Non-parallel-flow effect on compressible boundary layer on a flat plate

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Abstract

Instability of boundary layers is an important aspect of laminar flow - turbulence transition. A local, simple and non-perturbative approach to spatial instability of three-dimensional compressible boundary layers, referred to as 'extended eigenvalue' or 'EEV' approach, is presented, in which both the streamwise variation of the mean flow and the shape distortion of the disturbance are taken into account. In the case of compressible Blasius boundary layer on a flat plate, the predictions are verified and confirmed by direct numerical simulation results, and the non-parallel effects on eigenvalues, eigenfunctions and neutral curves for different Mach numbers are discussed.

Keywords: non-parallel flow; linear stability theory; compressible flow; boundary layers; eigenvalue problem.

1. Introduction

Instability of boundary layers is an important aspect of transition, and the problem of instability and laminar-turbulent transition of boundary layers has been studied extensively for many decades because of its fundamental importance in classical physics and practical relevance to many technologies. Since disturbances at early stage are pretty weak, linear stability theory (LST) has been developed and widely applied [1], in which the local parallel-flow assumption is applied and the streamwise variation and the normal velocity are ignored. It is recognized that non-parallelism is more significant for oblique waves [2] and cross-flow vortices [3].

Non-parallel-flow effects on the growth rate consist of two parts: (1) the direct contribution of the streamwise variation of the mean flow, and (2) the influence of the streamwise distortion of the disturbance. Based on the technique of WKB/multi-scale expansion, many methods have been developed to calculate the non-parallel-flow correction for incompressible flow, in which the non-parallel theory given by Gaster [4] is regarded as the best. However those methods are perturbative in that non-parallel needs to be a higher-order correction to the leading-order prediction by the parallel-flow approximation. The approach of parabolized stability equations (PSE)[3] has become a popular tool for analyzing instability of weakly non-parallel flows, which is an initial value problem and has the limitation of weak elliptic effect for compressible flow. Direct numerical simulation (DNS) [5] stands as a powerful tool for studying instability of spatial boundary layers, while the computation time is long and the cost is high. A non-perturbative

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approach to spatial instability of weakly incompressible non-parallel shear flows is proposed by Huang & Wu [6], in which both direction and indirect effects of non-parallelism are taken into account by expanding the local mean flow and the shape function as Taylor series, leading to a sequence of extended eigen-value (EEV) problems.

Most of the studies of non-parallel effects are focus on the incompressible flow, and the non-parallel-flow effect on spatial instability of compressible flow is still unclear. In this paper, we will extend our method from the incompressible flow to the compressible flow, and apply it to compressible Blasius boundary layer to study the non-parallel-flow effect on spatial instability.

2. Formulation

The flow of compressible boundary layer is characterized by a Reynolds number \( R \) and a Mach number \( M \), as
\[
R = \rho_\infty U_\infty \delta^\star / \mu_\infty, \quad M = U_\infty / c_\infty,
\]
where \( \rho, U, \mu \) and \( c \) note the density, velocity, viscous coefficient and sound speed respectively, \( \delta \) the displacement thickness of the boundary layer, the superscript \( * \) a dimensional quantity, the subscript \( \infty \) the free-stream condition. The governing equations are the dimensionless three-dimensional compressible N-S equations, in which the viscosity coefficient \( \mu \) is determined by Sutherland’s law and the conductivity coefficient \( \kappa \) is calculated by \( \mu \) with a constant Prandtl number \( Pr = 0.72 \).

In order to study its stability, the 2D mean flow \( \bar{Q}(x, y) = (\bar{\rho}, \bar{u}_y, \bar{T}) \) is perturbed by a 3D small amplitude disturbance \( Q'(x, y, z, t) = (\rho', u'_y, T') \). Substituting the total flow field \( Q(x, y, z, t) = \bar{Q} + Q' \) into the dimensionless three-dimensional compressible N-S equations and linearizing about the mean flow, we obtain the linear N-S equations for the perturbation, written in matrix form
\[
T \frac{\partial Q'}{\partial t} + A Q' + \sum_{i=1,3} B_i \frac{\partial Q'}{\partial x_i} + \sum_{i,j=1,3} C_{ij} \frac{\partial^2 Q'}{\partial x_i \partial x_j} = 0,
\]
where \( T, A, B, \) and \( C_{ij} \) are the coefficient matrices constructed by the mean flow \( \bar{Q} \) and its derivatives.

According to the concept of normal mode, the perturbation \( Q'(x, y, z, t) \) can be written as
\[
Q'(x, y, z, t) = \phi(x, y) \exp \left( i\left( \int \alpha(x) \, dx + \beta z - \omega t \right) \right) + c.c.,
\]
where \( i^2 = -1, c.c. \) stands for complex conjugate, the frequency \( \omega \) and the spanwise wavenumber \( \beta \) are constants, but the streamwise wavenumber \( \alpha \) and shape function \( \phi \) depend on \( x \).

Substitution of Eq.(3) into Eq.(2) yields the linear stability equations
\[
\hat{A} \phi + \hat{B} \frac{\partial \phi}{\partial x} + \hat{C} \frac{\partial^2 \phi}{\partial x^2} = 0,
\]
where
\[
\hat{A} = A - i\omega T + i\alpha B_1 + DB_2 + \alpha^2 C_{11} + D^2 C_{22} - \beta^2 C_{33} + i\alpha D(C_{12} + C_{21}) + i\beta D(C_{23} + C_{32}) - \alpha \beta (C_{13} + C_{31}),
\]
\[
\hat{B} = B_1 + 2i\alpha C_{11} + D(C_{12} + C_{21}) + i\beta (C_{13} + C_{31}), \quad \hat{C} = C_{11}, \quad D = \partial / \partial y.
\]

The matrix \( \hat{A} = \hat{A}_p + \hat{A}_n \) include the contribution of the parallel part \( \hat{A}_p \) and non-parallel part \( \hat{A}_n \) of the mean flow.

For a parallel mean flow, or under the local parallel-flow assumption, the streamwise variations of the mean flow, the shape function \( \phi \) and the wavenumber \( \alpha \) are absent or ignored, and consequently the linear stability equations Eq.(4) simplify to
\[
[\hat{A}_p]_{x=a} \phi(x_a, y) = 0,
\]
which holds at an arbitrary location \( x_a \), and can be further reduced to an ordinary differential equation, namely the classical O-S equation. Eq.(7) with the usual homogeneous boundary conditions poses an eigenvalue problem. The eigenvalue problem is to find the complex eigenvalue \( \alpha(x_a) \) and the corresponding eigenvector \( \phi(x_a, y) \) at a location \( x_a \) for a given \( \beta \) and \( \omega \). The eigenvector \( \phi(x_a, y) \) is usually normalized by \( \bar{u}(x_a, y_m) \), where \( y_m \) is the location at which \( |\bar{u}| \) reaches its maximum. This is what linear stability analysis is about, and the prediction will be labelled as LST.
In the case of a non-parallel flow, the streamwise variations of both the mean flow and the shape function are taken into account in the linear stability equations Eq.(4). Unfortunately this set of partial differential equations is elliptic, and is extremely difficult to solve. In order to simplify the problem, the local variation with respect to \( x \) in the vicinity of an arbitrary point \( x_a \) is approximated by a Taylor expansion, namely,

\[
\phi(x, y) \exp \left[ i \int \alpha(x) dx \right] = \left\{ \phi_0(y) + (x - x_a) \phi_1(y) + \frac{(x - x_a)^2}{2} \phi_2(y) + \cdots \right\} \exp \left[ i \int \alpha(x_a) dx \right],
\]

in which the streamwise variation of \( \alpha \) has been absorbed into the \( O((x - x_a)^2) \) term in the expansion of the shape function. Similarly, the variation of the mean flow is accounted for by expanding it as Taylor series. Correspondingly, the operator expands as

\[
\hat{A}(x) = \hat{A}(x_a) + (x - x_a) \frac{\partial \hat{A}}{\partial x} + \frac{(x - x_a)^2}{2} \frac{\partial^2 \hat{A}}{\partial x^2} + \cdots,
\]

and similar expansions are performed for \( \hat{B} \). Introducing the approximations of (8) and (9) into Eq.(4) and retaining the first a few terms, we obtain a sequence of extended eigenvalue problems. The method and the solution will be referred to as EEV\( n \), where \( n \) refers to the order at which the Taylor series are truncated. The extension to higher orders would allow for predictions with increasing accuracy.

When the non-parallel contribution of the mean flow and the first two terms on the right-hand side of Eq.(8) are corrected \( \alpha \), each eigenvalue problem can be solved for the whole domain.

The growth rate of a physical disturbance quantity \( q' \) is defined as its logarithmic derivative \( \partial \ln q'/\partial x \). For a parallel flow, all disturbance quantities grow or decay at exactly the same rate, which is \( -\alpha \), the imaginary part of the eigenvalue of the O-S equation. For a non-parallel mean flow, the growth rate depends on the eigenvalue \( \alpha \), the choice of the quantity \( q' \) and the location \( y \). Once \( \alpha \) and \( q' \) are found by solving the extended eigenvalue problem, the corrected wavenumber \( \tilde{\alpha} \) is given by

\[
\tilde{\alpha}(y) = \alpha + \hat{\alpha}(y), \quad \hat{\alpha}(y) = \frac{\partial \ln \hat{q}(y)}{\partial \tilde{x}},
\]

where \( \hat{\alpha} \) stands for the correction to the wavenumber due to the distortion of the shape function \( \hat{q}(y) \). Usually, the streamwise velocity \( \hat{u} \) of the perturbation is chosen, and \( y \) is taken to be the peak position of \( |\hat{u}| \) since \( \hat{u} \) is the quantity that can be measured rather accurately. Unless stated otherwise, this is how \( \tilde{\alpha} \) will be calculated.

3. Results

In figure 1(a), the growth rate of weak compressible flow \((M = 0.1)\) calculated by EEV1 are compared with those predicted by LST, DNS [5] and the non-parallel theory [4], where \( F \) is a non-dimensional frequency, defined as \( F = \omega/R \times 10^6 \). There is an appreciable difference between EEV1 and LST, but the EEV1 prediction is in good agreement with the DNS result of Fasel & Konzelmann [5] and the theoretical result of Gaster [4]; the latter itself has been confirmed by DNS [5] and PSE [3]. In figure 1(b), we compare the predictions by LST and EEV1 with our own DNS [7] for a compressible flow \((M = 4.5)\). LST differs appreciably from DNS, but the results from EEV1 are almost identical to DNS, indicating that the non-parallel-flow effects are captured by EEV1.

Figure 2 shows the variation of the local growth rate \( -\alpha(y) \) with transverse position \( y \), where the eigenfunctions, our DNS results and the experimental data from Kachanov & Michalke [8] are also plotted in order to help interpret
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Fig. 1. Comparison of the growth rate $-\tilde{\alpha}_i$ by different method. (a) $M = 0.1, F = 140$; (b) $M = 4.5, F = 117$.

Fig. 2. Variation of the local growth rate $-\tilde{\alpha}_i(y)$ with the transverse position $y$. (a) $M = 0.1, R = 1129, F = 94.1$; (b) $M = 4.5, R = 17000, F = 25$, first model; (c) $M = 4.5, R = 8000, F = 275$, second model. The DNS is our own [7], the experimental data are from Kachanov & Michalke [8].

Fig. 3. Comparison of the neutral curves. (a) $M = 0.1$, non-parallel theory by Gaster [4], DNS by Fasel & Konzelmann [5]; (b) $M = 4.5$, first model; (b) $M = 4.5$, second model.

The results. While LST gives a growth rate independent of $y$, the growth rates obtained by EEV1 vary considerably with $y$ in the boundary layer for both $M = 0.1$ and $M = 4.5$. The predictions by EEV1 are in excellent agreement with our DNS results, and have the same tendency with the experimental data.

Figure 3 plots the neutral curves predicted by LST and EEV1, in which the results for $M = 0.1$ given by the non-parallel theory [4] and DNS [5] are also shown for comparison. As is illustrated, for $M = 0.1$, the result by EEV1 overlaps with those by the non-parallel theory [4] and DNS [5], while the difference between LST and EEV1 is remarkable. In figure 3(a) with $M = 0.1$, the non-parallel-flow enhances the wave near the upper branch of the neutral curve but stable the wave near the lower branch of the neutral curve. The critical Reynolds number $R_c$ predicted by EEV1 is about $R_c = 520$ as LST does, but the corresponding neutral mode has a higher frequency. The non-parallel-flow effect on the first model for $M = 4.5$ in figure 3(b) is similar, however the critical Reynolds number $R_c$ predicted by EEV1 is about 6500, which is much smaller than that about 8000 by LST. The frequency of the
corresponding neutral mode predicted by EEV1 reaches $F = 80$, which is near two times larger than that by LST. The neutral curve of the second model with $M = 4.5$ given by EEV1 in figure 3(c) almost overlaps with that by LST for $R > 6000$, indicating that the non-parallel-flow has weak effect on the wave near the second model neutral curve for large Reynolds number. The wave of the second model neutral curve for $R < 6000$ is inhibited due to the non-parallel-flow.

Figure 4 displays the growth rates of the most unstable wave predicted by LST and EEV1, where the growth rate is normalized by the local displacement thickness of boundary layer with its corresponding Reynolds number $R$. It can be seen that, the non-parallel-flow has little effect on the growth rate of the most unstable wave for $M = 0.1$ in figure 4(a), but strongly affects the growth rate of the most unstable wave for $M = 4.5$. In figure 4(b) the growth rate of the most unstable wave of the first model for $M = 4.5$ and $R = 12000$ predicted by EEV1 is about $0.2 \times 10^{-2}$, which is twice as that by LST, indicating that the most unstable wave of the first model is intensely enhanced due to the non-parallel-flow for a relative small Reynolds number. On the contrary, it is found in figure 4(c) that the most unstable wave of the second model for $M = 4.5$ is stabilized by the non-parallel-flow.

4. Summaries

In summary, we present a local, simple and non-perturbative approach, named as EEVn, to spatial instability of three-dimensional compressible boundary layers is presented. Because both the streamwise variation of the mean flow and the shape distortion of the disturbance are taken into account, the non-parallel-flow effects are captured by EEVn, and the prediction of EEVn is in good agreement with DNS and experimental results. The non-parallelism effect on the boundary layer instability for supersonic flow is much large than that for weak compressible flow, and should not be ignored any more. The non-parallel-flow with $M = 4.5$ on a flat plate plays different role on the different model waves, in particular, it stables the second model wave but enhances the first model wave.

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