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On the Cop Number of a Graph*

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The cop number $c(G)$ of a graph G is an invariant connected with the genus and the girth. We prove that for a fixed k there is a polynomial-time algorithm which decides whether $c(G) \leq k$. This settles a question of T. Andreae. Moreover, we show that every graph is topologically equivalent to a graph with $c \leq 2$. Finally we consider a pursuit-evasion problem in Littlewood's miscellany. We prove that two lions are not always sufficient to catch a man on a plane graph, provided the lions and the man have equal maximum speed. We deal both with a discrete motion (from vertex to vertex) and with a continuous motion. The discrete case is solved by showing that there are plane graphs of cop number 3 such that all the edges can be represented by straight segments of the same length.

1. INTRODUCTION

A *graph* G consists of a finite non-empty set $V(G)$ of *vertices*, together with a set $E(G)$ of unordered pairs of distinct vertices called *edges*.

Given a connected graph G and a positive integer k we consider the following game with perfect information between two players $\mathscr{C}$ (cops) and $\mathscr{R}$ (robber). (We follow the description of [A86].) First $\mathscr{C}$ places k cops $C_1, \ldots, C_k$ at some vertices of G. Then $\mathscr{R}$ places a robber R at some

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vertex of G. Afterwards $\mathscr{C}$ and $\mathscr{R}$ move alternatively beginning with $\mathscr{C}$. A move of $\mathscr{C}$ consists of choosing a (possibly empty) subset of the set of cops and moving each cop of the subset along an edge to an adjacent vertex. Similarly, when $\mathscr{R}$ is moving, R either stays at his present vertex or he is moved to an adjacent vertex. Throughout it is allowed that two or more cops are on the same vertex. $\mathscr{C}$ wins if he catches R, i.e., he manages after a finite number of moves to put at least one cop on the same vertex as R. $\mathscr{R}$ wins if he avoids this forever.

The *cop number* $c(G)$ of a connected graph G is the least number of cops such that $\mathscr{C}$ has a winning strategy. We define the cop number of a non-connected graph as the maximum of the cop numbers of its connected components.

Many variants of this game have been studied. Most of them deal with the case of a continuous game in which the players move along a continuous path in some connected manifold. Of this kind is the classical game of the lion and the man in the arena (cf. [Bo86]) which we will consider in the last part of this paper. It should be noted that many variants of the game do not admit a perfect information. For example [Pa76] (see also [Ch91, MHGJP88]) considers a continuous game on a graph imbedded in $\mathbb{R}^3$ where the cops do not know the position of the robber.

It is shown in [AF84] that there are graphs of arbitrary high cop number, but every planar graph has cop number ≤ 3. In [Qu85] this result is extended by showing that $c(G) \leq 3 + 2\gamma(G)$, where $\gamma(G)$ is the genus of G. $c(G)$ is also connected with the girth and the minimum degree $\delta(G)$ of a graph. In [Fr87] it is shown that if G has girth at least $8t - 3$, then $c(G) > (\delta(G) - 1)^t$. See also [A86].

1.1. *Preliminary Notions*

A *play* is a finite or infinite sequence of legal moves played alternatively by $\mathscr{C}$ and $\mathscr{R}$. A *strategy for* $\mathscr{C}$ can be formalized as a function f which tells $\mathscr{C}$ the next move to do given the current positions of the cops and the robber and all the previous moves of $\mathscr{C}$ and $\mathscr{R}$. A *strategy for* $\mathscr{R}$ is defined similarly. A *winning strategy for* $\mathscr{C}$ is a strategy f such that in every play in which $\mathscr{C}$ follows f the robber will be caught after a finite number of moves regardless of $\mathscr{R}$'s moves. A *winning strategy for* $\mathscr{R}$ is a strategy g such that, if $\mathscr{R}$ follows g, he avoids capture forever regardless of $\mathscr{C}$'s moves. It can be proved that one of the two players has a winning strategy. Actually one of the two players has a winning strategy which takes into account only the current positions of the cops and the robber and not the previous moves. We call such strategies *without memory* and the general ones *with memory*. Although dispensable in principle, the consideration of strategies with memory simplifies some proofs (e.g., Theorem 3.1).

2. An Algorithm to Compute $c(G)$

We show that for fixed k, there is a polynomial time algorithm to decide whether $c(G) \leq k$. This settles a question of [A86]. More precisely we have:

Theorem 2.1. *There is an algorithm to determine whether $c(G) \leq k$ in time polynomial in $|V(G)|^{k+1}$, hence in time polynomial in $|V(G)|$ and exponential in k.*

Remark 2.2. Since the cops must obviously avoid repeated positions, if $\mathscr{C}$ has a winning strategy, then $\mathscr{C}$ has a strategy to win in $\leq |V|^{k+1}$ moves (equal to the number of possible configurations of k cops and one robber on the graph $G = (V, E)$). Thus a (not efficient) way to test whether $c(G) \leq k$ is to list all the possible pairs of strategies without memory (f, g), and run the game for at most $|V|^{k+1}$ alternate moves with $\mathscr{C}$ following f and $\mathscr{R}$ following g. If we find f which wins against all the competitor strategies g in $\leq |V|^{k+1}$ moves, then $c(G) \leq k$; otherwise $c(G) > k$. Since there are at most $|V^k|^{|V^{k+1}|}$ strategies without memory, this procedure gives an exponential algorithm in $|V^{k+1}|$ to test whether $c(G) \leq k$. This is not good enough for Theorem 2.1.

To prove Theorem 2.1 we proceed as follows.

Definition 2.3. Let States $= V^k \times V$ (where k is the number of cops). A state $s \in$ States codes the current positions of the cops and the robber, but does not say whose turn it is to move. We define a 0-1-state as an element of the set States $\times \{0, 1\}$. If $s \in$ States, the 0-1-state $(s, 0) \in$ States $\times \{0, 1\}$ codes the information that s is the current state and that it is $\mathscr{R}$'s turn to move. $(s, 1)$ says that s is the current state and that it is $\mathscr{C}$'s turn to move.

Note that a move of $\mathscr{C}$ (or $\mathscr{R}$) is uniquely determined by a pair of states (s, s') consisting of the state before and after the move. (Of course not every pair of states codes a permissible move.)

Definition 2.4. Let $W_0 \subseteq$ States $\times \{0, 1\}$ be the set consisting of all the pairs (s, i) such that $i \in \{0, 1\}$ and s codes a state in which at least one cop is in the same vertex as R (the robber is caught). Inductively define

W_{n+1} as follows:

1. $W_n \subseteq W_{n+1}$;

2. if (s, s') codes a move for the cops with $(s', 0) \in W_n$, then $(s, 1) \in W_{n+1}$;

3. if for every $s' \in$ States, (s, s') either does not code a permissible move of $\mathscr{R}$ or codes a move of $\mathscr{R}$ with $(s', 1) \in W_n$, then $(s, 0) \in W_{n+1}$;

4. nothing else belongs to W_{n+1}.

Define $W = \bigcup_{n \in \omega} W_n$.

By a routine argument about games with perfect informations we have:

LEMMA 2.5. *W_n coincides with the set B_n of those 0-1-states starting from which $\mathscr{C}$ can win in $\leq n$ moves if he plays correctly (counting both the moves of $\mathscr{C}$ and those of $\mathscr{R}$).*

Proof. We first prove that $W_n \subseteq B_n$. This is clear if $n = 0$. Suppose the inclusion holds for n and let us prove it for $n + 1$. So let $(s, i) \in W_{n+1}$. We can as well assume $(s, i) \in W_{n+1} - W_n$ otherwise we can apply immediately the induction hypothesis.

Suppose first that $i = 1$. Then by definition of W_{n+1} there exists $s' \in$ States such that (s, s') codes a move of $\mathscr{C}$ and $(s', 0) \in W_n$. Thus in the situation coded by $(s, 1)$ let $\mathscr{C}$ make the move (s, s') and apply the induction hypothesis.

Suppose now that $i = 0$. Then for every possible move (s, s') of $\mathscr{R}$, $(s', 1) \in W_n$. So regardless of $\mathscr{R}$'s move (starting from the situation coded by $(s, 0)$), we can apply the inductive hypothesis after $\mathscr{R}$'s move.

We have thus proved that $W_n \subseteq B_n$. The opposite inclusion is proved in a similar way. For instance, if $(s, 1) \in B_{n+1} - B_n$, there must be a move (s,s') of $\mathscr{C}$ which leads to a situation $(s', 0)$ from which $\mathscr{C}$ can win in $\leq n$ moves. By induction $(s', 0) \in W_n$, thus $(s, 1) \in W_{n+1}$ as desired. We leave the end of the proof to the reader. Q.E.D.

LEMMA 2.6. 1. *If there exists $s \in$ States such that $(s, 0) \notin W = \bigcup_n W_n$, then the robber has a winning strategy f.*

2. *If for all $s \in$ States, $(s, 0) \in W$, then the cops have a winning strategy g.*

Proof. Consider the following variant of the game: the robber $\mathscr{R}$ has the advantage of choosing the initial positions of both R and $C_1, \dots, C_k$, namely $\mathscr{R}$ chooses the initial state $s \in$ States. The game then proceeds as before. Note that the advantage is irrelevant since in a first phase of the game the cops can move to their favorite positions before starting the chase. So a player has a winning strategy in the original game iff it has a winning strategy in the modified game. So let us play with the modified rules and suppose there exists a state s with $(s, 0) \notin W$. Then the robber can win by choosing s as the initial state and choosing each successive move so that the resulting state s' is such that $(s', 0) \notin W$. This can always be achieved regardless of $\mathscr{C}$'s moves by definition of W. This proves part 1. To prove part 2 we just apply Lemma 2.5. Q.E.D.

To finish the proof of Theorem 2.1 we must give an efficient algorithm to test whether $(s, 0) \in W$.

LEMMA 2.7. *$W = W_r$, where $r = 2|V|^{k+1}$.*

Proof. Note that the sequence $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W$ is strictly increasing until it reaches a fixed point, and afterwards it is constant. Since all these sets are contained in States $\times \{0, 1\}$, a set of cardinality $2|V|^{k+1}$, the desired result follows. Q.E.D.

LEMMA 2.8. *There is a polynomial-time algorithm which transforms a list of the elements of W_n into a list of the elements of W_{n+1}.*

Proof. This is clear from the definition of W_{n+1}. Note that here we have used the specific nature of the cops–robber game. Q.E.D.

LEMMA 2.9. *There is an algorithm to determine whether $(s, 0) \in W$ in time polynomial in $|V|^{k+1}$, namely in time polynomial in $|V|$ and exponential in k.*

Proof. We can iterate the previous lemmas $r = 2|V|^{k+1}$ times to construct a list of the elements of W. Since each iteration step takes polynomial time in $|V|^{k+1}$, the desired result follows. Q.E.D.

Theorem 2.1 follows immediately from the preceding lemmas. Note that the above argument yields not only an algorithm to test whether $c(G) \leq k$, but also an efficient way to construct a winning strategy for $\mathscr{C}$ in case $c(G) \leq k$. Similarly if $c(G) > k$ we can construct an escaping strategy for $\mathscr{R}$ in time polynomial in $|V|$ (for fixed k).

3. RETRACTS

We will use "retracts" of graphs to show that every graph is homeomorphic (i.e., equal up to nodes of degree two) to one of cop numbers ≤ 2 (Theorem 4.2).

A graph H is said to be an *induced subgraph* of a graph G, $H < G$, if $V(H) \subseteq V(G)$ and $E(H) = E(G)|V(H)$.

The *difference* $G_1 - G_2$ of two graphs with $G_2 < G_1$ is defined as the induced subgraph of G_1 with set of nodes $V(G_1) - V(G_2)$.

The *distance* $d(u, v)$ between two vertices u, v of G is the length of the shortest path joining u and v. So a connected graph may be regarded as a finite metric space.

A *homomorphism* f: $G_1 \to G_2$ is a map f: $V(G_1) \to V(G_2)$ such that $(x,y) \in E(G_1) \to (f(x), f(y)) \in E(G_2) \vee f(x) = f(y)$. In other words a homomorphism is a map which does not increase the distances.

If G_2 is an induced subgraph of G_1, a *retract* ϕ: $G_1 \to G_2$ is defined as a homomorphism of G_1 onto G_2 which is the identity on G_2.

Retracts on a "geodesic path" were used by [AF84, Qu85] to give upper bounds on $c(G)$ in terms of the genus and by [A86] to give upper bounds in terms of a "forbidden minor." The following theorem can be used to give lower bounds on $c(G)$.

THEOREM 3.1. *If G_1 is connected and ϕ: $G_1 \to G_2$ is a retract, then* $c(G_2) \leq c(G_1)$.

Proof. Suppose k cops have a winning strategy f_1 in G_1. We must define a winning strategy f_2 for k cops in G_2. Consider a robber R which escapes in G_2. (By abuse of notation we identify the robber $\mathscr{R}$ with its position R.) Then a team of k cops moving in G_1, call them the G_1-cops, has a strategy to catch R (as R can be considered as a robber in G_1 which happens not to leave the subgraph G_2). Now the strategy of the G_2-cops that we are looking for is simply to act as the images via the retract ϕ of corresponding G_1-cops. Since the G_1-cops will eventually capture R, and R is always in G_2, the G_2-cops will also capture R at the same time (as ϕ is the identity on G_2). Q.E.D.

Note that the strategy f_2 that we constructed is a strategy with memory, even if we had started with a strategy f_1 without memory. In fact to apply f_2 we must know the current positions of the G_1-cops, which may not be uniquely determined by the current positions of the G_2-cops due to the fact that the retract ϕ is not necessarily injective. Note, however, that the current positions of the G_1-cops can be reconstructed by knowing all the previous moves of the robber R. So f_2 can indeed be construed as a strategy with memory.

The next theorem is a useful tool to prove upper bounds on $c(G)$.

THEOREM 3.2. *If G_1 is connected and ϕ: $G_1 \to G_2$ is a retract, then* $c(G_1) \leq \max\{c(G_2), c(G_1 - G_2) + 1\}$.

Proof. Let $k + 1 = \max\{c(G_2), c(G_1 - G_2) + 1\}$. We must define a winning strategy for $k + 1$ cops in G_1. This is done as follows.

In a first phase of the play the cops, instead of running after the real robber R, chase its image $\phi(R)$ in the subgraph G_2. Note that the "moves" of $\phi(R)$ in G_2 (which are induced by those of R in G_1), are permissible since ϕ is a homomorphism. Since $k + 1 \geq c(G_2)$, sooner or later one of the cops, say C_1, will capture $\phi(R)$. So at this time $C_1 = \phi(R)$. If at this stage R is in G_2, then $C_1 = \phi(R) = R$ and the robber has lost. Otherwise the play enters in a second phase.

In the second phase C_1 has the special role of preventing the robber to leave the subgraph $G_1 - G_2$. This can be achieved by prescribing to C_1 to follow the image of R via the retract ϕ, namely $C_1 = \phi(R)$ holds after each move of C_1 in the second phase. Again this is possible since ϕ is a homomorphism. Since ϕ is the identity on G_2, C_1 will capture R as soon as R tries to leave the subgraph $G_1 - G_2$. Therefore we can assume that from now on R is confined in $G_1 - G_2$. Since $c(G_1 - G_2) \leq k$, the remaining k G_1-cops, with the exclusion of C_1, have now a winning strategy to capture R; namely, first they position themselves in the connected component of $G_1 - G_2$ where the robber is (using the fact that G_1 is connected), and then they begin the chase. Q.E.D.

COROLLARY 3.3. *If G_1 is connected and ϕ: $G_1 \to G_2$ is a retract with $c(G_1 - G_2) \leq k$, then $c(G_1) \leq k + 1$ iff $c(G_2) \leq k + 1$.*

We have thus bounded the cop number of G_1 in terms of the cop numbers of the two subgraphs G_2 and $G_1 - G_2$. This suggests the following definition:

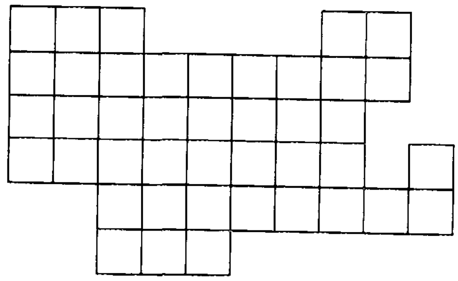
DEFINITION 3.4. A graph G is *1-contractible* if it is a tree. By induction on k and the number of vertices we define G to be *$(k + 1)$-contractible* if G is connected and either it is k-contractible or there is a retract ϕ: $G \to G'$ for some proper induced subgraph $G' < G$ such that G' is $(k + 1)$-contractible and the connected components of $G - G'$ are k-contractible.

THEOREM 3.5. *If G is k-contractible, then $c(G) \leq k$.*

Proof. By an obvious inductive argument using Corollary 3.3 plus the fact that every tree has cop number one. Q.E.D.

EXAMPLE 3.6. The graph G of Fig. 1 is two-contractible, so $c(G) \leq 2$ (actually $= 2$).

To see that a square grid like the one of Fig. 1 is two-contractible one reasons by induction on the number of squares showing that the operation of "adding one square" S (where S is a cycle of length four) transforms a two-contractible graph G into a two-contractible graph $G' = G \cup S$. For this to work it is important that the "surface of attachment" $S \cap G$ is a connected subgraph of the square S, so that there is a retract from G' to G. In this manner one can generate any square grid "without holes". Similar results hold for higher dimensional grids.



$c = 2$

FIGURE 1

4. HOMEOMORPHIC GRAPHS

DEFINITION 4.1. We recall that two graphs G and G' are *homeomorphic* if they can be obtained from the same graph H by repeated bisections of edges through the insertion of new vertices of degree 2.

Intuitively this means that G and G' are homeomorphic in the usual topological sense when embedded in R^3.

THEOREM 4.2. *Every connected graph G is homeomorphic to a graph G′ of cop number* ≤ 2. *Moreover, G′ can be obtained from G by repeated bisections of edges.*

Proof. By induction on the number of edges of G. If G is a tree then $c(G) = 1$ and we are done. Otherwise G has an edge (a, b) such that the removal of (a, b) from $E(G)$ yields a graph $H = G - (a, b)$ which is still connected. By induction hypothesis, by successive bisections of edges, H can be reduced to a graph H' with $c(H') \leq 2$. Since H is connected, so is H'; hence in H' there is a simple path P connecting a and b. Let G' be the graph obtained from H' by adding a new path P' connecting a and b of the same length as P. We can imagine that P' is obtained through repeated bisections of the (deleted) edge (a, b). Hence G' can be obtained from G by repeated bisections of edges and is therefore homeomorphic to G. Since P and P' have the same length there is a retract ϕ: $G' \rightarrow H'$ sending the path P' onto the path P and leaving everything else fixed. The difference $G' - H'$ is a path, so it is a graph of cop number 1. By Corollary 3.3, $c(G') \leq 2$ iff $c(H') \leq 2$. We can thus conclude $c(G') \leq 2$ as desired. Q.E.D.

Remark 4.3. An inspection of the proof reveals that we have in fact proved that every graph is topologically equivalent to a two-contractible graph.

Corollary 4.4. *The operation of bisecting an edge can both increase and decrease the cop number of a graph.*

Proof. It is known that there are graphs of arbitrarily high cop number (cf. [AF84]). We have seen that by repeated bisections the cop number of an arbitrary graph can be always reduced to two. So a *single* bisection can certainly decrease the cop number. On the other hand, a single bisection can increase the cop number, as shown by the example of a triangle (cop number 1) and a square (cop number 2). Q.E.D.

Corollary 4.5. *The contraction of an edge to a point can both increase and decrease the cop number.*

Proof. Just note that if G' is obtained from G by bisecting one edge, then G can be obtained from G' by contracting one edge. Q.E.D.

5. The Lion and the Man in the Arena: The Discrete Case

In [Bo86] we find the following description of the problem of the lion and man in the arena: "A lion and a man in a closed circular arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal?" In 1952 A. S. Besicovitch discovered that no matter what the lion does, the man can escape capture. However, two lions can catch the man. H. T. Croft [Cr64] showed more generally that n lions can catch a man in an n-cop-dimensional ball although $n - 1$ cannot (assuming the man has to run along a path of uniformly bounded curvature). Littlewood's discussion [Bo86] ends with the following problem: "Can two lions catch a man in a bounded (planar) area with rectifiable lakes?" This problem was recently discussed in [St92]. We begin by analyzing a discrete analogue of the problem.

Definition 5.1. A planar graph G is called *equilateral*, if G can be represented as a plane graph in such a way that all its edges are straight segments of the same length. We call such a representation of G a *plane equilateral representation*. Not all planar graphs admit such a representation.

Note that in an equilateral plane graph the cop–robber game is the discrete analogue of the game described by Littlewood. The equilaterality condition corresponds to the fact that the man and the lions have equal maximum speed; in fact in a equilateral plane graph the man and the lions

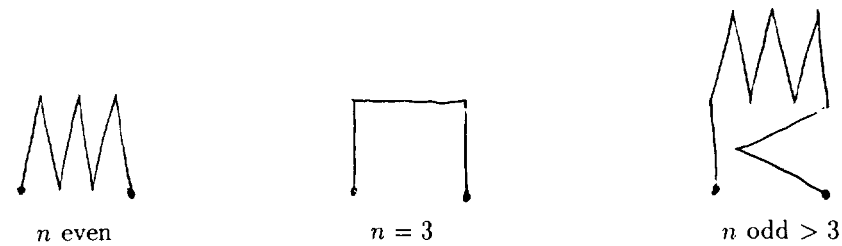


FIGURE 2

can travel at most a unit distance in a unit of time. So the equilaterality condition is a realistic hypothesis in real-life chasing problems.

The next theorem came as a surprise, since we had previously conjectured the opposite. The theorem gives a negative answer to the discrete two-lions problem.

THEOREM 5.2. *There is a planar equilateral graph of cop number* > 2 (*hence of cop number exactly three*).

We need some lemmas and definitions. By a *simple polygonal* we mean a simple curve $P: [0, 1] \to \mathbf{R}^2$ together with a finite partition of $[0, 1]$ such that P is linear on each piece of the partition. So each piece of the partition determines a *segment* of P. The next easy lemma says that we can draw arbitrarily long simple polygonals composed of unit segments inside a rectangular area of fixed size.

LEMMA 5.3. *Given two distinct points x and y on the plane at* (*euclidean*) *distance* $\varepsilon > 0$ *and given an integer* $n > 0$, *there exists a simple polygonal P with end-points x and y such that*: (1) *P is composed of n straight segments of length* ε; (2) *P is contained in a closed rectangle based on the segment joining x and y and of height* 2ε.

Proof. See Fig. 2. Q.E.D.

We call such a P an *n-gadget of base* ε.

DEFINITION 5.4. Let G be a graph and let $G^{[n]}$ be the graph obtained from G by replacing each edge of G by a path of length n. Note that $G^{[n]}$ is homeomorphic to G.

LEMMA 5.5. *Let G be a planar graph. Then for any sufficiently big n,* $G^{[n]}$ *is a planar equilateral graph.*

Proof. It is clear that every planar graph G can be embedded in the plane in such a way that every edge $(a, b) \in E(G)$ is represented by a polygonal path $P_{a,b}$ in $\mathbf{R}^2$ connecting a and b, where each such polygonal $P_{a,b}$ consists of consecutive straight segments of the same length $\varepsilon > 0$. If all these polygonals have the same number n of segments we have an equilateral embedding of $G^{[n]}$ in the plane. Otherwise we reason as follows. By repeated bisections (i.e., by refining the partition of the polygonal) we can replace ε by a much smaller $\varepsilon' > 0$ without changing the shape (i.e., the image) of the polygonals, but only increasing the number of their segments. Now removing a ε'-segment from each polygonal and replacing the removed segments by suitables n-gadgets of base ε' (for various n's), we can transform the polygonals into new polygonals having the same number of segments. Moreover, if ε' was sufficiently small, we can arrange it so that any two gadgets are at a distance much greater than ε', hence at a distance much greater than their diameter. In particular, the gadgets are pairwise disjoint and therefore the new polygonals do not intersect (except possibly at end-points). Thus we obtain an equilateral embedding of $G^{[n]}$ for all sufficiently big n's. Q.E.D.

It is known that lower bounds on $c(G)$ can be given in terms of the girth (length of a minimal cycle) of G and the minimal degree of a vertex $\delta(G)$. Frankl [Fr87] proved that if G has girth at least $8t - 3$, then $c(G) > (\delta(G) - 1)^t$. Unfortunately such results do not give any information if the graph has vertexes of degree ≤ 2 as it is the case for the graph $G^{[n]}$. However, we can prove the following lower bound on $c(G^{[n]})$.

THEOREM 5.6. *If n is odd, $c(G^{[n]}) \geq c(G)$.*

Proof. Let $k = c(G)$. So a robber has a winning strategy against $k - 1$ cops in G and we must find a similar winning strategy for a robber in $G^{[n]}$ against $k - 1$ cops (if $k = 1$ there is nothing to prove). The idea is to "copy" the strategy that we have in G in a consistent manner. To do this we use the fact that there are two maps ι: $V(G) \to V(G^{[n]})$ and ψ: $V(G^{[n]}) \to V(G)$ such that:

1. $d_{G^{[n]}}(x, y) \leq n \to d_G(\psi(x), \psi(y)) \leq 1$,
2. $d_G(x, y) \leq 1 \to d_{G^{[n]}}(\iota(x), \iota(y)) \leq n$.

ι: $V(G) \to V(G^{[n]})$ is the natural map such that the vertices in $V(G^{[n]}) - \iota[V(G)]$ are the new vertices of degree two added to G to obtain $G^{[n]}$.

To define ψ, recall that by the definition of $G^{[n]}$ if $(a, b) \in E(G)$, then there is a path $P_{a,b}$ of length n in $G^{[n]}$ joining $\iota(a)$ with $\iota(b)$. Since n is odd $P_{a,b}$ has an even number of vertices which can be obviously partitioned in two sets: those which are closer to the end-point $\iota(a)$ and those which are closed to the other end-point $\iota(b)$. The map ψ sends the vertices of $P_{a,b}$ which are closer to $\iota(a)$ to the vertex $a \in V(G)$, and those which are closer to $\iota(b)$ to the vertex $b \in V(G)$. This defines a map ψ with the desired properties.

The winning strategy of the robber R in $G^{[n]}$ is articulated into consecutive "megamoves", where a megamove consists in going from $\iota(x)$ to $\iota(y)$ for a suitable edge $(x, y) \in E(G)$ by making n consecutive moves in $G^{[n]}$ along the path of length n connecting $\iota(x)$ and $\iota(y)$ (regardless of the moves of $\mathscr{C}$ in the meantime). (We also allow the empty megamove from $\iota(x)$ to $\iota(x)$ consisting of n consecutive empty moves.)

It only remains to define how the choice of the vertex $y \in V(G)$ is made, assuming that R is in position $\iota(x)$ at the beginning of the megamove. Just before making a megamove the robber computes the ψ-images of the current positions of $R, C_1, \dots, C_{k-1}$. In G, the "image robber" $\phi(R)$ can use his winning strategy against $k - 1$ cops in positions $\psi(C_1), \dots, \psi(C_{k-1})$ to move to some vertex $y \in V(G)$. This is the y we are looking for.

It remains to prove that this is a winning strategy in $G^{[n]}$ (if we choose suitably the first move). The crucial point to observe is that, by property 1, any legal play in $G^{[n]}$ determines via ψ (computed at the end of each megamove) a corresponding legal play in G.

Suppose for a contradiction that one of the cops, say C_i, captures R in $G^{[n]}$. Then C_i can follow R until the end of his megamove. The ψ-image of this play is then a play in G in which the image robber has been caught and yet has followed his winning strategy. Contradiction. Q.E.D.

The above lemma is to be contrasted with Corollary 4.4: bisecting an edge can both increase or decrease the cop number, but splitting all the edges in an odd number of pieces does not decrease the cop number.

Proof of Theorem 5.2. Immediate from the previous lemmas and the fact that there exists a planar graph G of cop number 3, for instance, is the dodecahedron [AF84]. Note that the dodecahedron is not planar equilateral, but we can transform it into a planar equilateral graph $G^{[n]}$ by Lemma 5.5. Moreover, we can choose n odd, so by Lemma 5.6, $c(G^{[n]}) \geq 3$. Q.E.D.

6. The Lion and the Man in the Arena: The Continuous Case

We come back to the original formulation of the man–lion(s) problem, as discussed at the beginning of the previous section, where the man and the lion(s) move with continuity in a bounded area of the plane with equal maximal speeds. Littlewood does not state what are the exact hypotheses on the motion of the man and the lions and on the shape of the area. We will show that if the motion of the man and the lion(s) is only assumed to be piecewise-smooth and by "area" we mean just an arbitrary subset of the plane such that the man and the lion can go from any point to any other point, then the problem has a negative answer; namely, a man can escape from two lions in a suitable area. The hypothesis of *piecewise* smoothness allows us to define the following metric.

DEFINITION 6.1. The *path metric* of a subset $M \subseteq \mathbf{R}^2$ is defined by letting $d_M(x, y)$ to be the infimum of the lengths of the piecewise smooth paths connecting x and y. We say that M is *good* if $d_M(x, y)$ is defined and finite for all $x, y \in M$ and that the infimum is attained.

Given a good M we can play the man–lion(s) game on M by assuming that the man and the lions travel along piecewise smooth paths with the same maximum speed. (We do not make may restriction on the acceleration or the curvature.)

We will prove:

THEOREM 6.2. *There is a good $M \subseteq \mathbf{R}^2$ such that a man can escape two lions in M (assuming a continuous motion with equal maximal speeds as stated above). Actually we can take M to be an equilateral plane graph.*

We need the following definition.

DEFINITION 6.3. Let M be good and let X be a finite subset of M. We say that X is *safe* if for every $x \in X$ and $y, z \in M - \{x\}$ there exists $x' \in X - \{x\}$ such that $d_M(x, x') < d_M(y, x')$ and $d_M(x, x') < d_M(z, x')$.

LEMMA 6.4. *Let M be good and suppose that M has a safe subset X. Then a man can escape two lions in M.*

Proof. Using the fact that X is safe, it is easy to see that starting from any $x \in X$ the man can escape to some $x' \in X$ without being intercepted by two lions initially in two points y and z belonging to $M - \{x\}$. Then he can repeat the procedure starting from x'. The finiteness of X is used to guarantee that there is a positive lower bound on the time that it takes to go from $x \in X$ to some different $x' \in X$, so that if this happens infinitely many times the man is never caught. Q.E.D.

LEMMA 6.5. *There exists a good $M \subseteq \mathbf{R}^2$ which has a safe subset X.*

Proof. Let G be the graph of the dodecahedron. Let n be so big that $G^{[n]}$ is planar equilateral. Let $M \subseteq \mathbf{R}^2$ be a plane equilateral representation of $G^{[n]}$. Note that every point of M has a neighborhood homeomorphic to $\mathbf{R}$ except for the image of the 12 vertices of the dodecahedron. We claim that the set X consisting of the 12 vertices is safe. The idea is that X equipped with the metric d_M is isometric, up to a change of scale $1:n$, to the dodecahedron G equipped with the graph-theoretical metric. It is easy to verify that the dodecahedron has the property that for every vertex x and any two other vertices y and z, there exists a vertex x' adjacent to x but not to y or z. The desired result follows. Q.E.D.

Note that in the above proof we have used some specific properties of the dodecahedron and not just the fact that the dodecahedron has a cop number three. In this respect the results obtained in the previous section about the cop number of planar equilateral graphs are more general. There we showed how to transform *any* planar graph of cop number three into a planar equilateral graph of cop number three.

Our example M can be thought of as a circular arena with 20 rectifiable lakes corresponding to the faces of the dodecahedron. Note, however, that M is one-dimensional. If one desires a two-dimensional example one can perhaps follow the same general idea by replacing the one-dimensional segments of M by two-dimensional thin lines. The difficulty is that we no longer have a finite safe subset $X \in M$ because the vertices are now replaced by thin spots.

We do not consider here restrictions on the acceleration or the curvature, although it might be that the same example (the equilateral dodecahedron) could be adapted with some smoothening.

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