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A note on compact gradient Yamabe solitons

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ABSTRACT

We will give a simple proof that the metric of any compact Yamabe gradient soliton (M, g) is a metric of constant scalar curvature when the dimension of the manifold $n \geqslant 3$.

Recently there has been a lot of work on the Yamabe flow on manifolds by S. Brendle [1,2], B. Chow [4], P. Daskalopoulos and N. Sesum [5], S.Y. Hsu [8], A. Burchard, R.J. Mccan and A. Smith [3], L. Ma and L. Cheng [6], M. Del Pino, M. Sáez [7] and others. A time dependent metric $g(\cdot, t)$ on a Riemannian manifold M is said to evolve by the Yamabe flow if the metric g satisfies

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij} \tag{1}$$

on M where R is the scalar curvature. Yamabe gradient solitons are special solutions of Yamabe flow (1). We say that a metric g_{ij} on a Riemannian manifold M is a Yamabe gradient soliton if there exist a smooth function $f : M \to \mathbb{R}$ and a constant $\rho \in \mathbb{R}$ such that

$$(R - \rho) g_{ij} = \nabla_i \nabla_j f \quad \text{on } M. \tag{2}$$

It is proved in [5] that the metric of any compact Yamabe gradient soliton (M, g) is a metric of constant scalar curvature. In this paper we will give a simple alternate proof of this interesting result.

The main theorem of this paper is the following.

Theorem 1. *Let (M, g) be an n-dimensional compact Yamabe gradient soliton with $n \geqslant 3$. Then (M, g) is a manifold of constant scalar curvature.*

Proof. As observed in p. 20 of [5] (cf. [4]) by (2) and a direct computation one has

$$(n-1)\Delta_g R + \frac{1}{2}\langle \nabla R, \nabla f \rangle_g + R(R - \rho) = 0. \tag{3}$$

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Tracing (2) over i, j,

$$n(R-\rho)=\Delta f \tag{4}$$

$$\Rightarrow \quad \int\limits_M (R-\rho)\,dV=\frac{1}{n}\int\limits_M \Delta f\,dV=0. \tag{5}$$

Integrating (3) over M by (4) we get

$$\begin{aligned}\int\limits_M R(R-\rho)\,dV &= -\frac{1}{2}\int\limits_M \langle\nabla R,\nabla f\rangle_g\,dV\\ &= \frac{1}{2}\int\limits_M R\Delta f\,dV\\ &= \frac{n}{2}\int\limits_M R(R-\rho)\,dV.\end{aligned} \tag{6}$$

Since $n\geqslant 3$, by (6),

$$\int\limits_M R(R-\rho)\,dV=0. \tag{7}$$

By (5) and (7),

$$\int\limits_M (R-\rho)^2\,dV=0.$$

Hence $R\equiv\rho$ on M and the theorem follows. □

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