \rangle, \quad i \in \mathbb{N},$$
$$\big(\Lambda_0^{\mathfrak{b}^\circ} \big| \Lambda_0^{\mathfrak{b}^\circ}\big)_c = \big(\Lambda_0^{\mathfrak{b}^\circ} \big| \rho_c\big)_c = 0.$$

One checks that $(\epsilon_i|\epsilon_j)_c = \delta_{ij}$, $(\Lambda_0^{\mathfrak{b}^\circ}|\epsilon_i)_c = -1$ for $i, j \in \mathbb{N}$, and $2(\rho_c|\alpha_i)_c = (\alpha_i|\alpha_i)_c$ for $i \in \mathbb{Z}_+$. Let $\{s_i^{\mathfrak{b}^\circ}\}_{i \in \mathbb{Z}_+}$ be the sequence defined by

$$s_i^{\mathfrak{b}^\circ} := \begin{cases} -1, & \text{if } i = 0, \\ 1, & \text{if } i \geqslant 1. \end{cases}$$

It follows that by defining $s_i^{\mathfrak{b}^\circ} s_j^{\mathfrak{b}^\circ} (\alpha_i^\vee | \alpha_j^\vee)_c := (\alpha_i|\alpha_j)_c$, we obtain a symmetric bilinear form on $\mathfrak{h}$. This form can be extended to a non-degenerate invariant super-symmetric bilinear form on $\mathfrak{b}^\circ_\infty$ such that

$$(e_i|f_j)_c = \delta_{ij}/s_i^{\mathfrak{b}^\circ}, \tag{A.8}$$

where e_i and f_j $(i, j \in \mathbb{Z}_+)$ denote the Chevalley generators with $[e_i, f_i] = \alpha_i^\vee$. Now we can define the Casimir operator Ω as in [4, Section 4.1].

Let $d = 2\ell$ be even. By [31, Theorem 5.3] there exists a $(\mathfrak{c}_\infty, \mathrm{O}(d))$-duality on $\mathfrak{F}^{-\frac{d}{2}}$. Now $\Lambda(\mathbb{C}^{\frac{d}{2}})$ is an irreducible $\mathrm{Pin}(d)$-module and hence we have a commuting action of $\mathfrak{c}_\infty$ and $\mathrm{Pin}(d)$ on $\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda(\mathbb{C}^{\frac{d}{2}})$. One can show that the action of $\mathfrak{c}_\infty$ on $\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda(\mathbb{C}^{\frac{d}{2}})$ extends to an action of $\mathfrak{b}^\circ_\infty$ that commutes with the action of $\mathrm{Pin}(d)$, and furthermore $\mathfrak{b}^\circ_\infty$ generates $\mathrm{End}(\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda(\mathbb{C}^{\frac{d}{2}}))^{\mathrm{Pin}(d)}$.

In the sequel we will need to have this commuting action in a more explicit form. For this let us introduce odd indeterminates $\xi_{\frac{1}{2}} := \{\xi^k_{\frac{1}{2}} \mid k = 1, \ldots, \ell\}$, and identify $\Lambda(\mathbb{C}^{\frac{d}{2}})$ with the Clifford superalgebra generated by $\xi_{\frac{1}{2}}$. The action of the Lie algebra $\mathfrak{so}(d)$ on $\Lambda(\mathbb{C}^{\frac{d}{2}})$ in terms of differential operators in $\xi^k_{\frac{1}{2}}$ is explicitly given by the summands involving ξ^k_0 in (A.1) and (A.2). This action combined with the action of $\mathfrak{so}(d)$ on $\mathfrak{F}^{-\frac{d}{2}}$ as in [31, (5.52)], gives the action of $\mathfrak{so}(d)$ on $\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda(\mathbb{C}^{\frac{d}{2}})$. The commuting action of $\mathfrak{b}^\circ_\infty$ is as follows. First the subalgebra $\mathfrak{c}_\infty$ acts on $\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda(\mathbb{C}^{\frac{d}{2}})$ only on the first factor as in [31, Section 5.2]. To complete the description we only need to give formulas for the action of the odd root vectors $I_{\xi_{\frac{1}{2}}i}$ and $\Delta_{\xi_{\frac{1}{2}}i}$, corresponding to the roots ϵ_i and $-\epsilon_i$, $i \in \mathbb{N}$, respectively. They are as follows:

$$I_{\xi_{\frac{1}{2}}i} := \sum_{k=1}^{\ell}\left(\xi^k_{\frac{1}{2}}\gamma^{+,k}_{-i+\frac{1}{2}} - \frac{\partial}{\partial\xi^k_{\frac{1}{2}}}\gamma^{-,k}_{-i+\frac{1}{2}}\right), \qquad \Delta_{\xi_{\frac{1}{2}}i} := \sum_{k=1}^{\ell}\left(\xi^k_{\frac{1}{2}}\gamma^{+,k}_{i-\frac{1}{2}} + \frac{\partial}{\partial\xi^k_{\frac{1}{2}}}\gamma^{-,k}_{i-\frac{1}{2}}\right).$$

We have the following:

Theorem A.3. *As a* $\mathfrak{b}^\circ_\infty \times \mathrm{Pin}(d)$*-module we have*

$$\mathfrak{F}^{-\frac{d}{2}} \otimes \Lambda\left(\mathbb{C}^{\frac{d}{2}}\right) \cong \bigoplus_{\lambda \in \mathcal{P}(\mathrm{Pin}(d))} L\left(\mathfrak{b}^\circ_\infty, \Lambda^{\mathfrak{b}^\circ}_-(\lambda)\right) \otimes V^\lambda_{\mathrm{Pin}(d)}, \tag{A.9}$$

where $\Lambda^{\mathfrak{b}^\circ}_-(\lambda) := -\frac{d}{2}\Lambda^{\mathfrak{b}^\circ}_0 + \sum_{k=1}^{\frac{d}{2}} \lambda_k \epsilon_k$.

Proof. One checks that the $\mathfrak{c}_\infty \times \mathfrak{so}(d)$-joint highest weight vectors inside $\mathfrak{F}^{-\frac{d}{2}}$ given in [31, (5.54)] are also $\mathfrak{b}^\circ_\infty$-highest weight vectors. For this it is enough to check that they are annihilated by the root vector $\Delta_{\xi_{\frac{1}{2}}1}$ (corresponding to the simple odd root $-\epsilon_1$). This, however, is easy. The duality then follows from the fact that this set of vectors exhaust all irreducible finite-dimensional $\mathrm{Pin}(d)$-highest weights that do not factor through $\mathrm{O}(d)$. □

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