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On a class of bivariate second-order linear partial difference equations and their monic orthogonal polynomial solutions

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ABSTRACT

In this paper we classify the bivariate second-order linear partial difference equations, which are admissible, potentially self-adjoint, and of hypergeometric type. Using vector matrix notation, explicit expressions for the coefficients of the three-term recurrence relations satisfied by monic orthogonal polynomial solutions are obtained in terms of the coefficients of the partial difference equation. Finally, we make a compilation of the examples existing in the literature belonging to the class analyzed in this paper, namely bivariate Charlier, Meixner, Kravchuk and Hahn orthogonal polynomials.

1. Introduction

In last years, the role of univariate orthogonal polynomials on applications in mathematics, science, engineering or computations has increased due to the deep knowledge both in the continuous and discrete settings [2,6,13,21,29]. The cases of two and more variables orthogonal polynomials have been studied much less, although their main definitions and simplest properties were considered more than 100 years ago [3,28]. A considerable number of results on the theory of orthogonal polynomials in several variables has been published in last years (see e.g. [4,7–10,14,22,25,28,33,34] and references therein), emphasizing the relationship between multivariate orthogonal polynomials and partial differential and difference equations. In this multivariate situation it is possible to deal also with continuous or discrete weight functions for the corresponding orthogonality relation.

From the classical point of view, the three-term recurrence relation is one of the keys in the analysis of orthogonal polynomials, giving efficient algorithms for applying them to practical problems [5]. In the multivariate case the orthogonal polynomials satisfy three-term recurrence relations [7, Chapter 3] and in the continuous case there exists an explicit way for computing the recurrence matrices [4]. But, as far as we know, there exist no general formulae in order to compute the matrices appearing in the recurrences in the multivariate discrete situation from the coefficients of the second-order partial difference equation satisfied by the orthogonal polynomials. The significance of partial difference equations is well illustrated in applications involving population dynamics with spatial migrations, chemical reactions, control systems, combinatorics or finite difference schemes [12,18,27,32].

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So, let us consider a bivariate second-order partial difference equation [24,36]

$$\sigma_{11}(x,y)\Delta_1\nabla_1 u(x,y)+\sigma_{12}(x,y)\Delta_1\nabla_2 u(x,y)+\sigma_{21}(x,y)\Delta_2\nabla_1 u(x,y) + \sigma_{22}(x,y)\Delta_2\nabla_2 u(x,y)+\tau_1(x,y)\Delta_1 u(x,y)+\tau_2(x,y)\Delta_2 u(x,y)+\lambda u(x,y)=0, \tag{1}$$

where σ_{ij} and τ_i are polynomials of at most total degree two and one, respectively. In a recent paper [24] we studied the orthogonal polynomial solutions of a second-order partial difference equation of hypergeometric type of two variables as (1). The Pearson systems for the orthogonality weight of the solutions and also for the difference derivatives of the solutions were presented. The orthogonality property in subspaces was treated in detail, which lead to an analog of the Rodrigues type formula for orthogonal polynomials of two discrete variables. A classification of the admissible equations as well as some examples related with bivariate Hahn, Kravchuk, Meixner, and Charlier families, and their algebraic and difference properties were explicitly given. The idea was to use the (column) vector representation [15,16]. Following [24], let $\mathbf{x}=(x,y)\in\mathbb{R}^2$, and let $\mathbf{x}^n$ ($n\in\mathbb{N}_0$) denote the column vector of the monomials $x^{n-k}y^k$, whose elements are arranged in graded lexicographical order (see [7, p. 32]):

$$\mathbf{x}^n=\left(x^{n-k}y^k\right),\quad 0\leqslant k\leqslant n,\ n\in\mathbb{N}_0. \tag{2}$$

Let $\{P^n_{n-k,k}(x,y)\}$ be a sequence of polynomials in the space Π^2_n of all polynomials of total degree at most n in two variables, $\mathbf{x}=(x,y)$, with real coefficients. Such polynomials are finite sums of terms of the form $ax^{n-k}y^k$, where $a\in\mathbb{R}$.

From now on $\mathbb{P}_n$ will denote the (column) polynomial vector

$$\mathbb{P}_n=\left(P^n_{n,0}(x,y),P^n_{n-1,1}(x,y),\dots,P^n_{1,n-1}(x,y),P^n_{0,n}(x,y)\right)^T. \tag{3}$$

Then, each polynomial vector $\mathbb{P}_n$ can be written in terms of the basis (2) as:

$$\mathbb{P}_n=G_{n,n}\mathbf{x}^n+G_{n,n-1}\mathbf{x}^{n-1}+\cdots+G_{n,0}\mathbf{x}^0, \tag{4}$$

where $G_{n,j}$ are matrices of size $(n+1)\times(j+1)$ and $G_{n,n}$ is a nonsingular square matrix of size $(n+1)\times(n+1)$.

A polynomial vector $\widehat{\mathbb{P}}_n$ is said to be monic if its leading matrix coefficient $\widehat{G}_{n,n}$ is the identity matrix (of size $(n+1)\times(n+1)$); i.e.:

$$\widehat{\mathbb{P}}_n=\mathbf{x}^n+\widehat{G}_{n,n-1}\mathbf{x}^{n-1}+\cdots+\widehat{G}_{n,0}\mathbf{x}^0. \tag{5}$$

Then, each of its polynomial entries $\widehat{P}^n_{n-k,k}(x,y)$ is of the form:

$$\widehat{P}^n_{n-k,k}(x,y)=x^{n-k}y^k+\text{terms of lower total degree}. \tag{6}$$

In what follows the "hat" notation $\widehat{\mathbb{P}}_n$ will represent monic polynomials.

The main aim of this paper is to give explicit formulae for computing the matrices in the three-term recurrence relations satisfied by the monic orthogonal polynomial solutions of an admissible potentially self-adjoint second-order partial difference equation of hypergeometric type, using vector matrix notation. Moreover, we classify the possible partial difference equations of this type, completing preliminary results [24,25] where the potentially self-adjointness was not considered in the classification. Thus, we have analyzed a class of second-order partial difference equations for which it is possible to compute a weight function ϱ and a family of monic polynomials such that this family is orthogonal with respect to ϱ in a certain domain G. All of these, weight function, polynomials and domain of orthogonality are given explicitly in terms of the coefficients of the second-order partial difference equation.

The existence of a recurrence relation for a vector of bivariate discrete orthogonal polynomial family can be established in more general settings than those considered here [35]. The following existence theorem proved in [7] can be applied for infinite or finite ($n=0,1,\dots,N$) sequences of polynomials (see Examples 4.1 and 4.2 of [35]) since we are using graded lexicographical order (2).

Theorem 1.1. *Let $\mathcal{L}$ be a positive definite moment linear functional acting on the space Π^2_n of all polynomials of total degree at most n in two variables, and $\{\mathbb{P}_n\}_{n\geqslant 0}$ be an orthogonal family with respect to $\mathcal{L}$. Then, for $n\geqslant 0$, there exist unique matrices $A_{n,j}$ of size $(n+1)\times(n+2)$, $B_{n,j}$ of size $(n+1)\times(n+1)$, and $C_{n,j}$ of size $(n+1)\times n$, such that*

$$x_j\mathbb{P}_n=A_{n,j}\mathbb{P}_{n+1}+B_{n,j}\mathbb{P}_n+C_{n,j}\mathbb{P}_{n-1},\quad j=1,2, \tag{7}$$

with the initial conditions $\mathbb{P}_{-1}=0$ and $\mathbb{P}_0=1$. Here the notation $x_1=x$, $x_2=y$ is used.

In the aforementioned hypothesis for the partial difference equation (1), in this paper we give explicit expressions for the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ appearing in the three-term recurrence relations (7), in terms of the coefficients of σ_{ij} and τ_i in (1). These matrices allow us to compute the monic orthogonal polynomial solutions of an admissible potentially self-adjoint second-order partial difference equation of hypergeometric type.

The structure of the paper is as follows. In Section 2 we give the background necessary for the results contained in this paper. The classification of admissible potentially self-adjoint second-order partial difference equations of hypergeometric type is presented in Section 3. In Section 4 the explicit expressions for the matrix coefficients in the three-term recurrence relations (5) are given in the monic case. Finally, in Section 5, we identify the examples of bivariate orthogonal polynomials existing in the literature following our classification. In this way, monic bivariate Charlier, Meixner, Kravchuk and Hahn orthogonal polynomials can be computed by using the explicit formulae given in Section 4.

2. Basic definitions and notations

Before beginning the analysis proper, in this section we review previous results already existing in the literature for this bivariate discrete situation coming from [17,24,25].

2.1. Admissible partial difference equations of hypergeometric type

The forward and backward difference operators acting on a function of two variables are defined as

$$\Delta_1 u(\mathbf{x}) = u(x+1, y) - u(x, y), \qquad \Delta_2 u(\mathbf{x}) = u(x, y+1) - u(x, y),$$
$$\nabla_1 u(\mathbf{x}) = u(x, y) - u(x-1, y), \qquad \nabla_2 u(\mathbf{x}) = u(x, y) - u(x, y-1).$$

Definition 2.1. Let σ_{ij} and τ_i be polynomials in x and y of at most total degree two and one, respectively. We say that Eq. (1) belongs to the hypergeometric class if the difference derivatives $u_\alpha(\mathbf{x}) = \Delta_1^r \Delta_2^s u(\mathbf{x})$ of the solutions $u = u(\mathbf{x})$ of (1) are also solutions of an equation of the same type as (1).

This concept was introduced by Lyskova as basic class in [1,19,20] in the multivariable continuous case. As a direct consequence of the definition, we obtain the following result [24]:

Lemma 2.2. *Eq.* (1) *belongs to the hypergeometric class if and only if*

$$\sigma_{11}(x, y) = \sigma_{11}(x) = a_{11}x^2 + d_{11}x + f_{11}, \qquad \sigma_{22}(x, y) = \sigma_{22}(y) = b_{22}y^2 + e_{22}y + f_{22},$$
$$\sigma_{12}(x, y) = c_{12}xy + d_{12}x + e_{12}y + f_{12}, \qquad \sigma_{21}(x, y) = c_{21}xy + d_{21}x + e_{21}y + f_{21},$$
$$\tau_1(x, y) = \tau_1(x) = s_{11}x + v_{11}, \qquad \tau_2(x, y) = \tau_2(y) = t_{22}y + v_{22}.$$

Note that when

$$\lambda = -r\Delta_1\tau_1(\mathbf{x}) - s\Delta_2\tau_2(\mathbf{x}) - rs\Delta_1\Delta_2\big[\sigma_{12}(\mathbf{x}) + \sigma_{21}(\mathbf{x})\big] - \frac{r(r-1)}{2}\Delta_1^2\sigma_{11}(\mathbf{x}) - \frac{s(s-1)}{2}\Delta_2^2\sigma_{22}(\mathbf{x}), \tag{8}$$

the equation has a particular solution which is a polynomial of total degree $r + s$.

Definition 2.3. The difference equation (1) belonging to the hypergeometric class will be called admissible if there exists a sequence $\{\lambda_n\}$ $(n = 0, 1, \ldots)$ such that for $\lambda = \lambda_n$, there are precisely $n + 1$ linearly independent solutions in the form of polynomials of total degree n and are no non-trivial solutions in the set of polynomials whose total degree is less than n.

From [24, Theorem 3.1], we have that Eq. (1) is admissible if and only if the following conditions hold true

$$b_{22} = a_{11}, \qquad t_{22} = s_{11}, \qquad 2a_{11} - c_{12} - c_{21} = 0. \tag{9}$$

Therefore,

$$\lambda_\ell = -\ell\big((\ell - 1)a_{11} + s_{11}\big), \tag{10}$$

and the numbers a_{11} and s_{11} must satisfy the following condition

$$a_{11}m + s_{11} \neq 0,$$

for any non-negative integer m in order to be (1) an admissible equation.

Remark 1. It is possible to reduce the number of parameters in Eq. (1) after a translation in the x and y variables in order to have $f_{12} = f_{21} = 0$, which shall be assumed in what follows.

2.2. Potentially self-adjoint difference operators

Let us introduce the second-order partial difference operator

$$\begin{aligned}\mathcal{D}u(\mathbf{x}) &= \sigma_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x}) + \sigma_{12}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x}) + \sigma_{21}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}) \\ &\quad + \sigma_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x}) + \tau_1(\mathbf{x})\Delta_1 u(\mathbf{x}) + \tau_2(\mathbf{x})\Delta_2 u(\mathbf{x}),\end{aligned} \tag{11}$$

which allows us to present Eq. (1) in the form

$$\mathcal{D}u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0, \tag{12}$$

where λ is a constant.

The adjoint operator $\mathcal{D}^\dagger$ of $\mathcal{D}$ is defined by

$$\mathcal{D}^\dagger u = \Delta_1\nabla_1(\sigma_{11}u) + \Delta_1\nabla_2(\sigma_{21}u) + \Delta_2\nabla_1(\sigma_{12}u) + \Delta_2\nabla_2(\sigma_{22}u) - \nabla_1(\tau_1 u) - \nabla_2(\tau_2 u). \tag{13}$$

The operator $\mathcal{D}$ is self-adjoint if $\mathcal{D}^\dagger = \mathcal{D}$.

Definition 2.4. The operator $\mathcal{D}$ is potentially self-adjoint in a domain G if there exists a positive real function $\varrho(\mathbf{x}) = \varrho(x, y)$ in this domain such that the operator $\varrho(\mathbf{x})\mathcal{D}$ is self-adjoint in the domain G.

In order that $\mathcal{D}$ be potentially self-adjoint, we multiply (1) through by a positive function $\varrho(\mathbf{x})$ in some domain G to be chosen later. Then, the operator is self-adjoint provided [24, Section 4] a Pearson type system is satisfied by $\varrho(\mathbf{x})$, which can be explicitly computed as

$$\varrho(x, y) = \kappa \prod_{i=y_0}^{y-1} \mathcal{G}_2(x, i) \prod_{j=x_0}^{x-1} \mathcal{G}_1(j, y_0), \tag{14}$$

where κ is a constant, the functions $G_i(x, y)$ are given by

$$\mathcal{G}_1(x, y) = \frac{\varpi_3(x, y)}{\varpi_1(x+1, y)}, \qquad \mathcal{G}_2(x, y) = \frac{\varpi_4(x, y)}{\varpi_2(x, y+1)}, \tag{15}$$

and the polynomials $\varpi_i(x, y)$ can be computed explicitly from the second-order partial difference equation (1) as

$$\varpi_1(x, y) = \sigma_{11}(x, y) + \sigma_{21}(x, y), \qquad \varpi_3(x, y) = \sigma_{11}(x, y) + \sigma_{12}(x, y) + \tau_1(x, y), \tag{16}$$

$$\varpi_2(x, y) = \sigma_{22}(x, y) + \sigma_{12}(x, y), \qquad \varpi_4(x, y) = \sigma_{21}(x, y) + \sigma_{22}(x, y) + \tau_2(x, y). \tag{17}$$

The expression (14) is the discrete analog of [28, (22), p. 134] and in order to obtain its expression, it is necessary to introduce the domain G as the set of distinct points in $\mathbb{R}^2$, where the functions $\mathcal{G}_1(x, y)$ and $\mathcal{G}_2(x, y)$ are non-negative, i.e. G is a simply connected domain bounded by a piecewise smooth curve

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \mathcal{G}_1(x, y) = \mathcal{G}_2(x, y) = 0\right\}. \tag{18}$$

The function $\varrho(x, y)$, is called the weight function in the domain G, which is determined up to a constant factor, and from the Pearson system we obtain the following condition which must satisfy the polynomial coefficients of the partial difference equation (1)

$$\begin{aligned}&\sigma_{21}(x+1, y)\frac{\varpi_3(x, y)}{\varpi_1(x+1, y)} + \sigma_{21}(x, y-1)\frac{\varpi_2(x, y)}{\varpi_4(x, y-1)} \\ &\quad - \sigma_{12}(x, y+1)\frac{\varpi_4(x, y)}{\varpi_2(x, y+1)} - \sigma_{12}(x-1, y)\frac{\varpi_1(x, y)}{\varpi_3(x-1, y)} = 0,\end{aligned} \tag{19}$$

where the polynomials $\varpi_i(x, y)$ are defined in (16)–(17).

As already analyzed in [24], in order to exist the weight function ϱ it is necessary that the coefficients of the second-order partial difference equation satisfy the coupling hypergeometric condition [24, Eq. (50)]

$$\varpi_1(x+1, y)\varpi_2(x+1, y+1) = \varpi_1(x+1, y+1)\varpi_2(x, y+1), \tag{20}$$

and

$$\mathcal{G}_2(x, y)\Delta_2\mathcal{G}_1(x, y) = \mathcal{G}_1(x, y)\Delta_1\mathcal{G}_2(x, y), \tag{21}$$

which is a discrete analogue of [19, Eq. (17)] and a consequence of the Pearson system for the weight function.

Thus, in what follows we shall deal with a potentially self-adjoint admissible second-order partial difference equation of hypergeometric type, i.e. Eq. (1) where the polynomials

$$\begin{cases} \sigma_{11}(x,y)=\sigma_{11}(x)=a_{11}x^2+d_{11}x+f_{11}, \\ \sigma_{22}(x,y)=\sigma_{22}(y)=a_{11}y^2+e_{22}y+f_{22}, \\ \sigma_{12}(x,y)=c_{12}xy+d_{12}x+e_{12}y, \\ \sigma_{21}(x,y)=c_{21}xy+d_{21}x+e_{21}y, \\ \tau_1(x,y)=\tau_1(x)=s_{11}x+v_{11}, \\ \tau_2(x,y)=\tau_2(y)=s_{11}y+v_{22}, \end{cases} \tag{22}$$

satisfy (19), (20) and (21), the Pearson system [24] holds, with ϱ computed from (14), $\lambda=\lambda_\ell$ is defined in (10) and $\lambda_\ell \neq 0$ for $\ell > 0$.

3. Classification of admissible potentially self-adjoint second-order partial difference equations of hypergeometric type

Let us analyze all the possible situations of admissible second-order partial difference equations of hypergeometric type (1) in the above hypothesis. Some preliminary results about classification have been obtained in [24], which are completed in this paper, since in the previous results the potentially self-adjointness condition was not considered. By using (14) it is possible to obtain appropriate weight functions in each case in a certain domain G which shall be presented in the examples given in Section 5. In doing so, the following condition must hold in (22)

$$c_{21}=a_{11}, \tag{23}$$

which implies $c_{12}=a_{11}$ by using (9), and the following four cases appear for an admissible potentially self-adjoint second-order partial difference equations of hypergeometric type.

3.1. Case 1

If $a_{11}=d_{12}=d_{21}=e_{21}=e_{12}=0$ in (22), then

$$\sigma_{11}(x)=xd_{11}+f_{11}, \qquad \sigma_{12}(x,y)=0, \qquad \sigma_{21}(x,y)=0, \qquad \sigma_{22}(y)=ye_{22}+f_{22},$$
$$\tau_1(x)=xs_{11}+v_{11}, \qquad \tau_2(y)=ys_{11}+v_{22}, \qquad \lambda_\ell=-\ell s_{11}.$$

Moreover, in this case the functions $\mathcal{G}_1(x,y)$ and $\mathcal{G}_2(x,y)$ defined in (15) are given by

$$\mathcal{G}_1(x,y)=\frac{x(d_{11}+s_{11})+f_{11}+v_{11}}{(x+1)d_{11}+f_{11}}, \qquad \mathcal{G}_2(x,y)=\frac{y(e_{22}+s_{11})+f_{22}+v_{22}}{(y+1)e_{22}+f_{22}},$$

which allows us to construct the orthogonality weight function by using (14).

3.2. Case 2

If $a_{11}=d_{12}=e_{21}=f_{11}=f_{22}=0$, $d_{21}\neq 0$, $e_{22}=d_{21}-s_{11}$, $d_{11}=e_{12}-s_{11}$, and $v_{11}=e_{12}v_{22}/d_{21}$ in (22), then

$$\sigma_{11}(x)=x(e_{12}-s_{11}), \qquad \sigma_{22}(y)=y(d_{21}-s_{11}), \qquad \sigma_{12}(x,y)=ye_{12}, \qquad \sigma_{21}(x,y)=xd_{21},$$
$$\tau_1(x)=xs_{11}+e_{12}v_{22}/d_{21}, \qquad \tau_2(y)=ys_{11}+v_{22}, \qquad \lambda_\ell=-\ell s_{11},$$

and the functions $\mathcal{G}_1(x,y)$ and $\mathcal{G}_2(x,y)$ are given by

$$\mathcal{G}_1(x,y)=\frac{e_{12}(d_{21}(x+y)+v_{22})}{(x+1)d_{21}(d_{21}+e_{12}-s_{11})}, \qquad \mathcal{G}_2(x,y)=\frac{d_{21}(x+y)+v_{22}}{(y+1)(d_{21}+e_{12}-s_{11})}.$$

3.3. Case 3

If $a_{11}=d_{21}=e_{12}=0$, $d_{12}\neq 0$, $f_{11}=-e_{21}v_{22}/d_{12}$, $f_{22}=-v_{22}$, $d_{11}=e_{21}$, $v_{11}=e_{21}v_{22}/d_{12}$, and $e_{22}=d_{12}$, then

$$\sigma_{11}(x)=e_{21}\left(x-\frac{v_{22}}{d_{12}}\right), \qquad \sigma_{12}(x,y)=xd_{12}, \qquad \sigma_{21}(x,y)=ye_{21}, \qquad \sigma_{22}(y)=yd_{12}-v_{22},$$
$$\tau_1(x)=xs_{11}+\frac{e_{21}v_{22}}{d_{12}}, \qquad \tau_2(y)=ys_{11}+v_{22}, \qquad \lambda_\ell=-\ell s_{11},$$

and

$$\mathcal{G}_1(x,y)=\frac{d_{12}x(d_{12}+e_{21}+s_{11})}{e_{21}(d_{12}(x+y+1)-v_{22})}, \qquad \mathcal{G}_2(x,y)=\frac{y(d_{12}+e_{21}+s_{11})}{d_{12}(x+y+1)-v_{22}}.$$

3.4. Case 4

If $a_{11} \neq 0$, $d_{11} = -d_{12} - d_{21} + e_{12} + e_{21} + e_{22}$,

$$f_{11} = \frac{e_{21}(-d_{12} + e_{12} + e_{22})}{a_{11}}, \qquad f_{22} = \frac{d_{12}(-d_{12} + e_{12} + e_{22})}{a_{11}},$$
$$v_{11} = \frac{e_{21}(d_{12} - e_{22}) + e_{12}(e_{22} + s_{11} - d_{21})}{a_{11}}, \qquad v_{22} = \frac{d_{21}(-d_{21} + e_{21} + e_{22} + s_{11})}{a_{11}} - f_{22},$$

and one of the following five conditions holds true

$$d_{12} = e_{21} = 0, \quad \text{or} \tag{24}$$
$$d_{21} = e_{12} = 0, \quad \text{or} \tag{25}$$
$$e_{21} = e_{12} = 0, \quad \text{or} \tag{26}$$
$$d_{21} = d_{12} = 0, \quad \text{or} \tag{27}$$
$$e_{12} = d_{21}, \qquad e_{21} = d_{12}, \quad \text{and} \quad d_{21} = a_{11} + d_{12}, \tag{28}$$

then

$$\sigma_{11}(x) = x^2 a_{11} + x(-d_{12} - d_{21} + e_{12} + e_{21} + e_{22}) + \frac{e_{21}(-d_{12} + e_{12} + e_{22})}{a_{11}},$$
$$\sigma_{12}(x, y) = xya_{11} + xd_{12} + ye_{12},$$
$$\sigma_{21}(x, y) = xya_{11} + xd_{21} + ye_{21},$$
$$\sigma_{22}(y) = y^2 a_{11} + ye_{22} + \frac{d_{12}(-d_{12} + e_{12} + e_{22})}{a_{11}},$$
$$\tau_1(x) = s_{11}x + \frac{e_{21}(d_{12} - e_{22}) + e_{12}(e_{22} + s_{11}) - d_{21}e_{12}}{a_{11}},$$
$$\tau_2(y) = s_{11}y + \frac{d_{21}(e_{21} + e_{22} + s_{11} - d_{21}) - d_{12}(e_{12} + e_{22} - d_{12})}{a_{11}},$$
$$\lambda_\ell = -\ell\big((\ell - 1)a_{11} + s_{11}\big).$$

Thus, the functions $\mathcal{G}_1(x, y)$ and $\mathcal{G}_2(x, y)$ defined in (15) are given by

$$\mathcal{G}_1(x, y) = \frac{(a_{11}x + e_{12})(a_{11}(x + y) - d_{21} + e_{21} + e_{22} + s_{11})}{((x + 1)a_{11} + e_{21})(a_{11}(x + y + 1) - d_{12} + e_{12} + e_{22})},$$
$$\mathcal{G}_2(x, y) = \frac{(a_{11}y + d_{21})(a_{11}(x + y) - d_{21} + e_{21} + e_{22} + s_{11})}{((y + 1)a_{11} + d_{12})(a_{11}(x + y + 1) - d_{12} + e_{12} + e_{22})}.$$

Remark 2. It is important to mention here that in comparison with [24] we have now added the condition (19) in the classification process.

4. Explicit expressions for the coefficients in the three-term recurrence relations: monic case

Let us consider a monic vector polynomial family $\{\widehat{\mathbb{P}}_n\}_{n\in\mathbb{N}_0}$ solution of (1) and orthogonal with respect to the weight (14)

$$\sum_{(x,y)\in G} \sum \mathbf{x}^m \, \widehat{\mathbb{P}}_n^T \varrho(x, y) = \begin{cases} 0 \in \mathcal{M}^{(m+1,n+1)}, & \text{if } n > m, \\ H_n \in \mathcal{M}^{(n+1,n+1)}, & \text{if } m = n, \end{cases} \tag{29}$$

where H_n (of size $(n+1) \times (n+1)$) is nonsingular, in an appropriate domain $G \subset \mathbb{R}^2$.

Let us first introduce the matrices $L_{n,j}$ of size $(n+1) \times (n+2)$

$$L_{n,1} = \begin{pmatrix} 1 & & \bigcirc & 0 \\ & \ddots & & \vdots \\ \bigcirc & & 1 & 0 \end{pmatrix} \quad \text{and} \quad L_{n,2} = \begin{pmatrix} 0 & 1 & & \bigcirc \\ \vdots & & \ddots & \\ 0 & \bigcirc & & 1 \end{pmatrix}, \tag{30}$$

so that

$$x\mathbf{x}^n = L_{n,1}\mathbf{x}^{n+1}, \qquad y\mathbf{x}^n = L_{n,2}\mathbf{x}^{n+1}. \tag{31}$$

Let us observe that

$$x^2\mathbf{x}^n = L_{n,1}L_{n+1,1}\mathbf{x}^{n+2}, \qquad y^2\mathbf{x}^n = L_{n,2}L_{n+1,2}\mathbf{x}^{n+2},$$
$$L_{n,2}L_{n+1,1} = L_{n,1}L_{n+1,2}, \tag{32}$$

and for $j = 1, 2$,

$$L_{n,j}L_{n,j}^T = I_{n+1}, \tag{33}$$

where I_{n+1} denotes the identity matrix of size $n + 1$.

From the definition of the forward and backward difference operators Δ and ∇, we obtain

$$\Delta_j\mathbf{x}^n = \sum_{k=1}^{n}\mathbb{E}_{n,j}^{k}\mathbf{x}^{n-k}, \qquad \nabla_j\mathbf{x}^n = \sum_{k=1}^{n}(-1)^{k+1}\mathbb{E}_{n,j}^{k}\mathbf{x}^{n-k},$$

where the entries of the matrices $\mathbb{E}_{n,j}^{r} = (e_{p,q,j}^{r})$ of size $(n+1)\times(n-r)$ are given by

$$e_{p,q,1}^{r}(n) = \begin{cases} \binom{n-p}{r}, & p = q, \\ 0, & p \neq q, \end{cases} \qquad e_{p,q,2}^{r}(n) = \begin{cases} \binom{p}{r}, & p = q + r, \\ 0, & p \neq q + r. \end{cases}$$

If we substitute the expansion (5) in (1), by equating the coefficients in $\mathbf{x}^{n-1}$ and $\mathbf{x}^{n-2}$ we obtain the following explicit expressions for the matrices $\widehat{G}_{n,n-1}$ and $\widehat{G}_{n,n-2}$:

$$\widehat{G}_{n,n-1} = \mathbb{S}_n\mathbb{F}_{n-1}^{-1}(\lambda_n), \tag{34}$$
$$\widehat{G}_{n,n-2} = (\mathbb{T}_n + \widehat{G}_{n,n-1}\mathbb{S}_{n-1})\mathbb{F}_{n-2}^{-1}(\lambda_n), \tag{35}$$

where the nonsingular matrix $\mathbb{F}_n(\lambda_\ell)$ is given by

$$\mathbb{F}_n(\lambda_\ell) = (\lambda_n - \lambda_\ell)\mathbb{I}_{n+1}, \tag{36}$$

λ_n is given in (10), $\mathbb{I}_{n+1}$ denotes the identity matrix of size $(n+1)\times(n+1)$, and the matrix $\mathbb{S}_n$ of size $(n+1)\times n$ is given in terms of the coefficients of the polynomials σ_{ij} and τ_i of the partial difference equation (1) given in (22) by

$$\mathbb{S}_n = \begin{pmatrix} s_{1,1} & & & & \bigcirc \\ s_{2,1} & s_{2,2} & & & \\ & \ddots & \ddots & & \\ & & s_{n-1,n-2} & s_{n-1,n-1} & \\ & & & s_{n,n-1} & s_{n,n} \\ & \bigcirc & & 0 & s_{n+1,n} \end{pmatrix} \quad (n \geqslant 1), \tag{37}$$

where, for $1 \leqslant i \leqslant n$,

$$s_{i,i} = (-i + n + 1)\left((n-i)\left(d_{11} + \frac{s_{11}}{2}\right) + (i-1)(e_{12} + e_{21}) + v_{11}\right),$$

$$s_{i+1,i} = i(d_{12} + d_{21})(n-i) + (i-1)i\left(e_{22} + \frac{s_{11}}{2}\right) + iv_{22}.$$

Moreover, the matrix $\mathbb{T}_n$ of size $(n+1)\times(n-1)$ is given by

$$\mathbb{T}_n = \begin{pmatrix} t_{1,1} & & & \bigcirc \\ t_{2,1} & t_{2,2} & & \\ t_{3,1} & t_{3,2} & t_{3,3} & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & t_{n-1,n-3} & t_{n-1,n-2} & t_{n-1,n-1} \\ \bigcirc & & t_{n,n-2} & t_{n,n-1} \\ & & 0 & t_{n+1,n-1} \end{pmatrix} \quad (n \geqslant 2), \tag{38}$$

where, for $1 \leqslant i \leqslant n-1$,

$$t_{i,i} = -\frac{1}{12}(i-n-1)(i-n)\big(a_{11}(i-n+1)(3i+n-6) + 2\big(3(i-1)(e_{21}-e_{12}) - 6f_{11} + s_{11}(i-n+1) - 3v_{11}\big)\big),$$
$$t_{i+1,i} = -\frac{1}{2}i(i-n)\big((i-n+1)\big((i-1)a_{11} - d_{12} + d_{21}\big) + (1-i)(e_{12}-e_{21})\big),$$
$$t_{i+2,i} = \frac{1}{12}i\big(\big(i^2-1\big)\big(a_{11}(-3i+4n-6) + 2s_{11}\big) + 6(i+1)\big((d_{12}-d_{21})(i-n+1) + 2f_{22} + v_{22}\big)\big).$$

Now, in this monic situation, it is possible to generalize the well-known explicit expressions for the coefficients in the three-term recurrence relation in the one variable case [21, p. 14] to the bivariate discrete case. This is done with the help of the auxiliary matrices $L_{n,j}$ defined in (30)–(31) and the following result proved in [4] in the continuous bivariate situation, which is also valid in the bivariate discrete situation since it is a consequence of the three-term recurrence relations (7).

Theorem 4.1. *In the monic case, the explicit expressions of the matrices $A_{n,j}$, $B_{n,j}$ and $C_{n,j}$ ($j = 1, 2$) appearing in* (7) *in terms of the values of the leading coefficients $\widehat{G}_{n,n-1}$ and $\widehat{G}_{n,n-2}$, obtained in* (34) *and* (35) *respectively, are given by*

$$\begin{cases} A_{n,j} = L_{n,j}, \quad n \geqslant 0, \\ B_{0,j} = -L_{0,j}\widehat{G}_{1,0}, \qquad B_{n,j} = \widehat{G}_{n,n-1}L_{n-1,j} - L_{n,j}\widehat{G}_{n+1,n}, \quad n \geqslant 1, \\ C_{1,j} = -(L_{1,j}\widehat{G}_{2,0} + B_{1,j}\widehat{G}_{1,0}), \\ C_{n,j} = \widehat{G}_{n,n-2}L_{n-2,j} - L_{n,j}\widehat{G}_{n+1,n-1} - B_{n,j}\widehat{G}_{n,n-1}, \quad n \geqslant 2, \end{cases} \tag{39}$$

where the matrices $L_{n,j}$ have been introduced in (30)*.*

It has some interest to remark here that, as described in [7], since

$$\operatorname{rank}(L_{n,j}) = n+1 = \operatorname{rank}(C_{n+1,j}), \quad j = 1, 2, \ n \geqslant 0, \tag{40}$$

the columns of the joint matrices

$$L_n = \big(L_{n,1}^T, L_{n,2}^T\big)^T \quad \text{and} \quad C_n = \big(C_{n,1}^T, C_{n,2}^T\big)^T,$$

of size $(2n+2) \times (n+2)$ and $(2n+2) \times n$ respectively, are linearly independent, i.e.

$$\operatorname{rank}(L_n) = n+2, \qquad \operatorname{rank}(C_n) = n. \tag{41}$$

Therefore, the matrix L_n has full rank so that there exists a unique matrix $D_n^\dagger$ of size $(n+2) \times (2n+2)$, called the generalized inverse of L_n:

$$D_n^\dagger = (D_{n,1}|D_{n,2}) = \big(L_n^T L_n\big)^{-1} L_n^T, \tag{42}$$

such that

$$D_n^\dagger L_n = I_{n+2}.$$

Moreover, using the left inverse $D_n^\dagger$ of the joint matrix L_n

$$D_n^\dagger = \begin{pmatrix} 1 & & & & 0 & & & \\ & 1/2 & & \bigcirc & 1/2 & & \bigcirc & \\ & & \ddots & & & \ddots & & \\ & \bigcirc & & 1/2 & \bigcirc & & 1/2 & \\ & & & & 0 & & & 1 \end{pmatrix},$$

we can write a recursive formula for the monic orthogonal polynomials

$$\widehat{\mathbb{P}}_{n+1} = D_n^\dagger\left[\begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n\right]\widehat{\mathbb{P}}_n - D_n^\dagger C_n \widehat{\mathbb{P}}_{n-1}, \quad n \geqslant 0, \tag{43}$$

with the initial conditions $\widehat{\mathbb{P}}_{-1} = 0$, $\widehat{\mathbb{P}}_0 = 1$, where $\otimes$ denotes the Kronecker product and

$$B_n = \big(B_{n,1}^T, B_{n,2}^T\big)^T, \qquad C_n = \big(C_{n,1}^T, C_{n,2}^T\big)^T, \tag{44}$$

are matrices of size $(2n+2) \times (n+1)$ and $(2n+2) \times n$, respectively. This recurrence (43) gives another presentation of [7, (3.2.10)], already presented in the bivariate discrete case in [25].

Therefore, from (43) it is possible to compute a monic orthogonal polynomial solution of a potentially self-adjoint admissible second-order partial difference equation of hypergeometric type (1).

Next we give the entries of the matrices $\mathbb{S}_n$ and $\mathbb{T}_n$ introduced in (37) and (38) respectively, in the four possible cases of the classification presented in Section 3.

4.1. Case 1

In this case, the coefficients of the matrices $\mathbb{S}_n$ defined in (37) are given by

$$s_{i,i} = (-i+n+1)\left((n-i)\left(d_{11}+\frac{s_{11}}{2}\right)+v_{11}\right),$$
$$s_{i+1,i} = i\left((i-1)\left(e_{22}+\frac{s_{11}}{2}\right)+v_{22}\right),$$

and the coefficients of the matrices $\mathbb{T}_n$ defined in (38) are given by

$$t_{i,i} = \frac{1}{6}(i-n-1)(n-i)\big(-6f_{11}+s_{11}(i-n+1)-3v_{11}\big),$$
$$t_{i+1,i} = 0,$$
$$t_{i+2,i} = \frac{1}{6}i(i+1)\big(6f_{22}+(i-1)s_{11}+3v_{22}\big).$$

4.2. Case 2

In this case, the coefficients of the matrices $\mathbb{S}_n$ defined in (37) are given by

$$s_{i,i} = (-i+n+1)\left(e_{12}\left(\frac{v_{22}}{d_{21}}+n-1\right)+\frac{1}{2}s_{11}(i-n)\right),$$
$$s_{i+1,i} = \frac{1}{2}i\big(2(n-1)d_{21}-(i-1)s_{11}+2v_{22}\big),$$

and the coefficients of the matrices $\mathbb{T}_n$ defined in (38) are given by

$$t_{i,i} = \frac{1}{6}(i-n-1)(i-n)\left(3e_{12}\left(\frac{v_{22}}{d_{21}}+i-1\right)+s_{11}(-i+n-1)\right),$$
$$t_{i+1,i} = \frac{1}{2}i(i-n)\big(d_{21}(-i+n-1)+(i-1)e_{12}\big),$$
$$t_{i+2,i} = \frac{1}{6}i(i+1)\big(-3d_{21}(i-n+1)+(i-1)s_{11}+3v_{22}\big).$$

4.3. Case 3

In this case, the coefficients of the matrices $\mathbb{S}_n$ defined in (37) are given by

$$s_{i,i} = (-i+n+1)\left(e_{21}\left(\frac{v_{22}}{d_{12}}+n-1\right)+\frac{1}{2}s_{11}(n-i)\right),$$
$$s_{i+1,i} = \frac{1}{2}i\big(2(n-1)d_{12}+(i-1)s_{11}+2v_{22}\big),$$

and the coefficients of the matrices $\mathbb{T}_n$ defined in (38) are given by

$$t_{i,i} = \frac{(i-n-1)(n-i)(d_{12}(3(i-1)e_{21}+s_{11}(i-n+1))+3e_{21}v_{22})}{6d_{12}},$$
$$t_{i+1,i} = \frac{1}{2}i(i-n)\big(d_{12}(i-n+1)-(i-1)e_{21}\big),$$
$$t_{i+2,i} = \frac{1}{6}i(i+1)\big(3d_{12}(i-n+1)+(i-1)s_{11}-3v_{22}\big).$$

4.4. Case 4

Let us assume that one of the five conditions (24)–(28) holds true. In this case, the coefficients of the matrices $\mathbb{S}_n$ defined in (37) are given by

$$s_{i,i}=(n-i+1)\biggl((n-i)\biggl(e_{12}+e_{21}+e_{22}-d_{12}-d_{21}+\frac{s_{11}}{2}\biggr)+(i-1)(e_{12}+e_{21}) \\ +f_{22}+\frac{e_{12}(s_{11}-d_{21})-d_{12}(-d_{12}+e_{12}-e_{21}+e_{22})+(e_{12}-e_{21})e_{22}}{a_{11}}\biggr),$$

$$s_{i+1,i}=i\biggl((d_{12}+d_{21})(n-i)+(i-1)\biggl(e_{22}+\frac{s_{11}}{2}\biggr)-f_{22}+\frac{d_{21}(e_{21}+e_{22}-d_{21}+s_{11})}{a_{11}}\biggr),$$

and the coefficients of the matrices $\mathbb{T}_n$ defined in (38) are given by

$$t_{i,i}=\frac{1}{12}(i-n-1)(i-n)\biggl(\frac{6(e_{12}(2e_{21}+e_{22}-d_{12}-d_{21}+s_{11})+(d_{12}-e_{21})(d_{12}-e_{22}))}{a_{11}} \\ +(n-i-1)\bigl(a_{11}(3i+n-6)+2s_{11}\bigr)+6\bigl((i-1)e_{12}-(i-1)e_{21}+f_{22}\bigr)\biggr),$$

$$t_{i+1,i}=i(i-n)\biggl(\frac{d_{12}(e_{12}+e_{22}-d_{12})}{a_{11}}+f_{22} \\ +\frac{1}{2}\bigl((n-i)\bigl((i-1)a_{11}-d_{12}+d_{21}\bigr)+(i-1)(-a_{11}+e_{12}-e_{21})+d_{12}-d_{21}\bigr)\biggr),$$

$$t_{i+2,i}=\frac{1}{12}i(i+1)\biggl(\frac{6d_{21}(e_{21}+e_{22}-d_{21}+s_{11})}{a_{11}} \\ +(i-1)\bigl(2s_{11}-a_{11}(3i-4n+6)\bigr)+6\bigl((d_{12}-d_{21})(i-n+1)+f_{22}\bigr)\biggr).$$

5. Examples

There exist different families of bivariate discrete orthogonal polynomials in the literature, introduced from the generalized Kampé de Fériet hypergeometric series [26] or as product of univariate orthogonal polynomials of a discrete variable (Charlier, Kravchuk, Meixner and Hahn [13]). In this section we summarize these families and we show that in the lattices considered in this paper all of them are solution of an admissible potentially self-adjoint second-order partial difference equation of hypergeometric type. Moreover, the monic polynomials can be computed from (43) by using the matrices explicitly given in Section 4.

5.1. Case 1

Let us consider

$$\sigma_{11}(x)=x,\qquad \sigma_{12}=\sigma_{21}=0,\qquad \sigma_{22}(y)=y,\qquad \tau_1(x)=a_1-x,\qquad \tau_2(y)=a_2-y.$$

In this situation, it is known that Eq. (1) has two orthogonal polynomial family solutions. On one hand, the family introduced by Tratnik [31] as a product of Meixner and Charlier polynomials

$$\tilde{u}_{n_1,n_2}(x,y)=M_{n_1}(x;x-y,-a_1/a_2)C_{n_2}(x+y-n_1;a_1+a_2),\quad 0\leqslant n_1+n_2, \tag{45}$$

is solution of (1). Moreover, it is simple to prove that the monic family obtained as a product of two monic Charlier polynomials

$$\hat{v}_{n_1,n_2}(x,y)=\widehat{C}_{n_1}(x;a_1)\widehat{C}_{n_2}(y;a_2),\quad 0\leqslant n_1+n_2, \tag{46}$$

is another polynomial solution of the same equation, belonging to Case 1 of the classification done in Section 3. Both families (45) and (46) are orthogonal with respect to the weight function which can be computed from (14)

$$\varrho^{a_1,a_2}(x,y)=\frac{a_1^x a_2^y}{\Gamma(x+1)\Gamma(y+1)},$$

in the domain G defined by $x\geqslant 0$ and $y\geqslant 0$, assuming that $a_1>0$ and $a_2>0$.

5.2. Case 2

In this case, it is possible to obtain orthogonal families in a bounded or unbounded domain.

5.2.1. Meixner type

Let us first consider

$$\sigma_{11}(x) = (1-a_2)x, \qquad \sigma_{12}(x,y) = a_1 y, \qquad \sigma_{21}(x,y) = a_2 x, \qquad \sigma_{22}(y) = (1-a_1)y,$$
$$\tau_1(x) = (a_1+a_2-1)x + \beta a_1, \qquad \tau_2(y) = (a_1+a_2-1)y + \beta a_2.$$

This choice in (1) implies that this equation belongs uniquely to the second case according to Section 3. There exist at least three polynomial solutions of the above equation. The non-monic bivariate Meixner polynomials [31] defined in terms of generalized Kampé de Fériet hypergeometric series

$$M_{n_1,n_2}^{\beta,a_1,a_2}(x,y) = (x+y+\beta)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{matrix} -:-n_1,-x;-n_2,-y \\ -n_1-n_2-x-y-\beta+1:-;- \end{matrix}\,\middle|\, a_1^{-1}, a_2^{-1}\right), \quad 0 \leqslant n_1+n_2, \tag{47}$$

the non-monic bivariate Meixner polynomials [31] defined in terms of a product of Meixner polynomials

$$\widetilde{M}_{n_1,n_2}^{\beta,a_1,a_2}(x,y) = M_{n_1}(x;-x-y,-a_1/a_2)M_{n_2}(x+y-n_1;\beta+n_1,a_1+a_2), \quad 0 \leqslant n_1+n_2, \tag{48}$$

and the monic bivariate Meixner polynomials [31] defined in terms of generalized Kampé de Fériet hypergeometric series

$$\widehat{M}_{n_1,n_2}^{\beta,a_1,a_2}(x,y) = \frac{a_1^{n_1}a_2^{n_2}}{(a_1+a_2-1)^{n_1+n_2}}(\beta)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{matrix} -:-n_1,-x;-n_2,-y \\ \beta:-;- \end{matrix}\,\middle|\, \frac{a_1+a_2-1}{a_1}, \frac{a_1+a_2-1}{a_2}\right), \quad 0 \leqslant n_1+n_2, \tag{49}$$

are solutions of (1) [23–25]. The matrices of the recurrence relations of this later family (monic) can be easily computed from Theorem 4.1.

We would like to emphasize that these three families of polynomials (47), (48) and (49) are orthogonal with respect to the weight function computed from (14)

$$\varrho^{\beta,a_1,a_2}(x,y) = \frac{a_1^x a_2^y \Gamma(x+y+\beta)}{\Gamma(x+1)\Gamma(y+1)\Gamma(\beta)},$$

in the domain G defined by $x \geqslant 0$ and $y \geqslant 0$, assuming that $a_1 > 0$, $a_2 > 0$ and $\beta > 0$.

5.2.2. Kravchuk type

Let us also consider

$$\sigma_{11}(x) = (p_1-1)x, \qquad \sigma_{12}(x,y) = p_1 y, \qquad \sigma_{21}(x,y) = p_2 x, \qquad \sigma_{22}(y) = (p_2-1)y,$$
$$\tau_1(x) = x - Np_1, \qquad \tau_2(y) = y - Np_2.$$

This choice in (1) implies that this equation belongs uniquely to the second case according to Section 3. There exist at least three polynomial solutions of the above equation. For $0 \leqslant n_1+n_2 \leqslant N$ the non-monic bivariate Kravchuk polynomials [31]

$$K_{n_1,n_2}^{p_1,p_2}(x,y;N) = (x+y-N)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{matrix} -:-n_1,-x;-n_2,-y \\ -n_1-n_2-x-y+N+1:-;- \end{matrix}\,\middle|\, \frac{p_1+p_2-1}{p_1}, \frac{p_1+p_2-1}{p_2}\right), \tag{50}$$

the non-monic bivariate Kravchuk polynomials [31] defined as a product of Kravchuk polynomials

$$\widetilde{K}_{n_1,n_2}^{p_1,p_2}(x,y;N) = \frac{(N-n_1)!}{N!(n_1-N)_{n_2}} \times K_{n_1}\big(x;p_1/(p_1+p_2),x+y\big)K_{n_2}(x+y-n_1;p_1+p_2,N-n_1), \quad 0 \leqslant n_1+n_2 \leqslant N, \tag{51}$$

and the monic bivariate Kravchuk polynomials [31]

$$\widehat{K}_{n_1,n_2}^{p_1,p_2}(x,y;N) = (-1)^{n_1+n_2}p_1^{n_1}p_2^{n_2}(N-n_1-n_2+1)_{n_1+n_2} \times F_{1:0;0}^{0:2;2}\left(\begin{matrix} -:-n_1,-x;-n_2,-y \\ -N:-;- \end{matrix}\,\middle|\, \frac{1}{p_1}, \frac{1}{p_2}\right), \quad 0 \leqslant n_1+n_2 \leqslant N, \tag{52}$$

are solutions of (1) [23–25]. The explicit form of the matrices of the recurrence relations of this later family (monic bivariate Kravchuk polynomials) can be easily computed from Theorem 4.1, and they coincide with the results already given in [25, Section 4.2].

These three families of polynomials (50), (51) and (52) are orthogonal with respect to the trinomial distribution obtained from (14)

$$\varrho^{N,p_1,p_2}(x,y)=\frac{p_1^x p_2^y(1-p_1-p_2)^{N-x-y}\Gamma(N+1)}{\Gamma(x+1)\Gamma(y+1)\Gamma(N-x-y+1)},$$

in the domain G defined by $x\geqslant 0$, $y\geqslant 0$, and $0\leqslant x+y\leqslant N$, where N is a positive integer and p_1 and p_2 are real numbers satisfying

$$p_1>0,\qquad p_2>0,\quad 0<p_1+p_2<1.$$

5.3. Case 3

This situation corresponds to symmetric situations of Case 2 with respect to the origin.

Let $d_{12}=-a_2$, $e_{21}=-a_1$, $s_{11}=a_1+a_2-1$ and $v_{22}=-a_2\beta$ in (22). Thus,

$$\sigma_{11}(x)=a_1(\beta-x),\qquad \sigma_{12}(x,y)=-a_2x,\qquad \sigma_{21}(x,y)=-a_1y,\qquad \sigma_{22}(y)=a_2(\beta-y),$$
$$\tau_1(x)=(a_1+a_2-1)x-a_1\beta,\qquad \tau_2(y)=(a_1+a_2-1)y-a_2\beta,$$

and Eq. (1) belongs uniquely to the third case according to Section 3. The polynomial families $M_{n_1,n_2}^{\beta,a_1,a_2}(-x,-y)$, $\widetilde{M}_{n_1,n_2}^{\beta,a_1,a_2}(-x,-y)$, and $\widehat{M}_{n_1,n_2}^{\beta,a_1,a_2}(-x,-y)$, defined in (47), (48), and (49), respectively, are solution of this equation. A general approach for symmetries of orthogonal families has been already presented in [30] from the hypergeometric representations and without considering the partial difference equation satisfied by the polynomials.

On the other hand, if we set $d_{12}=p_2$, $e_{21}=p_1$, $s_{11}=-1$ and $v_{22}=-Np_2$, we obtain

$$\sigma_{11}(x,y)=p_1(N+x),\qquad \sigma_{12}(x,y)=p_2x,\qquad \sigma_{21}(x,y)=p_1y,\qquad \sigma_{22}(x,y)=p_2(N+y),$$
$$\tau_1(x,y)=-x-Np_1,\qquad \tau_2(x,y)=-y-Np_2$$

and Eq. (1) belongs uniquely to the third case according to Section 3. The polynomial families $K_{n_1,n_2}^{p_1,p_2}(-x,-y;N)$, $\widetilde{K}_{n_1,n_2}^{p_1,p_2}(-x,-y;N)$, and $\widehat{K}_{n_1,n_2}^{p_1,p_2}(-x,-y;N)$, defined in (50), (51), and (52) respectively, are solution of this equation.

5.4. Case 4

Let us consider as polynomial coefficients of the partial difference equation (1)

$$\sigma_{11}(x)=x(x-\beta-\gamma+M-2),\qquad \sigma_{22}(y)=y(y-\alpha-\gamma+M-2),$$
$$\sigma_{12}(x,y)=y(\alpha+x+1),\qquad \tau_1(x)=x(\alpha+\beta+\gamma+3)+(\alpha+1)M,$$
$$\sigma_{21}(x,y)=x(\beta+y+1),\qquad \tau_2(y)=y(\alpha+\beta+\gamma+3)+(\beta+1)M.$$

It is clear that this equation belongs uniquely to the fourth case according to Section 3. The above equation has a monic orthogonal polynomial solution defined by means of

$$\hat{u}_{n,m}(x,y)=\frac{(\alpha+1)_n(\beta+1)_m(M)_{n+m}}{(n+m+\alpha+\beta+\gamma+2)_{n+m}}\times F_{1:1;1}^{1:2;2}\left(\begin{matrix}n+m+\alpha+\beta+\gamma+2:-n,-x;-m,-y\\ M:\alpha+1;\beta+1\end{matrix}\,\middle|\,1,1\right).\tag{53}$$

Let $M=-N-1$, where N is a positive integer and $0\leqslant n+m\leqslant N+1$. Then, the polynomials (53) defined in [31] as (monic) Hahn polynomials are orthogonal with respect to

$$\varrho(x,y)=\frac{\Gamma(x+\alpha+1)\Gamma(y+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(x+1)\Gamma(y+1)}\frac{(N+2-x-y)_{x+y}}{(N+c+2-x-y)_{x+y}},\tag{54}$$

in the domain G defined by $x\geqslant 0$, $y\geqslant 0$, $x+y\leqslant N+1$, assuming that α, β and γ are real parameters such that $\alpha>-1$, $\beta>-1$ and $\alpha+\beta+\gamma\geqslant -2$. The matrices of the recurrence relations satisfied by these polynomials (53) can be computed from Theorem 4.1 and they coincide with the matrices published in [25, Section 4.1].

Moreover, if we set $M=N+1$ in the difference equation, where N is a positive integer, then we found in [24] that the Hahn family introduced by Tratnik in [31]

$$u_{n,m}(x,y)=(-1)^n(N+1+x+y)_{n+m}\times F_{1:1;1}^{1:2;2}\left(\begin{matrix}-n-m-\gamma:-n,-x;-m,-y\\ -n-m-N-x-y:\alpha+1;\beta+1\end{matrix}\,\middle|\,1,1\right),\tag{55}$$

is non-monic polynomial solutions of (1) orthogonal with respect to

$$\varrho(x, y) = \frac{\Gamma(x+\alpha+1)\Gamma(y+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(x+1)\Gamma(y+1)} \frac{(N+1)_{x+y}}{(N-c+1)_{x+y}}, \tag{56}$$

in the domain G defined by $x \geqslant 0$, $y \geqslant 0$, assuming that α, β and γ are real parameters such that $\alpha > -1$, $\beta > -1$ and $\gamma < 0$.

Finally, if we set $M = -N + 1$ in the difference equation, then we have found in [23] that the Hahn polynomials introduced by Karlin and McGregor [11] and Tratnik [31] as a product of Hahn polynomials

$$H_{n_1,n_2}^{\alpha,\beta,\gamma}(x, y; N) = \frac{\binom{x+y}{n_1}}{\binom{N-1}{n_1}} Q_{n_2}(x+y-n_1; \alpha+\beta+2n_1+1, \gamma, N-n_1) Q_{n_1}(x; \alpha, \beta, x+y+1) \tag{57}$$

for $0 \leqslant n_1 + n_2 \leqslant N - 1$, are non-monic solutions of (1). These polynomials are orthogonal with respect to the weight function obtained from (14)

$$\varrho(x, y) = \frac{\Gamma(N)\Gamma(\alpha+x+1)\Gamma(\beta+y+1)\Gamma(\alpha+\beta+\gamma+3)\Gamma(\gamma+N-x-y)}{\Gamma(x+1)\Gamma(y+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(N-x-y)\Gamma(\alpha+\beta+\gamma+N+2)}, \tag{58}$$

in the domain G defined by $x \geqslant 0$, $y \geqslant 0$ and $0 \leqslant x + y \leqslant N - 1$, assuming that α, β and γ are real parameters such that $\alpha > -1$, $\beta > -1$ and $\gamma > -1$.

Finally, we would like to mention here that our interest is not to compute the matrices for all the cases (due to the length and since some of them have been previously published without giving a general approach), but to emphasize the information contained in this class of partial difference equation.

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