## Trees

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To the memory of Raymond Stora


#### Abstract

An algebraic formalism, developed with V. Glaser and R. Stora for the study of the generalized retarded functions of quantum field theory, is used to prove a factorization theorem which provides a complete description of the generalized retarded functions associated with any tree graph. Integrating over the variables associated to internal vertices to obtain the perturbative generalized retarded functions for interacting fields arising from such graphs is shown to be possible for a large category of space-times. © 2016 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP ${ }^{3}$.


## 1. Introduction

The general properties of generalized retarded $n$-point functions in general field theory were discovered and studied by several authors in the 1960s [12-15,7,1,2,4]. In particular [4] gives support properties which lead to the full primitive domain of analyticity in momentum space. In the 1970s a new presentation, inspired by perturbation theory, was described in [9,10]. It is based on an algebraic structure which has some interest in itself, and is well-adapted to not necessarily Minkowskian space-times. This paper presents an application of this algebraic formalism for which Raymond Stora always had a liking - to a small problem in perturbation theory. In perturbation theory the time-ordered or retarded functions of interacting fields are obtained by integrating graphs in the variables attached to some internal vertices (see Sect. 5 for more de-

[^0]tails). In Minkowski space-time this integration is always feasible in the absence of zero masses (see e.g. [8]), but not always when they occur. This is even true in the case of tree graphs. It was remarked by J. Bros, in the few-vertex case, that it is still possible to obtain the retarded function associated to a tree graph. Here we will see (Sects. 4 and 5) that the generalized retarded functions for the interacting fields associated to a tree graph can always be obtained by integrating the variables attached to the internal vertices over a bounded region, provided one deals with a space-time in which the double-cones are bounded. This is based on a factorization theorem (Corollary 3.1) which follows naturally from the algebraic formalism mentioned above, and which also yields a complete description of the generalized retarded functions associated to any tree graph (Sect. 4).

Let $\mathcal{X}$ denote a "space-time". This is a smooth manifold which can be the $d$-dimensional Minkowski space $\left(M_{d}\right)$, or the $d$-dimensional de Sitter space $\left(d S_{d}\right)$, or the universal cover of the $d$-dimensional Anti-de Sitter space $\left(\widetilde{A d S_{d}}\right)$, or even a more general space-time. We suppose that a closed reflexive relation denoted $x \leq y$ (or equivalently $y \geq x$ ) is defined in $\mathcal{X}$. This need not be an order relation, but it is in the three examples mentioned above. The relation $x \lesssim y$ (or equivalently $y \gtrsim x$ ) is defined as the negation of $y \leq x$. If $A$ and $B$ are subsets of $\mathcal{X}$, we denote

$$
\begin{equation*}
A \lesssim B \Longleftrightarrow B \gtrsim A \Longleftrightarrow(\forall x \in A \quad \forall y \in B, \quad x \lesssim y) \tag{1}
\end{equation*}
$$

Thus (1) means that there is no $x \in A$ and no $y \in B$ such that $x \geq y$. If $x \in \mathcal{X}$, the future (resp. past) set of $x$ is the (closed) set of all $y$ such that $y \geq x$ (resp. $y \leq x$ ). If $A \subset \mathcal{X}$, the future (resp. past) set of $A$ is the union of the future (resp. past) sets of all the elements of $A$. The condition (1) means that $B$ does not intersect the past set of $A$, or, equivalently, that $A$ does not intersect the future set of $B$. If $x, y \in \mathcal{X}$ the set $\{z \in \mathcal{X}: y \leq z \leq x\}$ is called the double-cone with vertices $x$ and $y$. If $\leq$ is an order relation, it is empty unless $y \leq x$. In Minkowski space, and in de Sitter space $d S_{d}$ viewed as a hyperboloid imbedded in a $(d+1)$-dimensional Minkowski space, $x \leq y \Leftrightarrow y \in x+\overline{V_{+}}$, and $(A \lesssim B) \Longleftrightarrow\left(A \cap\left(B+\overline{V_{+}}\right)=\emptyset\right)$, where, as usual, $V_{+}=$ $\left\{\left(x^{0}, \vec{x}\right): x^{0}>|\vec{x}|\right\}=-V_{-}$and $\overline{V_{+}}=\left\{\left(x^{0}, \vec{x}\right): x^{0} \geq|\vec{x}|\right\}=-\overline{V_{-}}$. In the Minkowski and de Sitter spaces, the double-cones are compact.

Let $X$ be a finite set of (distinct) indices with cardinal denoted $|X|$. By $\mathcal{X}^{X}$ we denote $\mathcal{X}^{|X|}$ or more precisely the set of maps $X \rightarrow \mathcal{X} . \mathcal{P}(X)$ denotes the set of subsets of $X . \mathcal{P}_{*}(X)$ denotes the set of proper subsets of $X$ i.e. $\mathcal{P}_{*}(X)=\{J \subset X: J \neq \emptyset$ and $J \neq X\}$. A proper sequence in $\mathcal{P}(X)$ is a sequence $\left\{J_{1}, \ldots, J_{v}\right\}$ of disjoint non-empty subsets of $X$ with union $X$. A linear system of generalized time-ordered functions (GTOF) in variables indexed by $X$ is a set of distributions on $\mathcal{X}^{X}$, indexed by the proper sequences in $\mathcal{P}(X)$, usually denoted

$$
\begin{equation*}
t_{J_{1}, \ldots, J_{v}} \text { or } t_{J_{1}, \ldots, J_{v}}^{c} \text { (connected, or truncated version) } \tag{2}
\end{equation*}
$$

and having the property of being symmetric in the variables with indices contained in any given $J_{k}$. In addition we suppose that for any $k \in X$ these distributions are $\mathcal{C}^{\infty}$ in the variable $x_{k}$ when smeared with smooth test-functions in the remaining variables (let us call this the property of partial regularity, or PR). In the Minkowski case this is a consequence of the translational invariance which we will always impose, together with the usual spectral assumptions, on these distributions. In the de Sitter or Anti-de Sitter case, PR is a consequence of the group invariance if it holds. In more general space-times it follows from the microlocal properties imposed by several authors as a substitute for translational invariance (see [5] and references therein).

Last but not least, the set of distributions $\left\{t_{J_{1}, \ldots, J_{v}}\right\}$ must have the property of causal factorization. This means that if $X=A \cup B, A \cap B=\emptyset$, then

$$
\begin{equation*}
t_{J_{1}, \ldots, J_{v}}-t_{J_{1} \cap A, J_{1} \cap B \ldots, J_{\nu} \cap A, J_{v} \cap B} \text { vanishes in the open set }\left\{x \in \mathcal{X}^{X}:\{x\}_{A} \gtrsim\{x\}_{B}\right\} \tag{3}
\end{equation*}
$$



Fig. 1. Factorization at an internal vertex.
The last notation means the set of all $x \in \mathcal{X}^{X}$ such that $x_{j} \gtrsim x_{k}$ for all $j \in A$ and all $k \in B$. (If $K$ is a set of indices, $\{x\}_{K}$ denotes $\left\{x_{j}: j \in K\right\}$.) Of course the term "causal factorization" does not mean that the distribution $t_{J_{1}, \ldots, J_{v}}$ actually factorizes, but refers to the property of causal factorization possessed by time-ordered products of local quantum fields. If time-ordered products $T\left(\left\{\phi_{k}\left(x_{k}\right)\right\}_{k \in K}\right)$, abbreviated as $T(K)$, have been defined, and $I \cup J=K, I \cap J=\emptyset$ then $T(K)-T(I) T(J)$ vanishes in $\left\{x:\{x\}_{I} \gtrsim\{x\}_{J}\right\}$, and the distributions

$$
\begin{equation*}
t_{J_{1}, \ldots, J_{v}}=\left(\Omega, T\left(J_{1}\right) \ldots T\left(J_{v}\right) \Omega\right) \tag{4}
\end{equation*}
$$

or their truncated versions

$$
\begin{equation*}
t_{J_{1}, \ldots, J_{v}}^{c}=\left(\Omega, T\left(J_{1}\right) \ldots T\left(J_{v}\right) \Omega\right)_{c} \tag{5}
\end{equation*}
$$

are examples of GTOF. See e.g. [10]. However this paper is concerned with GTOF which exhibit another type of (actual) factorization, described in the next subsection.

### 1.1. Factorization at an index

We consider a linear system of GTOF in variables indexed by a finite set $X=A \cup B \cup\{1\}$, with $1 \notin A \neq \emptyset, 1 \notin B \neq \emptyset, A \cap B=\emptyset$, which factorize as symbolized by Fig. 1 .

This means that the linear system of GTOF $t^{c}$ satisfies

$$
\begin{equation*}
t_{J_{1}, \ldots, J_{v}}^{c}\left(\{x\}_{X}\right)=t_{J_{1} \cap(A \cup 1), \ldots, J_{\nu} \cap(A \cup 1)}^{1, c}\left(\{x\}_{A \cup 1}\right) t_{J_{1} \cap(B \cup 1), \ldots, J_{\nu} \cap(B \cup 1)}^{2, c}\left(\{x\}_{B \cup 1}\right) \tag{6}
\end{equation*}
$$

where the $\left\{t^{1, c}\right\}$ (resp. $\left\{t^{2, c}\right\}$ ) are a linear system of GTOF in variables indexed by $A \cup\{1\}$ (resp. $B \cup\{1\}$ ). In the case when the $t^{c}$ are connected (or truncated) vacuum expectation values of products of time-ordered products of local fields, this takes the form

$$
\begin{align*}
& \left(\Omega, T\left(J_{1}\right) \ldots T\left(J_{v}\right) \Omega\right)_{c} \\
& \quad=\left(\Omega, T\left(J_{1} \cap(A \cup 1)\right) \ldots T\left(J_{v} \cap(A \cup 1)\right) \Omega\right)_{1, c} \\
& \quad \times\left(\Omega, T\left(J_{1} \cap(B \cup 1)\right) \ldots T\left(J_{v} \cap(B \cup 1)\right) \Omega\right)_{2, c} \tag{7}
\end{align*}
$$

This makes sense because of the PR property: smearing with test-functions in the variables indexed by $A$ and $B$ yields a product of two $\mathcal{C}^{\infty}$ functions of $x_{1}$. In addition the product (6) itself has the property PR. This is obvious if the distinguished variable is $x_{1}$. If the distinguished variable is $x_{j}$, with $j \in A$, we first smear in the variables indexed by $B$. The second factor then becomes a $\mathcal{C}^{\infty}$ function of $x_{1}$ and smearing the variables indexed by $\{1\} \cup A \backslash\{j\}$ finally yields a $\mathcal{C}^{\infty}$ function of $x_{j}$. In the Minkowskian case, we can take as variables the two independent groups $\left\{x_{j}-x_{1}: j \in A\right\}$ and $\left\{x_{j}-x_{1}: j \in B\right\}$, so that (6), (7) are really tensor products of distributions.

The main example of this situation is the system of GTOF associated to a perturbative tree graph with vertices labelled by $X$, in which 1 is an internal vertex, and $A \cup\{1\}$ and $B \cup\{1\}$ are the sets of vertices of two subtrees. A more general example is given by any perturbative graph which is splittable at an internal point (see again Fig. 1). In this case, we are really dealing with Wick monomials of free or generalized free fields [3,11,5,17,18], each field indexed by $j \in A$
being of the form : $\phi^{m_{j}}\left(x_{j}\right): \otimes 1$, each field indexed by $k \in B$ being of the form $1 \otimes: \psi^{m_{k}}\left(x_{k}\right)$ :, and the field labelled by 1 being : $\phi^{a}\left(x_{1}\right): \otimes: \psi^{b}\left(x_{1}\right):$, operating in a tensor product of two Fock spaces.

In the Minkowskian case, given a distribution

$$
\begin{align*}
& F\left(\left\{x_{j}^{\prime}\right\}_{j \in A},\left\{x_{k}^{\prime \prime}\right\}_{k \in B}, x_{1}\right)=f\left(\left\{x_{j}^{\prime}-x_{1}\right\}_{j \in A},\left\{x_{k}^{\prime \prime}-x_{1}\right\}_{k \in B}\right)  \tag{8}\\
& \quad=F_{1}\left(\left\{x_{j}^{\prime}\right\}_{j \in A}, x_{1}\right) F_{2}\left(\left\{x_{k}^{\prime \prime}\right\}_{k \in B}, x_{1}\right)  \tag{9}\\
& \quad=f_{1}\left(\left\{x_{j}^{\prime}-x_{1}\right\}_{j \in A}\right) f_{2}\left(\left\{x_{k}^{\prime \prime}-x_{1}\right\}_{k \in B}\right), \tag{10}
\end{align*}
$$

we have, for the Fourier transforms:

$$
\begin{align*}
& \widetilde{F}\left(\left\{p^{\prime}\right\}_{A},\left\{p^{\prime \prime}\right\}_{B}, p_{1}\right)=\delta\left(p_{A}^{\prime}+p_{B}^{\prime \prime}+p_{1}\right) \widetilde{f}\left(\left\{p^{\prime}\right\}_{A},\left\{p^{\prime \prime}\right\}_{B}\right),  \tag{11}\\
& \widetilde{f}\left(\left\{p^{\prime}\right\}_{A},\left\{p^{\prime \prime}\right\}_{B}\right)=\widetilde{f}_{1}\left(\left\{p^{\prime}\right\}_{A}\right) \widetilde{f}_{2}\left(\left\{p^{\prime \prime}\right\}_{B}\right),  \tag{12}\\
& \widetilde{F}_{1}\left(\left\{p^{\prime}\right\}_{A}, p_{1}\right)=\delta\left(p_{A}^{\prime}+p_{1}\right) \widetilde{f}_{1}\left(\left\{p^{\prime}\right\}_{A}\right),  \tag{13}\\
& \widetilde{F}_{2}\left(\left\{p^{\prime \prime}\right\}_{B}, p_{1}\right)=\delta\left(p_{B}^{\prime \prime}+p_{1}\right) \widetilde{f}_{2}\left(\left\{p^{\prime \prime}\right\}_{B}\right), \tag{14}
\end{align*}
$$

where we denote

$$
\begin{equation*}
\left\{p^{\prime}\right\}_{A}=\left\{p_{j}^{\prime}\right\}_{j \in A}, \quad\left\{p^{\prime \prime}\right\}_{B}=\left\{p_{k}^{\prime \prime}\right\}_{k \in B}, \quad p_{J}^{\prime}=\sum_{j \in J} p_{j}^{\prime} \tag{15}
\end{equation*}
$$

## 2. Generalized retarded operators and functions

This section is a summary of a part of [10] which will be applied in the subsequent sections to factorizing systems.

### 2.1. The algebra of sequences $\mathcal{A}(X)$

In this subsection $X$ denotes a finite set such that $|X| \geq 2$. We denote $\mathcal{P}(X)$ the set of subsets of $X$. We denote $\mathcal{P}_{*}(X)$ the set of proper subsets of $X$, i.e.

$$
\begin{equation*}
\mathcal{P}_{*}(X)=\{J \subset X: J \neq \emptyset, \quad J \neq X\} \tag{16}
\end{equation*}
$$

Definition 2.1. A proper sequence in $\mathcal{P}(X)$ is a sequence $\left\{J_{1}, \ldots, J_{\nu}\right\}$, where $\nu=1,2, \ldots,|X|$, the $J_{k}$ are non-empty, disjoint subsets of $X$ with union $X$. A multiplication law is defined for proper sequences by

$$
\begin{align*}
& \left\{A_{1}, \ldots, A_{n}\right\}\left\{B_{1}, \ldots, B_{m}\right\}= \\
&  \tag{17}\\
& \quad\left\{A_{1} \cap B_{1}, \ldots, A_{n} \cap B_{1}, \ldots, A_{1} \cap B_{m}, \ldots, A_{n} \cap B_{m}\right\} \bmod \emptyset
\end{align*}
$$

Here "mod $\emptyset$ " means: "omit every occurrence of the empty set". The algebra $\mathcal{A}(X)$ (called the algebra of sequences in $\mathcal{P}(X)$ ) is the vector space of all formal complex linear combinations of proper sequences in $\mathcal{P}(X)$, equipped with the multiplication generated by (17).

The multiplication is associative and $\mathcal{A}(X)$ has all the usual properties of an algebra. It has a unit, namely $\{X\}$. Any proper sequence $c$ in $\mathcal{P}(X)$ is an idempotent, i.e. $c c=c$. When no ambiguity arises, we denote $\hat{I}=\{I, X \backslash I\}$ for any proper subset $I$ of $X$. Since $\hat{I}$ is a proper sequence, $\hat{I} \hat{I}=\hat{I}$.

The following lemma is easy to prove (see [10] p. 19).

Lemma 2.1. Let $I \in \mathcal{P}_{*}(X), a, b \in \mathcal{A}(X), \hat{I}=\{I, X \backslash I\} \in \mathcal{A}(X)$. If a is such that $(1-\hat{I}) a=0$, then also $(1-\hat{I}) b a=0$.

Definition 2.2. A geometrical cell associated to $X$ is one of the connected components of

$$
\begin{equation*}
\left\{s=\{s\}_{X} \in \mathbf{R}^{|X|}: s_{X}=0, \quad s_{J} \neq 0 \forall J \in \mathcal{P}_{*}(X)\right\} \tag{18}
\end{equation*}
$$

Here $\{s\}_{X}$ denotes the set of variables $\left\{s_{j}: j \in X\right\}$, and, for every subset $J$ of $X, s_{J} \stackrel{\text { def }}{=} \sum_{j \in J} s_{j}$.
A picture of the geometrical cells for $X=\{1,2,3,4\}$ is given in Fig. A. 4 in Appendix A.

Definition 2.3. A paracell associated to $X$ is a non-empty subset $\mathcal{S}$ of $\mathcal{P}_{*}(X)$ such that if $I \in \mathcal{S}$ and $J \in \mathcal{S}$, then $I \cap J \in \mathcal{S}$ or $I \cup J \in \mathcal{S}$. In this case $\mathcal{S}^{\prime}=\{J \subset X: X \backslash J \in \mathcal{S}\}$ is also a paracell, and $\mathcal{S} \cap \mathcal{S}^{\prime}=\emptyset$. The two paracells $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are said to be opposite.

There are no paracells associated to $X$ unless $|X| \geq 2$.
It follows from this definition that if $I \in \mathcal{S}$ and $J \in \mathcal{S}$, then $I \cap J=\emptyset \Rightarrow I \cup J \in \mathcal{S}$, and $I \cup J=X \Rightarrow I \cap J \in \mathcal{S}$.

Definition 2.4. A precell associated to $X$ is a paracell $\mathcal{S}$ associated to $X$ such that, for every $J \in \mathcal{P}_{*}(X), J \in \mathcal{S}$ or $X \backslash J \in \mathcal{S}$. In other words $\mathcal{S} \cup \mathcal{S}^{\prime}=\mathcal{P}_{*}(X)$, where $\mathcal{S}^{\prime}$ denotes the paracell opposite to $\mathcal{S}$. In this case $\mathcal{S}^{\prime}$ is also a precell.

The following are trivial consequences of the definition, assembled in a lemma for future reference.

Lemma 2.2. Let $\mathcal{S}$ be a precell.
(i) if $J \in \mathcal{S}$ then $X \backslash J \notin \mathcal{S}$;
(ii) if $J \in \mathcal{P}_{*}(X)$ and $J \notin \mathcal{S}$ then $X \backslash J \in \mathcal{S}$;
(iii) if $J \in \mathcal{S}, K \in \mathcal{S}$, and $J \cap K=\emptyset$, then $J \cup K \in \mathcal{S}$;
(iv) if $J \in \mathcal{S}, K \in \mathcal{S}$, and $J \cup K=X$, then $J \cap K \in \mathcal{S}$.
(v) if $K \cup L=J \in \mathcal{S}$ and $K \cap L=\emptyset$, then $K \in \mathcal{S}$ or $L \in \mathcal{S}$.

Conversely, if $\mathcal{S}$ is a subset of $\mathcal{P}_{*}(X)$ having the properties (i)-(iv), and if $\mathcal{S}^{\prime}$ denotes $\{J \subset$ $X: X \backslash J \in \mathcal{S}\}$, then $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are opposite precells relative to $X$.

To prove (v) we note that the assertion is obvious if either $K$ or $L$ is empty. Otherwise they must both belong to $\mathcal{P}_{*}$, and cannot both belong to $\mathcal{S}^{\prime}$ since that would imply $J \in \mathcal{S}^{\prime}$, hence at least one of them belongs to $\mathcal{S}$. Note that this assertion does not hold for general paracells.

Definition 2.5. A cell associated to $X$ is a precell $\mathcal{S}$ such that there exist $|X|$ real numbers $\left\{s_{j}: j \in X\right\}$ satisfying

$$
\begin{equation*}
\sum_{j \in X} s_{j}=0, \quad s_{J} \stackrel{\text { def }}{=} \sum_{j \in J} s_{j}>0 \quad \forall J \in \mathcal{S} . \tag{19}
\end{equation*}
$$

In other words a cell associated to $X$ is a precell $\mathcal{S}$ such that there is a geometrical cell $\mathcal{C}_{\mathcal{S}}$ such that, for every $J \in \mathcal{S}$, and every $s \in \mathcal{C}_{\mathcal{S}}, s_{J}>0$.
$\mathcal{S}$ and $\mathcal{C}_{\mathcal{S}}$ are then uniquely determined by each other and $\mathcal{S}^{\prime}$ is the cell associated to the geometrical cell opposite to $\mathcal{C}_{\mathcal{S}}$, i.e. $\mathcal{C}_{\mathcal{S}^{\prime}}=-\mathcal{C}_{\mathcal{S}}$. It is clear that every geometrical cell determines a cell in this way.

If $Y$ and $X$ are disjoint non-empty finite sets, and $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are opposite paracells associated to $X$, we denote

$$
\begin{align*}
Y \downarrow \mathcal{S} & =\left\{J \in \mathcal{P}_{*}(Y \cup X): J \cap X=X \text { or } J \cap X \in \mathcal{S}\right\}  \tag{20}\\
Y \uparrow \mathcal{S}^{\prime} & =\left\{J \in \mathcal{P}_{*}(Y \cup X): J \cap X=\emptyset \text { or } J \cap X \in \mathcal{S}^{\prime}\right\} \tag{21}
\end{align*}
$$

It is easily verified that $Y \downarrow \mathcal{S}$ and $Y \uparrow \mathcal{S}^{\prime}$ are opposite paracells associated to $Y \cup X$. If $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are opposite precells (resp. cells) associated to $X$, then $Y \downarrow \mathcal{S}$ and $Y \uparrow \mathcal{S}^{\prime}$ are opposite precells (resp. cells) associated to $X \cup Y$. In case $Y$ has only one element $j$ we abbreviate $\{j\} \uparrow \mathcal{S}$ to $j \downarrow \mathcal{S}$.

If $X$ has only one element (denoted $j$ ), there are no paracells associated to $X$, but we define

$$
\begin{align*}
& Y \downarrow j=\left\{J \in \mathcal{P}_{*}(Y \cup X): j \in J\right\},  \tag{22}\\
& Y \uparrow j=\left\{J \in \mathcal{P}_{*}(Y \cup X): j \notin J\right\} . \tag{23}
\end{align*}
$$

$Y \downarrow j$ and $Y \uparrow j$ are opposite cells, corresponding respectively to the geometrical cell

$$
\begin{equation*}
\left\{s=\{s\}_{Y \cup X} \in \mathbf{R}^{|Y|+1}: s_{j}+\sum_{k \in Y} s_{k}=0, \quad s_{k}<0 \forall k \in Y\right\} \tag{24}
\end{equation*}
$$

and its opposite.
If $Y=\left(j_{1}, j_{2}, \ldots, j_{p}\right)$,

$$
\begin{equation*}
Y \uparrow \mathcal{S}=j_{1} \uparrow j_{2} \uparrow \ldots j_{p} \uparrow \mathcal{S}, \quad Y \downarrow \mathcal{S}=j_{1} \downarrow j_{2} \downarrow \ldots j_{p} \downarrow \mathcal{S} \tag{25}
\end{equation*}
$$

This is independent of the order of the $j_{k}$ 's.
If $|X|>1$ and $j \in X$, then $\{j\} \in Y \uparrow \mathcal{S}$ (resp. $\{j\} \in Y \downarrow \mathcal{S}$ ) if and only if $\{j\} \in \mathcal{S}$. If $X=\{j\}$ then $\{j\} \in Y \downarrow j$ and $\{j\} \notin Y \uparrow j$.

If $X$ has two elements, say $X=(1,2)$ then the only paracells associated to $X$ are $\{\{1\}\}=1 \uparrow$ $2=2 \downarrow 1$ and its opposite $\{\{2\}\}=1 \downarrow 2=2 \uparrow 1$. They are cells corresponding respectively to the geometrical cells

$$
\begin{equation*}
\left\{s_{1}, s_{2}: s_{1}+s_{2}=0, s_{1}>0\right\} \text { and }\left\{s_{1}, s_{2}: s_{1}+s_{2}=0, s_{1}<0\right\} \tag{26}
\end{equation*}
$$

Cells of the form $j_{1} \uparrow \ldots j_{k-1} \uparrow j_{k}$ are called Steinmann monomials (there are cells which are not of this form). In fact the operations $Y \uparrow$ and $Y \downarrow$ can be defined as linear maps of the whole algebra $\mathcal{A}(X)$ into $\mathcal{A}(Y \cup X)$. See [10] for details.

Definition 2.6. Let $\mathcal{S}$ be a paracell in $X$, and $\mathcal{S}^{\prime}$ the opposite paracell. A $v$-chain associated to $\mathcal{S}^{\prime}$ is a sequence $\left\{J_{1}, \ldots, J_{v}\right\}$ of $v$ disjoint, non-empty subsets of $X$ such that $J_{1} \cup \ldots \cup J_{v}=X$ and that, for every $r<v$,

$$
\begin{equation*}
I_{r} \stackrel{\text { def }}{=} J_{1} \cup \ldots \cup J_{r} \in \mathcal{S}^{\prime} \tag{27}
\end{equation*}
$$

This requires $1 \leq v \leq|X|$, and, of course, if $v=1$ the condition (27) is empty.
We denote $\mathcal{M}_{\nu}\left(\mathcal{S}^{\prime}\right)$ the set of all $v$-chains associated to $\mathcal{S}^{\prime}$. In particular $\mathcal{M}_{1}\left(\mathcal{S}^{\prime}\right)=\{\{X\}\}$.

Thus, in particular, $\mathcal{M}_{\nu}\left(\mathcal{S}^{\prime}\right) \subset \mathcal{A}(X)$. For any paracell $\mathcal{S}$, with opposite paracell $\mathcal{S}^{\prime}$, we denote

$$
\begin{equation*}
\mathcal{U}_{\mathcal{S}}=\sum_{\nu=1}^{|X|} \sum_{\left\{J_{1}, \ldots, J_{v}\right\} \in \mathcal{M}_{v}\left(\mathcal{S}^{\prime}\right)}(-1)^{\nu-1}\left\{J_{1}, \ldots, J_{v}\right\} \tag{28}
\end{equation*}
$$

The following lemmas are proved in [10].
Lemma 2.3. Let $\mathcal{S}^{\prime}$ be a paracell in $X$, and $\left\{I_{1}, \ldots, I_{N}\right\}$ be an arbitrary ordering of all the elements of $\mathcal{S}^{\prime}$. Then, for any permutation $\pi$ of $(1, \ldots, N)$,

$$
\begin{equation*}
\left(1-\hat{I}_{1}\right) \ldots\left(1-\hat{I}_{N}\right)=\left(1-\hat{I}_{\pi(1)}\right) \ldots\left(1-\hat{I}_{\pi(N)}\right) \tag{29}
\end{equation*}
$$

In other words the lhs of (29) does not depend on the chosen ordering.
Lemma 2.4. Let $\mathcal{S}$ be a paracell with opposite paracell $\mathcal{S}^{\prime}$. Then

$$
\begin{equation*}
\forall I \in \mathcal{S}^{\prime} \quad \hat{I} \mathcal{U}_{\mathcal{S}}=0 \tag{30}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\forall I \in \mathcal{S}^{\prime} \quad(1-\hat{I}) \mathcal{U}_{\mathcal{S}}=\mathcal{U}_{\mathcal{S}} \tag{31}
\end{equation*}
$$

Lemma 2.5. With the notations of Lemma 2.3,

$$
\begin{equation*}
\mathcal{U}_{\mathcal{S}}=\left(1-\hat{I}_{1}\right) \ldots\left(1-\hat{I}_{N}\right) \tag{32}
\end{equation*}
$$

### 2.2. Generalized retarded operators and functions

Given a set of time-ordered products for fields indexed by $X$, (resp. a linear system of GTOF indexed by $X$ ), we can define a linear map of the algebra $\mathcal{A}(X)$ into the operator-valued distributions (resp. the distributions) by defining, for each proper sequence $a=\left\{J_{1}, \ldots, J_{\nu}\right\}$

$$
\begin{align*}
& \mathbf{T} a=T\left(J_{1}\right) \ldots T\left(J_{v}\right),  \tag{33}\\
& \text { resp. } \mathbf{t} a=t_{J_{1}, \ldots, J_{v}}^{c} \tag{34}
\end{align*}
$$

This extends by linearity to the whole $\mathcal{A}(X)$ since the proper sequences are (by definition) a basis of this vector space. If $b$ is an element of $\mathcal{A}(X)$, we will also denote $t_{b}^{c}=\mathbf{t} b$.

In particular we may associate to every paracell $\mathcal{S}$, with opposite paracell $\mathcal{S}^{\prime}$

$$
\begin{equation*}
R_{\mathcal{S}}=\mathbf{T} \mathcal{U}_{\mathcal{S}}=\sum_{\nu=1}^{|X|}(-1)^{\nu-1} \sum_{\left\{J_{1}, \ldots, J_{\nu}\right\} \in \mathcal{M}_{\nu}\left(\mathcal{S}^{\prime}\right)} T\left(J_{1}\right) \ldots T\left(J_{\nu}\right) \tag{35}
\end{equation*}
$$

resp.

$$
\begin{equation*}
r_{\mathcal{S}}=\mathbf{t} \mathcal{U}_{\mathcal{S}}=\sum_{\nu=1}^{|X|}(-1)^{\nu-1} \sum_{\left\{J_{1}, \ldots, J_{v}\right\} \in \mathcal{M}_{\nu}\left(\mathcal{S}^{\prime}\right)} t_{J_{1}, \ldots, J_{v}}^{c} \tag{36}
\end{equation*}
$$

In the special case when $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are cells, the $R_{\mathcal{S}}$ (resp. $r_{\mathcal{S}}$ ) are the generalized retarded operators (resp. generalized retarded functions, abbreviated to GRF). In this special case, in the last equation, the $t_{J_{1}, \ldots, J_{v}}^{c}$ can be all replaced by their non-truncated versions without affecting
the result (in other words the GRF are naturally truncated. For this well-known fact see [1,2,4, 7, 10,13-15]).

The property of causal factorization (3) can be reexpressed as follows: if $I^{\prime}$ is a proper subset of $X$ and $a$ is a proper sequence in $\mathcal{P}(X)$, then

$$
\begin{equation*}
\mathbf{T}\left(1-\hat{I}^{\prime}\right) a\left(\operatorname{resp.} \mathbf{t}\left(1-\hat{I}^{\prime}\right) a\right) \text { vanishes in the open set }\left\{x \in \mathcal{X}^{X}:\{x\}_{I^{\prime}} \gtrsim\{x\}_{X \backslash I^{\prime}}\right\} \tag{37}
\end{equation*}
$$

This, of course, extends to any $a \in \mathcal{A}(X)$. For every precell $\mathcal{S}$ opposite to $\mathcal{S}^{\prime}$, and every $I^{\prime} \in \mathcal{S}^{\prime}$, the identity $\left(1-\hat{I}^{\prime}\right) \mathcal{U}_{\mathcal{S}}=\mathcal{U}_{\mathcal{S}}$ and eqs. (35), (36), and (37) imply that

$$
\begin{equation*}
R_{\mathcal{S}} \text { and } r_{\mathcal{S}} \text { vanish in } \bigcup_{I \in \mathcal{S}}\left\{\{x\}_{X}:\{x\}_{I} \lesssim\{x\}_{X \backslash I}\right\}, \tag{38}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\text { support } R_{\mathcal{S}} \text { and support } r_{\mathcal{S}} \subset \bigcap_{I \in \mathcal{S}}\left\{\{x\}_{X}: \exists j \in I, \exists k \in X \backslash I \text { s.t. } x_{k} \leq x_{j}\right\} \tag{39}
\end{equation*}
$$

As the simplest example, if $X=(1,2)$ the only paracells are, as we have seen, $\mathcal{S}=\{\{1\}\}=1 \uparrow$ $2=2 \downarrow 1$ and $\mathcal{S}^{\prime}=\{\{2\}\}=1 \downarrow 2=2 \uparrow 1$, and

$$
\begin{equation*}
\text { support } R_{1 \uparrow 2} \subset\left\{\left(x_{1}, x_{2}\right): x_{1} \geq x_{2}\right\}, \quad \text { support } R_{1 \downarrow 2} \subset\left\{\left(x_{1}, x_{2}\right): x_{1} \leq x_{2}\right\} \tag{40}
\end{equation*}
$$

Of course the same holds for the $r_{\mathcal{S}}$.

## 3. Factorization

### 3.1. Some algebra

In applying Sect. 2 to the case of a linear system of time-ordered functions which factorize as in Subsect. 1.1, there is a first non-trivial fact to prove, namely that, given a cell $\mathcal{S}$ relative to $X=A \cup B \cup\{1\}$, there are cells $(\mathcal{S} \| 1, A)$ relative to $A \cup\{1\}$ (opposite to $\left(\mathcal{S}^{\prime} \| 1, A\right)$ ) and $(\mathcal{S} \| 1, B)$ relative to $B \cup\{1\}$ (opposite to $\left(\mathcal{S}^{\prime} \| 1, B\right)$ ), such that

$$
\begin{equation*}
r_{\mathcal{S}}=r_{(\mathcal{S} \| 1, A)} r_{(\mathcal{S} \| 1, B)}, \tag{41}
\end{equation*}
$$

and similarly with operators, if we make the suitable commutation assumptions.
We suppose, as above, that $X=A \cup B \cup\{1\}$, with $A$ and $B$ disjoint and non-empty and $1 \notin A \cup B$. The tensor product $\mathcal{A}(A \cup\{1\}) \otimes \mathcal{A}(B \cup\{1\})$ is defined in the standard way, i.e. if $c, c^{\prime}$ are proper sequences in $\mathcal{P}(A \cup\{1\}), f, f^{\prime}$ are proper sequences in $\mathcal{P}(B \cup\{1\})$, then $(c \otimes f)\left(c^{\prime} \otimes f^{\prime}\right)=\left(c c^{\prime}\right) \otimes\left(f f^{\prime}\right)$, and this extends by linearity. We define a linear map Fac of $\mathcal{A}(X)$ into $\mathcal{A}(A \cup\{1\}) \otimes \mathcal{A}(B \cup\{1\})$ by defining $\mathbf{F a c}(c)$ for an arbitrary proper sequence $c=\left\{J_{1}, \ldots, J_{\nu}\right\}$ in $\mathcal{P}(X)$ as follows:

$$
\begin{align*}
\operatorname{Fac}(c) & =c_{A} \otimes c_{B}, \\
c_{A} & =\left\{J_{1} \cap(A \cup\{1\}), \ldots, J_{v} \cap(A \cup\{1\})\right\} \bmod \emptyset, \\
c_{B} & =\left\{J_{1} \cap(B \cup\{1\}), \ldots, J_{v} \cap(B \cup\{1\})\right\} \bmod \emptyset . \tag{42}
\end{align*}
$$

Again this extends by linearity to all of $\mathcal{A}(X)$. From this formula and the definition of the multiplication in $\mathcal{A}(X)$ it immediately follows that if $c$ and $f$ are two proper sequences in $\mathcal{P}(X)$, with $\operatorname{Fac}(c)=c_{A} \otimes c_{B}, \boldsymbol{\operatorname { F a c }}(f)=f_{A} \otimes f_{B}$, then $\operatorname{Fac}(c f)=c_{A} f_{A} \otimes c_{B} f_{B}$, and hence

$$
\begin{equation*}
\boldsymbol{\operatorname { F a c }}(c f)=\boldsymbol{\operatorname { F a c }}(c) \boldsymbol{\operatorname { F a c }}(f) \tag{43}
\end{equation*}
$$

holds for any $c, f \in \mathcal{A}(X)$.

Lemma 3.1. Let $\mathcal{S}$ and $\mathcal{S}^{\prime}$ be two opposite precells relative to $X$. We denote

$$
\begin{align*}
(\mathcal{S} \| 1, A)= & \left\{J \in \mathcal{P}_{*}(A \cup\{1\}): J \subset A, J \in \mathcal{S}\right\} \\
& \cup\left\{J \in \mathcal{P}_{*}(A \cup\{1\}): 1 \in J, A \backslash J \in \mathcal{S}^{\prime}\right\},  \tag{44}\\
\left(\mathcal{S}^{\prime} \| 1, A\right)= & \left\{J \in \mathcal{P}_{*}(A \cup\{1\}): J \subset A, J \in \mathcal{S}^{\prime}\right\} \\
& \cup\left\{J \in \mathcal{P}_{*}(A \cup\{1\}): 1 \in J, A \backslash J \in \mathcal{S}\right\},  \tag{45}\\
(\mathcal{S} \| 1, B)= & \left\{J \in \mathcal{P}_{*}(B \cup\{1\}): J \subset B, J \in \mathcal{S}\right\} \\
& \cup\left\{J \in \mathcal{P}_{*}(B \cup\{1\}): 1 \in J, B \backslash J \in \mathcal{S}^{\prime}\right\},  \tag{46}\\
\left(\mathcal{S}^{\prime} \| 1, B\right)= & \left\{J \in \mathcal{P}_{*}(B \cup\{1\}): J \subset B, J \in \mathcal{S}^{\prime}\right\} \\
& \cup\left\{J \in \mathcal{P}_{*}(B \cup\{1\}): 1 \in J, B \backslash J \in \mathcal{S}\right\} . \tag{47}
\end{align*}
$$

Then
(i) $(\mathcal{S} \| 1, A)$ and $\left(\mathcal{S}^{\prime} \| 1, A\right)$ are opposite precells relative to $A \cup\{1\}$, and $(\mathcal{S} \| 1, B)$ and $\left(\mathcal{S}^{\prime} \| 1, B\right)$ are opposite precells relative to $B \cup\{1\}$;
(ii) If $\mathcal{S}$ is a cell then $(\mathcal{S} \| 1, A),\left(\mathcal{S}^{\prime} \| 1, A\right),(\mathcal{S} \| 1, B)$ and $\left(\mathcal{S}^{\prime} \| 1, B\right)$ are cells.

Proof. (i) It suffices to prove that under the hypotheses of the lemma $(\mathcal{S} \| 1, A)$ and $\left(\mathcal{S}^{\prime} \| 1, A\right)$ are opposite precells relative to $A \cup\{1\}$. Suppose first that $J$ and $J^{\prime}$ are complementary proper subsets of $A \cup\{1\}$. If $J \in(\mathcal{S} \| 1, A)$ it is immediate that $J^{\prime} \in\left(\mathcal{S}^{\prime} \| 1, A\right)$, and that $J^{\prime} \notin(\mathcal{S} \| 1, A)$. If $J \notin(\mathcal{S} \| 1, A)$ and $J \subset A$ then $J \notin \mathcal{S}$ so $J \in \mathcal{S}^{\prime}$ hence $J \in\left(\mathcal{S}^{\prime} \| 1, A\right)$. If $J \notin(\mathcal{S} \| 1, A)$ and $1 \in J$, then $J^{\prime} \subset A$ and $J^{\prime} \notin \mathcal{S}^{\prime}$, hence $J^{\prime} \in \mathcal{S}$, hence again $J \in\left(\mathcal{S}^{\prime} \| 1, A\right)$. Thus $(\mathcal{S} \| 1, A)$ possesses the properties (i) and (ii) of Lemma 2.2. Assume now that $J \in(\mathcal{S} \| 1, A), K \in(\mathcal{S} \| 1, A)$ and $J \cap K=\emptyset$. Let $J^{\prime}=(A \cup\{1\}) \backslash J$ and $K^{\prime}=(A \cup\{1\}) \backslash K$ If $J \subset A$ and $K \subset A$ then $J \in \mathcal{S}$, $K \in \mathcal{S}$ and $J \cup K \in \mathcal{S}$ and $J \cup K \subset A$, so that $J \cup K \in(\mathcal{S} \| 1, A)$. If $J \subset A$ and $1 \in K$ then $J \in \mathcal{S}$ and $J \subset K^{\prime} \in \mathcal{S}^{\prime}$. Therefore $K^{\prime} \backslash J=K^{\prime} \cap J^{\prime} \in \mathcal{S}^{\prime}$ hence $K^{\prime} \cap J^{\prime} \in\left(\mathcal{S}^{\prime} \| 1, A\right)$ which, as we have seen before, implies $J \cup K \in(\mathcal{S} \| 1, A)$. Thus $(\mathcal{S} \| 1, A)$ possesses the property (iii) of Lemma 2.2. So does $\left(\mathcal{S}^{\prime} \| 1, A\right)$ since its definition is symmetrical to that of $(\mathcal{S} \| 1, A)$, and this implies that both of them possess the property (iv). By Lemma 2.2, this finishes the proof of (i). (ii) Let $s$ be a point of the geometrical cell associated to $\mathcal{S}$. Let $s_{j}^{\prime}=s_{j}$ for every $j \in A, s_{1}^{\prime}=$ $s_{1}+s_{B}$. Then $s_{A \cup\{1\}}^{\prime}=0, s_{J}^{\prime}>0$ for every $J \subset A$ such that $J \in \mathcal{S}$, or every $J \subset A \cup\{1\}$ such that $1 \in J$ and $A \backslash J \in \mathcal{S}^{\prime}$, i.e. for every $J \in(\mathcal{S} \| 1, A)$. Therefore $s_{K}^{\prime}<0$ for every $K \in\left(\mathcal{S}^{\prime} \| 1, A\right)$. Since this accounts for all proper subsets of $A \cup\{1\},(\mathcal{S} \| 1, A)$ and $\left(\mathcal{S}^{\prime} \| 1, A\right)$ are opposite cells, and similarly for $(\mathcal{S} \| 1, B)$ and $\left(\mathcal{S}^{\prime} \| 1, B\right)$.

We will prove:
Theorem 3.1. Under the above assumptions (including in particular $\mathcal{S}$ and $\mathcal{S}^{\prime}$ being opposite precells)

$$
\begin{equation*}
\operatorname{Fac}\left(\mathcal{U}_{\mathcal{S}}\right)=\mathcal{U}_{(\mathcal{S} \| 1, A)} \otimes \mathcal{U}_{(\mathcal{S} \| 1, B)} \tag{48}
\end{equation*}
$$

Proof. Let $K \in\left(\mathcal{S}^{\prime} \| 1, A\right)$, and let $K^{\prime}=(A \cup\{1\}) \backslash K$. We claim that $\left(\left\{K, K^{\prime}\right\} \otimes 1\right) \boldsymbol{F a c}\left(\mathcal{U}_{\mathcal{S}}\right)=$ 0 . There are two cases to consider.

Case 1: $K \subset A$ and $K \in \mathcal{S}^{\prime}$. Let $I=K \in \mathcal{S}^{\prime}, I^{\prime}=X \backslash I$. Then $\operatorname{Fac}\left(\left\{I, I^{\prime}\right\}\right)=\left\{K, K^{\prime}\right\} \otimes 1$. By Lemma 2.4, $\left\{I, I^{\prime}\right\} \mathcal{U}_{\mathcal{S}}=0$, and applying Fac gives $\left(\left\{K, K^{\prime}\right\} \otimes 1\right) \operatorname{Fac}\left(\mathcal{U}_{\mathcal{S}}\right)=0$.

Case 2: $1 \in K$ and $K^{\prime} \in \mathcal{S}$. Let $I=X \backslash K^{\prime} \in \mathcal{S}^{\prime}$. Again $\left(\left\{K, K^{\prime}\right\} \otimes 1\right) \boldsymbol{\operatorname { F a c }}\left(\mathcal{U}_{\mathcal{S}}\right)=\boldsymbol{\operatorname { F a c }}(\{I, X \backslash$ $\left.I\} \mathcal{U}_{\mathcal{S}}\right)=0$.

Similarly, for $L \in\left(\mathcal{S}^{\prime} \| 1, B\right), L^{\prime}=(B \cup\{1\}) \backslash L$, we have $\left(1 \otimes\left\{L, L^{\prime}\right\}\right) \operatorname{Fac}\left(\mathcal{U}_{\mathcal{S}}\right)=0$. As a consequence

$$
\begin{equation*}
\operatorname{Fac}\left(\mathcal{U}_{\mathcal{S}}\right)=\prod_{K \in\left(\mathcal{S}^{\prime} \| 1, A\right)}\left(1-\left\{K, K^{\prime}\right\}\right) \otimes \prod_{L \in\left(\mathcal{S}^{\prime} \| 1, B\right)}\left(1-\left\{L, L^{\prime}\right\}\right) \operatorname{Fac}\left(\mathcal{U}_{\mathcal{S}}\right) . \tag{49}
\end{equation*}
$$

Here $K^{\prime}$ stands for $A \cup\{1\} \backslash K$, and $L^{\prime}$ stands for $B \cup\{1\} \backslash L$. Note that the order of factors in the two products is irrelevant by Lemma 2.3. Let $(-1)^{\nu-1} c=(-1)^{\nu-1}\left\{J_{1}, \ldots, J_{v}\right\}$ be one of the terms in the expansion (28) of $\mathcal{U}_{\mathcal{S}}$ with $v>1$. Then $J_{1} \in \mathcal{S}^{\prime}$. This implies that $K=$ $J_{1} \cap(A \cup\{1\}) \in\left(\mathcal{S}^{\prime} \| 1, A\right)$ or $L=J_{1} \cap(B \cup\{1\}) \in\left(\mathcal{S}^{\prime}| | 1, B\right)$. Indeed suppose first that $1 \notin J_{1}$. Then $J_{1} \cap(A \cup\{1\})=J_{1} \cap A$ and $J_{1} \cap(B \cup\{1\})=J_{1} \cap B$ are disjoint subsets with union $J_{1} \in \mathcal{S}^{\prime}$. At least one of them must belong to $\mathcal{S}^{\prime}$ (Lemma 2.2 (v)). Suppose now that $1 \in J_{1}$. Then $K^{\prime}=$ $(A \cup\{1\}) \backslash K=A \backslash K$ and $L^{\prime}=(B \cup\{1\}) \backslash L=B \backslash L$ are disjoint sets with union $X \backslash J_{1} \in \mathcal{S}$, and at least one of them belongs to $\mathcal{S}$. Therefore $\left(1-\left\{K, K^{\prime}\right\}\right)$ occurs in $\prod_{K \in\left(\mathcal{S}^{\prime}| | 1, A\right)}\left(1-\left\{K, K^{\prime}\right\}\right)$ or $\left(1-\left\{L, L^{\prime}\right\}\right)$ occurs in $\prod_{L \in\left(\mathcal{S}^{\prime} \| 1, B\right)}\left(1-\left\{L, L^{\prime}\right\}\right)$. Suppose e.g. that $K=J_{1} \cap(A \cup\{1\}) \in$ $\left(\mathcal{S}^{\prime} \| 1, A\right)$ and hence $\left(1-\left\{K, K^{\prime}\right\}\right)$ occurs in $\prod_{K \in\left(\mathcal{S}^{\prime}| | 1, A\right)}\left(1-\left\{K, K^{\prime}\right\}\right)$. We have

$$
\begin{equation*}
c_{A}=\left\{K, J_{2} \cap(A \cup\{1\}), \ldots, J_{v} \cap(A \cup\{1\})\right\} \quad \bmod \emptyset \tag{50}
\end{equation*}
$$

All the sets $J_{k} \cap(A \cup\{1\})$ with $k>1$ are contained in $K^{\prime}$ hence $\left(1-\left\{K, K^{\prime}\right\}\right) c_{A}=0$. Similarly if $L \in\left(\mathcal{S}^{\prime} \| 1, B\right),\left(1-\left\{L, L^{\prime}\right\}\right) c_{B}=0$. Applying Lemma 2.1 (or Lemma 2.3) we see that the contribution of $c$ to the rhs of (49) vanishes. There remains only the contribution of $\{X\}$, and this proves the theorem.

Remark 3.1. The above theorem does not extend to the case when $\mathcal{S}$ and $\mathcal{S}^{\prime}$ are arbitrary opposite paracells. For example if $1 \in I \in \mathcal{P}_{*}(X)$ and $I^{\prime}=X \backslash I$, we can define a paracell $\mathcal{S}=\{I\}$ with $\mathcal{S}^{\prime}=\left\{I^{\prime}\right\}$ as the opposite paracell. Then $\mathcal{U}_{\mathcal{S}}=\{X\}-\left\{I^{\prime}, I\right\}$ and

$$
\begin{equation*}
\boldsymbol{F a c}\left(\mathcal{U}_{\mathcal{S}}\right)=\{A \cup\{1\}\} \otimes\{B \cup\{1\}\}-\left\{I^{\prime} \cap A, I \cap(A \cup\{1\})\right\} \otimes\left\{I^{\prime} \cap B, I \cap(B \cup\{1\})\right\} \tag{51}
\end{equation*}
$$

Moreover if we suppose $I \not \subset A, I^{\prime} \not \subset A, I \not \subset B, I^{\prime} \not \subset B$, then $(\mathcal{S} \| 1, A),\left(\mathcal{S}^{\prime} \| 1, A\right),(\mathcal{S} \| 1, B)$, and $\left(\mathcal{S}^{\prime} \| 1, B\right)$ as defined by (44)-(47) are empty.

A system of GTOF which factorizes as in (6) satisfies

$$
\begin{equation*}
\mathbf{t} c=\left(\mathbf{t}^{1} c_{A}\right)\left(\mathbf{t}^{2} c_{B}\right) \tag{52}
\end{equation*}
$$

for every proper sequence $c \in \mathcal{A}(X)$, with $c_{A}$ and $c_{B}$ given by (42). This can be reexpressed symbolically as

$$
\begin{equation*}
\mathbf{t} c=\mathbf{t}^{1} \otimes \mathbf{t}^{2} \mathbf{F a c} c \tag{53}
\end{equation*}
$$

for every $c \in \mathcal{A}(X)$. As a consequence,
Corollary 3.1. Given a system of GTOF which factorize as in (6), the associated GRF also factorize:

$$
\begin{equation*}
r_{\mathcal{S}}=r_{(\mathcal{S} \| 1, A)} r_{(\mathcal{S} \| 1, B)}, \tag{54}
\end{equation*}
$$

for every cell $\mathcal{S}$ relative to $X$, with opposite cell $\mathcal{S}^{\prime}$, and with the notations (44)-(47).

For future use we need an explicit description of $(\mathcal{S} \| 1, A),\left(\mathcal{S}^{\prime} \| 1, A\right),(\mathcal{S} \| 1, B),\left(\mathcal{S}^{\prime} \| 1, B\right)$ in the case when $\mathcal{S}=1 \downarrow \mathcal{T}, \mathcal{S}^{\prime}=1 \uparrow \mathcal{T}^{\prime}$, where $\mathcal{T}$ and $\mathcal{T}^{\prime}$ are two opposite cells relative to $A \cup B$. This means

$$
\begin{align*}
& \mathcal{S}=1 \downarrow \mathcal{T}=\left\{J \in \mathcal{P}_{*}(X): J=A \cup B \text { or } J \cap(A \cup B) \in \mathcal{T}\right\}  \tag{55}\\
& \mathcal{S}^{\prime}=1 \uparrow \mathcal{T}^{\prime}=\left\{J \in \mathcal{P}_{*}(X): J=\{1\} \text { or } J \cap(A \cup B) \in \mathcal{T}^{\prime}\right\} \tag{56}
\end{align*}
$$

According to our definitions,

$$
\begin{align*}
&(1 \downarrow \mathcal{T} \| 1, A)=\{J \subset A \cup\{1\}: J \subset A \text { and } J \in \mathcal{T}\} \cup \\
&\left\{J \subset A \cup\{1\}: 1 \in J \text { and } A \backslash J \in \mathcal{T}^{\prime}\right\},  \tag{57}\\
&\left(1 \uparrow \mathcal{T}^{\prime} \| 1, A\right)=\left\{J \subset A \cup\{1\}: J \subset A \text { and } J \in \mathcal{T}^{\prime}\right\} \cup \\
&\{J \subset A \cup\{1\}: 1 \in J \text { and } A \backslash J \in \mathcal{T}\}, \tag{58}
\end{align*}
$$

and the same with $B$ instead of $A$.

### 3.2. Analyticity in momentum space in the Minkowskian case

In this subsection ${ }^{1}$ we assume the $d$-dimensional Minkowskian case. All the linear sets of generalized time-ordered functions we consider are supposed to possess the standard translational and spectral properties. Supposing that we have linear sets $t^{c}, t^{1, c}, t^{2, c}$, of GTOF having the factorization property (6), we consider all the GRF $r_{\mathcal{S}}$ associated to $t^{c}$. Their Fourier-Laplace transforms are all branches of a single function $H$, holomorphic in a domain $D_{X}$ of

$$
\begin{equation*}
\mathcal{L}_{X}=\left\{k=\left(\{k\}_{A},\{k\}_{B}, k_{1}\right) \in \mathbf{C}^{d|X|}: k_{1}+k_{A}+k_{B}=0\right\} \tag{59}
\end{equation*}
$$

(Recall that for any $J \subset X,\{k\}_{J}$ stands for the set of variables indexed by the elements of $J$, while $k_{J}=\sum_{j \in J} k_{j}$.) We denote $\mathcal{L}_{X}^{(r)}=\mathcal{L}_{X} \cap \mathbf{R}^{d|X|}$. For each cell $\mathcal{S}$, the Laplace transform of $r_{\mathcal{S}}$ is the restriction of $H$ to the tube

$$
\begin{equation*}
\mathcal{L}_{X}^{(r)}+i K_{\mathcal{S}}=\left\{k=p+i q \in \mathcal{L}_{X}: \forall I \in \mathcal{S}, \quad q_{I} \in V_{+}\right\} . \tag{60}
\end{equation*}
$$

The Fourier transform $\widetilde{r}_{\mathcal{S}}$ of $r_{\mathcal{S}}$ (with $\delta\left(p_{X}\right)$ removed) is thus the boundary value of $H$ from the tube (60).

We denote

$$
\begin{equation*}
H\left(\{k\}_{A},\{k\}_{B}, k_{1}\right)=h\left(\{k\}_{A},\{k\}_{B}\right) . \tag{61}
\end{equation*}
$$

There also exist two functions $H_{1}$ and $H_{2}$, the momentum-space analytic functions associated to the GTOF $t^{1, c}$ and $t^{2, c}$ (appearing in (6)), depending on variables labelled by $A \cup\{1\}$ and $B \cup\{1\}$, respectively, such that

$$
\begin{equation*}
H_{1}\left(\{k\}_{A},-k_{A}\right)=h_{1}\left(\{k\}_{A}\right), \quad H_{2}\left(\{k\}_{B},-k_{B}\right)=h_{2}\left(\{k\}_{B}\right) \tag{62}
\end{equation*}
$$

and we have

$$
\begin{equation*}
h\left(\{k\}_{A},\{k\}_{B}\right)=h_{1}\left(\{k\}_{A}\right) h_{2}\left(\{k\}_{B}\right) \tag{63}
\end{equation*}
$$

[^1]This can be rewritten as

$$
\begin{equation*}
H\left(\{k\}_{A},\{k\}_{B}, k_{1}\right)=H_{1}\left(\{k\}_{A}, k_{1}+k_{B}\right) H_{2}\left(\{k\}_{B}, k_{1}+k_{A}\right) . \tag{64}
\end{equation*}
$$

While this factorization follows from Corollary 3.1, it can also be understood without invoking Theorem 3.1 or Corollary 3.1. Indeed the domain $D_{X}$ contains the real open set $\mathcal{R}=\left\{p \in \mathcal{L}_{X}^{(r)}\right.$ : $\left.p_{I}^{2}<0 \forall I \in \mathcal{P}_{*}(X)\right\}$, and $H$ coincides there with the Fourier transform of $t_{X}^{c}$ (with $\delta\left(p_{X}\right)$ removed). The latter factorizes as indicated in (11)-(15), and this implies (64) by analytic continuation.

If $q$ is in the cone $K_{\mathcal{S}}$, for every $I \subset A$ which belongs to $\mathcal{S}$, (resp. to $\mathcal{S}^{\prime}$ ), $q_{I} \in V_{+}$(resp. $q_{I} \in V_{-}$). This suffices to prescribe the "sign" of $q_{J}$ for every proper subset $J$ of $A \cup\{1\}$, and we see that the point $q=\left(\{q\}_{A}, q_{1}+q_{B}\right) \in \mathcal{L}_{A \cup\{1\}}^{(r)}$ is in the cone $K_{(\mathcal{S} \| 1, A)}$, where $(\mathcal{S} \| 1, A)$ is precisely the cell associated to $A \cup\{1\}$ which is given by (44). Therefore the boundary value of $H_{1}\left(\{k\}_{A}, k_{1}+k_{B}\right)$ from the tube $\mathcal{L}_{X}^{(r)}+i K_{S}$ is the Fourier transform of $r_{(\mathcal{S} \| 1, A)}$ (without $\delta$ function). Similarly the boundary value of $H_{2}\left(\{k\}_{B}, k_{1}+k_{A}\right)$ from the same tube is the Fourier transform of $r_{(\mathcal{S} \| 1, B)}$. Thus, using (64),

$$
\begin{align*}
\widetilde{r}_{S}\left(\{p\}_{A},\{p\}_{B}, p_{1}\right) & =\lim _{q \in K_{\mathcal{S}}, q \rightarrow 0} H\left(\{p+i q\}_{A},\{p+i q\}_{B}, p_{1}+i q_{1}\right) \\
& =\widetilde{r}_{(\mathcal{S} \| 1, A)}\left(\{p\}_{A}, p_{1}+p_{B}\right) \widetilde{r}_{(\mathcal{S} \| 1, B)}\left(\{p\}_{B}, p_{1}+p_{A}\right) . \tag{65}
\end{align*}
$$

Using (8)-(15) we recover the identity

$$
\begin{equation*}
r_{\mathcal{S}}=r_{(\mathcal{S} \| 1, A)} r_{(\mathcal{S} \| 1, B)} \tag{66}
\end{equation*}
$$

obtained by algebraic arguments in the preceding subsection.
We can now ask about restricting $H$ (more precisely the restriction of $H$ to the tube $\mathcal{L}_{X}^{(r)}+$ $i K_{\mathcal{S}}$ ) to the manifold $\left\{k: k_{1}=0\right\}$.

Let us assume first that $\mathcal{S}=(A \cup B) \downarrow 1$. In this case we see immediately that if $q \in K_{\mathcal{S}}$, then $q_{A}$ and $q_{B}$ must both be in $V_{-}$hence $k_{A}+k_{B}$ can never vanish in the tube $\mathcal{L}_{X}^{(r)}+i K_{\mathcal{S}}$ and it is therefore not possible to restrict the restriction of $H$ to this tube to $\left\{k: k_{1}=0\right\}$ or $\widetilde{r}_{\mathcal{S}}$ to $\left\{p: p_{1}=0\right\}$.

We now take $\mathcal{S}=1 \downarrow \mathcal{T}, \mathcal{S}^{\prime}=1 \uparrow \mathcal{T}^{\prime}$, where $\mathcal{T}$ and $\mathcal{T}^{\prime}$ are two opposite cells relative to $A \cup B$. In this case $(1 \downarrow \mathcal{T} \| 1, A)$ and $\left(1 \uparrow \mathcal{T}^{\prime} \| 1, A\right)$ are given in (57) and (58), and similarly for $(1 \downarrow \mathcal{T} \| 1, B)$ and $\left(1 \uparrow \mathcal{T}^{\prime} \| 1, B\right)$.

If $k=p+i q \in \mathcal{L}_{A \cup B}^{(r)}+i K_{\mathcal{T}}$, then $\left(\{k\}_{A}, k_{B}\right) \in \mathcal{L}_{A \cup 1}+i K_{(1 \downarrow \mathcal{T} \| 1, A)}$ and $\left(\{k\}_{B}, k_{A}\right) \in$ $\mathcal{L}_{B \cup 1}+i K_{(1 \downarrow \mathcal{T} \| 1, B)}$. It is therefore possible to restrict the restriction of $H$ to $\mathcal{L}_{X}^{(r)}+i K_{1 \downarrow \mathcal{T}}$ to the submanifold $\left\{k_{1}=0\right\}$. We note that the actual reason is that $A$ and $B$ are complementary in $A \cup B$, hence $k_{A}$ and $k_{B}$ must have opposite "signs" in $\mathcal{T}$.

Momentum space analyticity thus provides, in the Minkowskian case, a shorter and clearer account of the factorization property, but the algebraic formalism is necessary to deal with more general space-times, even relatively simple ones such as de Sitter space-time.

### 3.3. Factorization and supports

We return to the general (not necessarily Minkowskian) case. Let again $X=A \cup B \cup\{1\}, A$, $B$ and $\{1\}$ disjoint, $A \neq \emptyset, B \neq \emptyset$, and suppose given a linear system of GTOF, indexed by $X$, which has the factorization property (6). Let $\mathcal{T}$ and $\mathcal{T}^{\prime}$ be opposite cells relative to $A \cup B$ and, as above, $\mathcal{S}=1 \downarrow \mathcal{T}, \mathcal{S}^{\prime}=1 \uparrow \mathcal{T}^{\prime}$. According to Corollary 3.1, under our assumptions (6),

$$
\begin{equation*}
r_{1 \downarrow \mathcal{T}}=r_{(1 \downarrow \mathcal{T} \| 1, A)} r_{(1 \downarrow \mathcal{T} \| 1, B)} \tag{67}
\end{equation*}
$$

where the cells $(1 \downarrow \mathcal{T} \| 1, A)$ (relative to $A \cup\{1\}$ ) and $(1 \downarrow \mathcal{T} \| 1, B)$ (relative to $B \cup$ $\{1\}$ ) are given by (57) and (58). Suppose first that $A \in \mathcal{T}$, and hence $B \in \mathcal{T}^{\prime}$. Then $A \in$ $(1 \downarrow \mathcal{T} \| 1, A)$, hence $\{1\}$ belongs to the opposite cell $\left(1 \uparrow \mathcal{T}^{\prime} \| 1, A\right), B \in\left(1 \uparrow \mathcal{T}^{\prime} \mid \| 1, B\right)$ hence $\{1\} \in(1 \downarrow \mathcal{T} \| 1, B)$. As a consequence, by (39),

$$
\begin{align*}
& \text { support } r_{(1 \downarrow \mathcal{T} \| 1, A)} \subset \bigcup_{j \in A}\left\{\{x\}_{A \cup\{1\}}: x_{1} \leq x_{j}\right\}  \tag{68}\\
& \text { support } r_{(1 \downarrow \mathcal{T} \| 1, B)} \subset \bigcup_{k \in B}\left\{\{x\}_{B \cup\{1\}}: x_{k} \leq x_{1}\right\},  \tag{69}\\
& \text { support } r_{1 \downarrow \mathcal{T}} \subset \bigcup_{j \in A} \bigcup_{k \in B}\left\{x: x_{k} \leq x_{1} \leq x_{j}\right\} \tag{70}
\end{align*}
$$

If $A \in \mathcal{T}^{\prime}$, and hence $B \in \mathcal{T}$, the roles of $A$ and $B$ are exchanged, and

$$
\begin{equation*}
\text { support } r_{1 \downarrow \mathcal{T}} \subset \bigcup_{j \in A} \bigcup_{k \in B}\left\{x: x_{j} \leq x_{1} \leq x_{k}\right\} \tag{71}
\end{equation*}
$$

Thus, in all cases, $r_{1 \downarrow \mathcal{T}}$ vanishes unless $x_{1}$ remains in a finite union of double-cones (depending on the other variables). If the space-time $\mathcal{X}$ is such that all double-cones are compact, it is possible to integrate $r_{1 \downarrow \mathcal{T}}$ in the variable $x_{1}$, obtaining a distribution (denoted $\hat{r}_{\mathcal{T}}$ ) in the remaining variables.

Recall that the relation $x_{k} \leq x_{j}$ need not be an order relation, and the above result would be valid even in e.g. the Buchholz-Fredenhagen framework [6] (in this case this relation depends on the indices $j$ and $k$ ).

### 3.4. Additional properties when $\leq$ is an order relation

The properties (38) and (39) have much stronger consequences in the cases when the relation $x_{k} \leq x_{j}$ is an order relation, e.g. the Minkowski and de Sitter spaces and the covering of the Anti-de Sitter space.

If $\leq$ is an order relation, and if $X$ and $Y$ are disjoint sets of indices, and $\mathcal{S}$ is a cell relative to $X$, then (see [10]) the support of $r_{Y \uparrow \mathcal{S}}$ is contained in that of $r_{\mathcal{S}}$, and

$$
\begin{align*}
& \text { support } r_{Y \downarrow \mathcal{S}} \subset\left\{\{y\}_{Y},\{x\}_{X}:\{x\}_{X} \in \text { support } r_{\mathcal{S}},\right. \\
& \left.\forall k \in Y \exists j \in X \text { s.t. } y_{k} \leq x_{j}\right\}  \tag{72}\\
& \text { support } r_{Y \uparrow \mathcal{S}} \subset\left\{\{y\}_{Y},\{x\}_{X}:\{x\}_{X} \in \text { support } r_{\mathcal{S}},\right. \\
& \left.\forall k \in Y \exists j \in X \text { s.t. } x_{j} \leq y_{k}\right\} \tag{73}
\end{align*}
$$

Furthermore if the variables indexed by $Y$ are kept fixed, the set of the $r_{Y \downarrow \mathcal{S}}$ (resp. $r_{Y \uparrow \mathcal{S}}$ ) (as $\mathcal{S}$ runs over all the cells associated to $X$ ) has all the linear properties of a set of GRF associated to $X$. In particular the distributions $\hat{r}_{\mathcal{S}}$ obtained in the preceding subsection by integrating over $x_{1}$ have all the linear properties of a set of GRF associated to $X \backslash\{1\}$.

From now on we shall restrict our attention to the cases when the relation $\leq$ is an order relation.


Fig. 2. Integration can be performed first on 3, then 2, then 1 .

### 3.5. Integrating over several variables

The factorization formula (48), applied to the case $\mathcal{S}=1 \downarrow \mathcal{T}, \mathcal{S}^{\prime}=1 \uparrow \mathcal{T}^{\prime}$, where $\mathcal{T}$ and $\mathcal{T}^{\prime}$ are opposite cells relative to $A \cup B$, (see eqs (55) and (56)) gave the formulae (57) and (58) for the cells $(1 \downarrow \mathcal{T} \| 1, A)$ and $\left(1 \uparrow \mathcal{T}^{\prime} \| 1, A\right)$. We now wish to specialize this to the case when $2 \in A, A \backslash\{2\}=C \neq \emptyset, \mathcal{V}$ and $\mathcal{V}^{\prime}$ are opposite cells relative to $C \cup B$, and

$$
\begin{align*}
& \mathcal{T}=2 \downarrow \mathcal{V}=\left\{J \in \mathcal{P}_{*}(A \cup B): J=C \cup B \text { or } J \cap(C \cup B) \in \mathcal{V}\right\},  \tag{74}\\
& \mathcal{T}^{\prime}=2 \uparrow \mathcal{V}^{\prime}=\left\{J \in \mathcal{P}_{*}(A \cup B): J=\{2\} \text { or } J \cap(C \cup B) \in \mathcal{V}^{\prime}\right\} . \tag{75}
\end{align*}
$$

Specializing (58) to this case gives:

$$
\begin{align*}
(1 \uparrow & \left.2 \uparrow \mathcal{V}^{\prime} \| 1, A\right)= \\
& \left\{J \in \mathcal{P}_{*}(A \cup\{1\}): J=\{2\} \text { or } J_{1}=J \backslash\{2\}\right. \text { satisfies : } \\
& \left.\left(J_{1} \subset C \text { and } J_{1} \in \mathcal{V}^{\prime}\right) \text { or }\left(1 \in J_{1} \text { and } C \backslash J_{1} \in \mathcal{V}\right)\right\} \\
& =2 \uparrow\left(1 \uparrow \mathcal{V}^{\prime} \| 1, C\right), \tag{76}
\end{align*}
$$

and hence

$$
\begin{equation*}
(1 \downarrow 2 \downarrow \mathcal{V} \| 1, A)=2 \downarrow(1 \downarrow \mathcal{V} \| 1, C) . \tag{77}
\end{equation*}
$$

If all double-cones are bounded, this makes it possible to integrate recursively the expression $r_{Y \uparrow \mathcal{T}}$ associated to a "tree" of the type symbolized in Fig. 2 over all variables labelled by $Y$, provided each $j \in Y$ is a "splitting vertex" for the "tree". We do not go into details on this subject, but will focus on the special case provided by actual tree graphs.

## 4. Actual trees

The first part of this section does not require the assumption that $\leq$ is an order relation. In this section we consider the linear system of GTOF associated with a tree: this is a connected and simply connected graph $Q$ whose vertices are labelled by a finite set of indices $X(|X| \geq 2)$, which we take as $\{1,2, \ldots,|X|\}$. Each line $\ell$ of the tree has two distinct ends (also called extremities) in $X$, denoted $b(\ell)$ and $e(\ell)$ with $b(\ell)<e(\ell)$. An extremity of $Q$ is a vertex which is the extremity of a single line. A vertex is called internal if it is not an extremity of the tree, i.e. it is an end to at least two distinct lines. A tree with more than one vertex always has at least two extremities. If a line $\ell$ of the tree $Q$ is deleted, the resulting graph is the union of two disjoint tree graphs. If their sets of vertices are denoted $Y$ and $Z$ we denote $Q \mid Y$ and $Q \mid Z$ the corresponding trees.

For each line $\ell$ of $Q$, let $F_{\ell}$ be a two-point distribution having the PR property. Then the product

$$
\begin{equation*}
G=\prod_{\ell} F_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) \tag{78}
\end{equation*}
$$

is well-defined and also has the PR property. This can be seen by induction on the number of vertices $|X|$ of $Q$ by the same argument as in subsect. 1.1.

The time-ordered function associated to the tree is

$$
\begin{equation*}
t_{X}\left(x_{1}, \ldots, x_{|X|}\right)=\prod_{\ell} t_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) \tag{79}
\end{equation*}
$$

where the product extends over all the lines of the tree $Q$, and $t_{\ell}(u, v)$ is the time-ordered twopoint function of some free or generalized free field $\phi_{\ell}$. Let $w_{\ell+}(u, v)=\left(\Omega, \phi_{\ell}(u) \phi_{\ell}(v) \Omega\right)=$ $w_{\ell-}(v, u)$. We assume that all the $t_{\ell}$ and $w_{\ell \pm}$ have the PR property. If $a=\left\{J_{1}, \ldots, J_{v}\right\}$ is a proper sequence in $\mathcal{P}(X)$,

$$
\begin{align*}
t_{a}\left(x_{1}, \ldots, x_{|X|}\right) & =t_{J_{1}, \ldots, J_{v}}\left(x_{1}, \ldots, x_{|X|}\right)=\prod_{\ell} F_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right), \\
F_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) & =t_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) \text { if } b(\ell) \text { and } e(\ell) \text { belong to the same } J_{k}, \\
F_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) & =w_{\ell+}\left(x_{b(\ell)}, x_{e(\ell)}\right) \text { if } b(\ell) \in J_{r}, e(\ell) \in J_{k} \text { with } r<k, \\
F_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right) & =w_{\ell-}\left(x_{b(\ell)}, x_{e(\ell)}\right) \text { if } b(\ell) \in J_{r}, e(\ell) \in J_{k} \text { with } r>k \tag{80}
\end{align*}
$$

Again the product (80) is well-defined because each of the two-point functions involved has the PR property and because we are dealing with a tree graph. According to preceding definitions

$$
\begin{align*}
& r_{e(\ell) \uparrow b(\ell)}\left(x_{b(\ell)}, x_{e(\ell)}\right)=t_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right)-w_{\ell+}\left(x_{b(\ell)}, x_{e(\ell)}\right), \\
& r_{e(\ell) \downarrow b(\ell)}\left(x_{b(\ell)}, x_{e(\ell)}\right)=t_{\ell}\left(x_{b(\ell)}, x_{e(\ell)}\right)-w_{\ell-}\left(x_{b(\ell)}, x_{e(\ell)}\right) . \tag{81}
\end{align*}
$$

The condition that the $t_{a}$ 's have the causal factorization property is that:

$$
\begin{equation*}
\text { support } r_{e(\ell) \uparrow b(\ell)} \subset\left\{(x, y) \in \mathcal{X}^{2}: x \leq y\right\}, \quad \text { support } r_{e(\ell) \downarrow b(\ell)} \subset\left\{(x, y) \in \mathcal{X}^{2}: x \geq y\right\} \tag{82}
\end{equation*}
$$

Lemma 4.1. Let $Q$ be a tree with vertices indexed by $X$ as above, and $\mathcal{S}$ be a cell associated to $X$, opposite to $\mathcal{S}^{\prime}$. For each line $\ell$ of $Q$ with ends $b(\ell)$ and $e(\ell)>b(\ell)$, let $B(\ell)$ and $E(\ell)$ be the sets of vertices of the two subtrees of $Q$ obtained by severing the line $\ell$ and such that $b(\ell) \in B(\ell), e(\ell) \in E(\ell)$. The GRF $r_{\mathcal{S}}$ associated to $Q$ (i.e. obtained from the GTOF defined in (80)) is a product over the lines of $Q$

$$
\begin{equation*}
r_{\mathcal{S}}=\prod_{\ell} r_{b(\ell) \mathfrak{\perp}(\ell)} . \tag{83}
\end{equation*}
$$

(i) If $e(\ell)$ is an extremity of the tree $Q$ (i.e. if $E(\ell)=\{e(\ell)\}$ ) the factor contributed by the line $\ell$ is $r_{e(\ell) \uparrow b(\ell)}$ if $\{e(\ell)\} \in \mathcal{S}$ while it is $r_{e(\ell) \downarrow b(\ell)}$ if $\{e(\ell)\} \in \mathcal{S}^{\prime}$.
(ii) More generally, the factor contributed by the line $\ell$ is $r_{b(\ell) \uparrow e(\ell)}$ if $B(\ell) \in \mathcal{S}$, and $r_{b(\ell) \downarrow e(\ell)}$ if $B(\ell) \in \mathcal{S}^{\prime}$.

Proof. Repeated application of the factorization theorem (Corollary 3.1) shows that the GRF associated to $Q$ also factorize as products of two-point GRF. We suppose $|X|>2$. Let $\mathcal{S}$ be a cell


Fig. 3. The tree $Q$.
associated to $X$. We may suppose without loss of generality that some line $\ell_{0}$ of the tree $Q$ has 1 and 2 as ends (i.e. $b\left(\ell_{0}\right)=1, e\left(\ell_{0}\right)=2$ ), and that 1 is not an extremity of $Q$. Severing $\ell_{0}$ produces two disjoint subtrees of $Q$, whose sets of vertices are denoted $B \cup\{1\}$ and $A$, respectively, with $A \cup B \cup\{1\}=X, A, B$, and $\{1\}$ disjoint, $B \neq \emptyset, 2 \in A$. By Corollary 3.1, the GRF $r_{\mathcal{S}}$ for $Q$ factorizes into

$$
\begin{equation*}
r_{\mathcal{S}}=r_{(\mathcal{S} \| 1, A)} r_{(\mathcal{S} \| 1, B)} \tag{84}
\end{equation*}
$$

where $r_{(\mathcal{S} \| 1, A)}$ is the GRF associated to the tree $Q \mid A \cup\{1\}$ and the cell $(\mathcal{S} \| 1, A)$, and $r_{(\mathcal{S} \| 1, B)}$ is the GRF associated to the tree $Q \mid B \cup\{1\}$ and the cell $(\mathcal{S} \| 1, B)$. If $A \in \mathcal{S}$ then $A \in(\mathcal{S} \| 1, A)$ hence $\{1\} \in\left(\mathcal{S}^{\prime} \| 1, A\right)$, while if $A \in \mathcal{S}^{\prime}$, then $\{1\} \in(\mathcal{S} \| 1, A)$ (see Fig. 3).

Assume first that 2 is an extremity of $Q$, that is, $A=\{2\}$. Then if $\{2\} \in \mathcal{S}$, we find $(\mathcal{S} \| 1, A)=$ $2 \uparrow 1$, while if $\{2\} \in \mathcal{S}^{\prime}$, we find $(\mathcal{S} \| 1, A)=1 \uparrow 2=2 \downarrow 1$. This proves part(i) of the lemma.

We now return to the general case ( 2 not necessarily an extremity of $Q$ ) and note that 1 is now an extremity of $Q \mid A \cup\{1\}$. By the above, the contribution of the line $\ell_{0}$ to $r_{(\mathcal{S} \| 1, A)}$, hence to the $r_{\mathcal{S}}$ associated to $Q$, is $r_{1 \uparrow 2}$ if $A \in \mathcal{S}^{\prime}$, and $r_{1 \downarrow 2}$ if $A \in \mathcal{S}$. This proves part(ii).

Another way to express the content of Lemma 4.1 is to represent the GRF $r_{\mathcal{S}}$ associated to the graph $Q$ and the cell $\mathcal{S}$ by orienting the lines of $Q$ : the line $\ell$ will be oriented (e.g. by drawing an arrow) from $b(\ell)$ to $e(\ell)$ if $E(\ell) \in \mathcal{S}$, and in the opposite direction if $B(\ell) \in \mathcal{S}$. After this, if a line joins the vertices $j$ and $k$ and its arrow points towards $k$, this line contributes $r_{k \uparrow j}$ and the support of $r_{\mathcal{S}}$ is contained in $\left\{x: x_{j} \leq x_{k}\right\}$, i.e. the arrow points to the future. Still supposing that 1 is an internal vertex, we now assume $\mathcal{S}=1 \downarrow \mathcal{T}$ where $\mathcal{T}$ is a cell relative to $X \backslash\{1\}$. Let $\ell_{0}, \ldots, \ell_{p}$ be the lines having 1 as an extremity. Since 1 is internal, $p \geq 1$. We assume that (in accordance with the notations of this subsection) $b\left(\ell_{j}\right)=1$ for $0 \leq j \leq p$, and the sets $E\left(\ell_{j}\right)$ are defined as in Lemma 4.1. These sets are disjoint and non-empty and $\bigcup_{0}^{p} E\left(\ell_{j}\right)=X \backslash\{1\}$. For any $j, \ell_{j}$ is oriented away from 1 if $E\left(\ell_{j}\right) \in \mathcal{S}$, and towards 1 if $E\left(\ell_{j}\right) \in \mathcal{S}^{\prime}$. By the definition of $1 \downarrow \mathcal{T}$, $E\left(\ell_{j}\right) \in \mathcal{S} \Leftrightarrow E\left(\ell_{j}\right) \in \mathcal{T}$ and $E\left(\ell_{j}\right) \in \mathcal{S}^{\prime} \Leftrightarrow E\left(\ell_{j}\right) \in \mathcal{T}^{\prime}$. At least one $E\left(\ell_{j}\right)$ belongs to $\mathcal{T}$, and at least another one, $E\left(\ell_{k}\right)$, belongs to $\mathcal{T}^{\prime}$. Hence at least one of the lines $\ell_{j}$ is directed away from 1 , giving a contribution $r_{1 \downarrow e\left(\ell_{j}\right)}$, and another one $\ell_{k}$ is directed towards 1 , giving a contribution $r_{1 \uparrow e\left(\ell_{k}\right)}$. Thus the support of $r_{\mathcal{S}}$ is contained in $\left\{\{x\}_{X}: x_{e\left(\ell_{k}\right)} \leq x_{1} \leq x_{e\left(\ell_{j}\right)}\right\}$. As mentioned before, this conclusion does not hold for general $\mathcal{S}$, and it is false for $\mathcal{S}=(X \backslash\{1\}) \downarrow 1$.

We again assume, from now on, that $\leq$ is an order relation. Let us suppose that $X=Y \cup Z$, $Y \cap Z=\emptyset, Y \neq \emptyset$ (say $1 \in Y$ ), $Z \neq \emptyset$, all vertices labelled by $Y$ being internal (some of the $Z$-vertices may also be internal, but $Z$ contains all the extremities of $Q$ ). Let $\mathcal{S}=Y \downarrow \mathcal{V}$, where $\mathcal{V}$ is a cell associated to $Z$. We assign a direction (arrow) to each line as described above. For each $j \in Y$ we have $\mathcal{S}=j \downarrow(Y \backslash\{j\}) \downarrow \mathcal{V})$. Hence we may apply to the index $j$ the same argument that was applied above to the case $j=1$. Therefore there is at least one line having $j$ as extremity and directed towards $j$ and another line directed away from $j$. If the other vertex $k$ of this line is in $Y$, we can continue this line by another one directed away from $k$ and so on until we reach a $Z$-vertex. We may proceed in the same manner downwards from $j$, and (because $\leq$ is an order relation) there are two indices $a \in Z$ and $b \in Z$ such that the support of $r_{\mathcal{S}}$ is contained in
$\left\{\{x\}_{X}: x_{a} \leq x_{j} \leq x_{b}\right\}$. Thus the variable indexed by any $Y$-vertex is confined to a double-cone with $Z$-vertices as its vertices. If the double-cones are all compact in $\mathcal{X}$ (as it is the case in the Minkowski and de Sitter space-times) one can integrate over all variables indexed by $Y$, and the result has all the properties of a GRF associated to $Z$ and the cell $\mathcal{V}$.

## 5. Perturbation theory: conclusion

In perturbation theory, solving the UV problem for a certain theory provides time-ordered products for Wick monomials in a finite number of free (or generalized free) fields. A connected Feynman graph ${ }^{2} G$ with vertices indexed by $X=\{1, \ldots, n\}$ represents one of the contributions to the (truncated) vacuum expectation value of a time-ordered product of $|X|$ Wick monomials of free (or generalized free) fields, e.g.

$$
\begin{equation*}
\left(\Omega, T\left(\psi_{1}\left(x_{1}\right) \ldots \psi_{n}\left(x_{n}\right)\right) \Omega\right)_{c} \tag{85}
\end{equation*}
$$

However we may assign a different free field $\phi_{\ell}$ to each line $\ell$ of $G$ (with $\left[\phi_{\ell}, \phi_{\ell^{\prime}}\right]=0$ if $\ell \neq \ell^{\prime}$ ) and redefine

$$
\begin{equation*}
\psi_{j}\left(x_{j}\right)=\prod_{\ell: j \text { is an end of } \ell} \phi_{\ell}\left(x_{j}\right) \tag{8}
\end{equation*}
$$

(We decorate with derivatives and additional indices as needed.) With this definition, $G$ represents the only contribution to (85). The construction of time-ordered products also provides a time-ordered product for any subset of the fields $\psi_{j}$, so that $G$ is associated to a full linear system of GTOF:

$$
\begin{align*}
& t_{J_{1}, \ldots, J_{v}}^{c}=\left(\Omega, T\left(J_{1}\right) \ldots T\left(J_{v}\right) \Omega\right)_{c},  \tag{87}\\
& J_{1} \cup \ldots J_{v}=X, \quad J_{j} \cap J_{k}=\emptyset \text { if } j \neq k,  \tag{88}\\
& T\left(J_{r}\right)=T\left(\prod_{j \in J_{r}} \psi_{j}\left(x_{j}\right)\right) . \tag{89}
\end{align*}
$$

From this one can obtain the contribution of the graph to all possible generalized retarded functions for these Wick monomials. A simple example of this situation has been seen in the case when $G$ is a tree graph in Sect. 4. (Of course we are adhering here to the most simplistic view of perturbation theory, and the reader might enjoy, by contrast, the point of view of [16] and references therein.)

The contribution of such a graph to the GTOF for the interacting fields is obtained by distinguishing two groups of variables indexed by $Y$ and $Z$ (where $Y \cup Z=X, Y \cap Z=\emptyset$ ). The $Y$ variables correspond to the interactions and the associated Wick monomials are assumed to have degree $>1$ (all the $Y$ vertices are internal). One should then perform the integration denoted symbolically

$$
\begin{equation*}
\hat{t}_{J_{1}, \ldots, J_{v}}^{c}=\int_{\mathcal{X}^{Y}} d Y t_{Y \downarrow\left\{J_{1}, \ldots, J_{v}\right\}}^{c} \tag{90}
\end{equation*}
$$

[^2]

Fig. A.4. The Steinmann Planet. The unit sphere in $\left\{\left(s_{1}, \ldots, s_{4}\right) \in \mathbf{R}^{4}: \sum_{j=1}^{4} s_{j}=0\right\}$ intersects the planes $\left\{s_{J}=0\right\}$ along great circles. The geometrical cells are the polyhedral open cones which are the connected components of the complement of the union of these planes.

Here $J_{1} \cup \ldots \cup J_{v}=Z$, and hatted quantities refer to interacting fields. In particular, if $\mathcal{T}$ is a cell relative to $Z$,

$$
\begin{equation*}
\hat{r}_{\mathcal{T}}=\int_{\mathcal{X}^{Y}} d Y r_{Y \downarrow \mathcal{T}} \tag{91}
\end{equation*}
$$

The weak adiabatic limit problem consists in making sense of the $Y$ integration. In the Minkowskian case it is easy (see e.g. [8]) when there are no zero masses.

What we have seen in Sect 4 is that, for tree graphs, if double-cones are bounded, it is always possible to obtain the contribution of the graph to the GRF of the interacting fields. Obtaining from this the GTOF (in particular the Wightman functions) requires a splitting of multiple commutators: for example for a 2-point function in the Minkowski case, given the commutator function $\hat{c}\left(x_{2}-x_{1}\right)$, find $\hat{w}_{+}\left(x_{2}-x_{1}\right)$ and $\hat{w}_{-}\left(x_{2}-x_{1}\right)$, respectively holomorphic in the future and past tubes, such that $\hat{w}_{+}-\hat{w}_{-}=\hat{c}$. In this case it is easy to find solutions, but this is not known for general $n$-point functions.

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## Appendix A. Some simple examples

Let $X=\{1,2,3,4\}$. All cells relative to $X$ are Steinmann monomials [14,15]. They are given by

$$
\begin{array}{cl}
a_{j}=k \uparrow m \uparrow n \uparrow j & r_{j}=k \downarrow m \downarrow n \downarrow j  \tag{A.1}\\
a_{j k}=k \downarrow m \uparrow n \uparrow j & r_{j k}=k \uparrow m \downarrow n \downarrow j
\end{array}
$$

where $(j, k, m, n)$ is any permutation of $(1,2,3,4)$. (More precisely $a_{j}, a_{j k}, r_{j}, r_{j k}$ denote the corresponding GRF, while $j \downarrow k \downarrow m \downarrow n$ denote cells. This will produce no confusion here.) Fig. A. 4 illustrates the 4-point geometrical cells.

Table A. 1
Factorization table for the first example.

| Cell | Support of cell | Factorization |
| :---: | :---: | :---: |
| $4 \downarrow 1 \downarrow 2 \downarrow 3\left(r_{3}\right)$ | $x_{j} \leq x_{3}$ for all $j=4,1,2$ |  |
| $4 \downarrow 1 \uparrow 2 \uparrow 3\left(a_{34}\right)$ | $x_{4} \leq x_{j}$ for $j=1$ or 2 or $3, x_{1} \geq x_{3}, x_{2} \geq x_{3}$ |  |
| $1 \uparrow 4 \downarrow 2 \downarrow 3\left(r_{31}\right)$ | $x_{1} \geq x_{j}$ for $j=4$ or 2 or $3, x_{4} \leq x_{3}, x_{2} \leq x_{3}$ |  |
| $1 \uparrow 2 \uparrow 3 \uparrow 4\left(a_{4}\right)$ | $x_{j} \geq x_{4}$ for all $j=1,2,3$ |  |
| $1 \downarrow 2 \uparrow 3 \uparrow 4\left(a_{41}\right)$ | $x_{1} \leq x_{j}$ for $j=4$ or 2 or $3, x_{2} \geq x_{4}, x_{3} \geq x_{4}$ |  |

Table A. 1 shows how this applies to the 4-point GRF associated with the very simple tree graph


By virtue of the symmetric role played by the indices 1,2 , and 3 , and the rules of the arrow calculus, all the cells not mentioned in the table can be obtained from those mentioned by a permutation of the indices $1,2,3$, or by reversing all arrows, or both.

We now consider $Z=Y \cup X, X=\{1,2,3,4\}, Y=\{0\}$ or $Y=\{0,5\}$ and the cells $0 \downarrow \mathcal{S}$ and $0 \downarrow 5 \downarrow \mathcal{S}$ respectively associated with the tree graphs:


Here $\mathcal{S}$ is a cell associated with $X$. Table A. 2 shows the factorizations and the resulting supports for a sample of the cells $\mathcal{S}$.

In particular the four-point generalized retarded functions for a scalar field with an interaction density $\mathcal{L}(x)=: \phi^{4}(x)$ : are given, in the first order of perturbation theory, by

$$
\begin{equation*}
\left(\Omega, \hat{R}_{\mathcal{S}}\left(\phi\left(x_{1}\right), \ldots, \phi\left(x_{4}\right)\right) \Omega\right)=\int_{\mathcal{X}}\left(\Omega, R_{0 \downarrow \mathcal{S}}\left(\mathcal{L}\left(x_{0}\right), \phi\left(x_{1}\right), \ldots, \phi\left(x_{4}\right)\right) \Omega\right) d x_{0} \tag{A.4}
\end{equation*}
$$

Table A. 2
Factorization table for Graphs 2 and 3.


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[^1]:    1 The contents of this subsection will not be used in the remainder of this paper. Readers unfamiliar with the subject of momentum-space analyticity, or whose memories of it have faded with time, might omit this subsection (with the possible exception of its last sentence).

[^2]:    2 We only consider graphs in which each line has two distinct ends.

