

**TOPOLOGICAL IMPROVEMENTS OF CATEGORIES OF STRUCTURED SETS**

Horst HERRLICH

*Feldhäuser Str. 69, 2804 Lilienthal, Fed. Rep. Germany*

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Categories of structured sets often fail to have some desirable properties. They may even fail to have any interesting decently behaved full subcategories. But, under some natural assumptions (and disregarding purely set theoretic problems concerning 'size'), it is always possible to embed them into nicely behaved topological categories, in particular each such category has:

- (1) a topological hull (= Mac Neille completion),
- (2) a cartesian closed topological hull (= Antoine-completion),
- (3) a hereditary topological hull,
- (4) a concrete quasitopos hull (= Wyler completion).

The purpose of this paper is to discuss these hulls and to provide several illuminating examples.

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construct	concrete category	topological category
function space	cartesian closedness	quasitopos
Mac Neille completion	topological hull	injective hull
Cartesian closed topological hull	topological quasitopos hull	

**Introduction**

The objects of this paper are categories of structured sets and structure-preserving maps (more formally: categories  $\mathbf{A}$  supplied with a forgetful functor  $U: \mathbf{A} \rightarrow \mathbf{Set}$ ) satisfying the following condition:

- (C) every constant map between  $\mathbf{A}$ -objects is an  $\mathbf{A}$ -morphism.

Such categories will be called *constructs*. Familiar constructs, such as **Top**, **Unif** and **Prox**, share an important categorical property: they are *topological*, i.e., they have sufficient initial and final structures. This property, even though being rather convenient, is considered by an increasing number of topologists and analysts (cf. the reference list B) not convenient enough. Additional convenience properties are required. The existence of a *quasitopos* structure is particularly desirable. This property splits naturally into two parts, each interesting in its own right: *cartesian closedness*, whose value has been recognized for more than 20 years, and *heredity*, whose usefulness has become apparent more recently. These properties will be described in Section 1.

Natural candidates for convenient constructs are full subconstructs and superconstructs of the classical constructs, in particular of **Top**. Unfortunately, constructs may fail to have any interesting convenient full subconstructs, e.g.:

(1) **Top** has only 2 full subconstructs, which are topological quasitopoi: they consist of all discrete spaces and, respectively, of all indiscrete spaces [24].

(2) **Top<sub>0</sub>**, the construct of  $T_0$ -spaces, has only one full subconstruct, which is topological: it consists of all discrete spaces.

(3) **Met**, the construct of metric spaces and non-expansive maps, has infinitely many topological (even quasitopoi) full subconstructs, but each of these contains only metric spaces whose underlying topologies are discrete.

It is more surprising that (modulo some foundational problems, discussed in Section 3.2) every construct can be fully embedded as a finally-dense subcategory into some convenient construct: e.g., its *largest finally tight extension*, which is a concrete quasitopos with productive quotient maps [25]. Although extensions of constructs follow the rule:

(BB) ‘bigger is better’; i.e. stronger convenience stipulations require bigger extensions,

they also follow the rule:

(SB) ‘smaller is better’; i.e. smaller extensions generally preserve more structure of the original construct.

Hence it seems desirable to find, for a given construct  $A$  and a given convenience condition  $P$ , a smallest extension  $P(A)$  of  $A$  satisfying  $P$ . Such an extension will be called a  $P$ -hull of  $A$ . For the convenience properties, mentioned above, such hulls exist (modulo-foundational problems, see section 3.2) for any reasonable construct. The properties of these hulls and various examples will be presented in Section 2, their construction in Section 3. Finally, Section 4 indicates that the mentioned results are special cases of far more general results.

Background material, definitions, examples and references up to 1981 can be found in the survey article [23]. Most references given there will not be repeated here.

## 1. Convenience properties

### 1.1. Topological constructs

A construct  $A$  is called *topological* provided it satisfies the following equivalent conditions:

- (1)  $A$  is initially complete, i.e., every source of the form  $(f_i: X \rightarrow UA_i)_{i \in I}$  has an initial lift  $(f_i: A \rightarrow A_i)_{i \in I}$ ,
- (2)  $A$  is finally complete, i.e., every sink of the form  $(f_i: UA_i \rightarrow X)_{i \in I}$  has a final lift  $(f_i: A_i \rightarrow A)_{i \in I}$ ,
- (3)  $A$  is an injective object in the quasicategory *Const* of constructs and concrete functors.

Most of the familiar constructs, consisting of ‘topological objects’, are topological, e.g., the constructs **Top**, **PrTop**, **PsTop**, **Conv**, **k-Top**, **Prox**, **Unif**, **Near**, **Mer**, **Bor** and **SynTop**, consisting of topological, pretopological, pseudotopological, convergence, compactly generated topological, proximity, uniform, nearness, merotopic, bornological and syntopogeneous spaces respectively. So are the constructs **Simp**, **Prost** and **Rere**, consisting of simplicial complexes, preordered sets and reflexive relations. Subconstructs of topological constructs, defined by separation axioms, like **Top<sub>0</sub>**, **Top<sub>1</sub>**, **Top<sub>2</sub>**, **Unif<sub>2</sub>** etc., generally fail to be topological, but usually have familiar constructs as their topological hulls.

Topological constructs have many pleasant properties, e.g., they have concrete limits and concrete colimits, and their fibres are complete lattices.

Recent references: [20, 25].

### 1.2. Cartesian closed topological constructs

A topological construct  $\mathbf{A}$  is *cartesian closed* provided it satisfies the following equivalent conditions:

- (1) for every  $\mathbf{A}$ -object  $A$  the endofunctor  $A \times \_ : \mathbf{A} \rightarrow \mathbf{A}$  has a right adjoint,
- (2)  $\mathbf{A}$  has decently behaved function spaces, i.e., for every pair  $(A, B)$  of  $\mathbf{A}$ -objects the set  $\text{Mor}_{\mathbf{A}}(A, B)$  can be supplied with the structure of an  $\mathbf{A}$ -object—usually denoted by  $B^A$ , and called a function space of a power object—such that for any  $\mathbf{A}$ -object  $C$  and any map  $f: A \times C \rightarrow B$  the following conditions are equivalent:
  - (a)  $f: A \times C \rightarrow B$  is an  $\mathbf{A}$ -morphism,
  - (b)  $f^*: C \rightarrow B^A$ , defined by  $(f^*(c))(a) = f(a, c)$ , is an  $\mathbf{A}$ -morphism,
- (3) in  $\mathbf{A}$  final epi-sinks are finitely productive, i.e., if  $(f_i: A_i \rightarrow A)_{i \in I}$  and  $(g_j: B_j \rightarrow B)_{j \in J}$  are final epi-sinks in  $\mathbf{A}$ , then so is  $(f_i \times g_j: A_i \times B_j \rightarrow A \times B)_{(i,j) \in I+J}$ ,
- (4)  $\mathbf{A}$  is a injective object in the quasicategory **Const<sub>p</sub>** of constructs with finite concrete products and finite product preserving concrete functors.

For small-fibred  $\mathbf{A}$  these conditions are equivalent to:

- (5)  $\mathbf{A}$  has the following properties:
  - (a) in  $\mathbf{A}$  quotient maps are finitely productive, i.e., if  $f: A \rightarrow B$  and  $g: C \rightarrow D$  are quotient maps in  $\mathbf{A}$ , then so is  $f \times g: A \times C \rightarrow B \times D$ ,
  - (b) in  $\mathbf{A}$  products commute with coproducts, i.e., for any  $\mathbf{A}$ -object  $A$  and any set-indexed family  $(B_i)_I$  of  $\mathbf{A}$ -objects the natural morphism  $\text{II}(A \times B_i) \rightarrow A \times \text{II}B_i$  is an isomorphism.

Of the familiar topological constructs, listed in 1.1, only **PsTop**, **Conv**, **k-Top**, **Bor**, **Simp**, **Prost** and **Rere** are cartesian closed.

Further examples: filter-generated merotopic spaces [16, 30] uniform limit spaces [31, 39], and Cauchy-spaces [15].

Recent references: [3, 11, 25, 28, 36, 37].

### 1.3. Hereditary topological constructs

A topological construct  $\mathbf{A}$  is called *hereditary* provided it satisfies the following equivalent conditions:

(1) in  $\mathbf{A}$  final sinks are hereditary, i.e., if  $(f_i: A_i \rightarrow A)_{i \in I}$  is a final sink in  $\mathbf{A}$ ,  $B$  is a subspace of  $A$ ,  $B_i$  is the subspace of  $A_i$  with underlying set  $f_i^{-1}[B]$ , and  $g_i: B_i \rightarrow B$  is the corresponding restriction of  $f_i$ , then  $(g_i: B_i \rightarrow B)_{i \in I}$  is a final sink in  $\mathbf{A}$  too;

(2) in  $\mathbf{A}$  final epi-sinks are hereditary;

(3) in  $\mathbf{A}$  partial morphisms are representable, i.e. every  $\mathbf{A}$ -object  $A$  can be embedded via the addition of a single point  $\infty_A$  into an  $\mathbf{A}$ -object  $A^+ = A \cup \{\infty_A\}$  such that the following holds:

For every partial morphism  $B \rightarrow A$ , i.e., for every  $\mathbf{A}$ -morphism  $f: C \rightarrow A$  from a subobject  $C$  of  $B$  into  $A$ , the unique function  $f^+: B \rightarrow A^+$ , defined by

$$f^+(b) = \begin{cases} f(b), & \text{if } b \in C \\ \infty_A, & \text{if } b \notin C \end{cases}$$

is an  $\mathbf{A}$ -morphism;

(4)  $\mathbf{A}$  is an injective object in the quasicategory  $\mathbf{Const}_h$  of constructs with subspaces and subspace preserving concrete functors.

For small-fibred  $\mathbf{A}$  these conditions are equivalent to:

(5) in  $\mathbf{A}$  coproducts and quotients are hereditary.

Of the familiar topological constructs, listed in 1.1, only **PrTop**, **PsTop**, **Conv**, **Mer**, **Bor**, **Simp** and **Rere** are hereditary. The only full subconstructs of **Top**, which are hereditary topological, are those consisting of all discrete resp. of all indiscrete spaces [24].

### 1.4. Concrete quasitopoi

A topological construct  $\mathbf{A}$  is a *quasitopos* provided it satisfies the following equivalent conditions:

(1)  $\mathbf{A}$  is cartesian closed and in  $\mathbf{A}$  partial morphisms are representable,

(2)  $\mathbf{A}$  is cartesian closed and hereditary,

(3) in  $\mathbf{A}$  final epi-sinks are universal, i.e., if  $(f_i: A_i \rightarrow A)_{i \in I}$  is a final epi-sink in  $\mathbf{A}$ ,  $f: B \rightarrow A$  is an  $\mathbf{A}$ -morphism and for each  $i \in I$  the diagram

$$\begin{array}{ccc} B_i & \xrightarrow{k_i} & A_i \\ g_i \downarrow & & \downarrow f_i \\ B & \xrightarrow{f} & A \end{array}$$

is a pullback in  $\mathbf{A}$ , then  $(g_i: B_i \rightarrow B)_{i \in I}$  is a final epi-sink in  $\mathbf{A}$ ,

(4) in  $\mathbf{A}$  colimits are universal,

(5) for each  $\mathbf{A}$ -object  $A$  the comma category  $\mathbf{A}/A$  is cartesian closed,

(6)  $\mathbf{A}$  is an injective object in the quasicategory  $\mathbf{Const}_{ph}$  of constructs with finite concrete products and subspaces and concrete functors preserving finite products and subspaces.

For small-fibred  $\mathbf{A}$ , these conditions are equivalent to:

(7) in  $\mathbf{A}$  quotients and coproducts are universal.

Of the familiar topological constructs, listed in Section 1.1, only **PsTop**, **Conv**, **Bor**, **Simp** and **Rere** are quasitopoi.

Alternative names, which have been used for topological constructs, which are quasitopoi, are concrete quasitopoi [22], strongly topological categories [24], and topological universes [34].

Recent references: [2, 3, 4, 25]

## 2. Hulls

In this section various *extensions* of a construct  $\mathbf{A}$ , i.e., full concrete embeddings  $E: \mathbf{A} \rightarrow P(\mathbf{A})$  are characterized. For simplicity we will assume that  $\mathbf{A}$  is a full subconstruct of  $P(\mathbf{A})$  and that  $E: \mathbf{A} \rightarrow P(\mathbf{A})$  is the corresponding inclusion.

### 2.1. Topological hulls

Every construct  $\mathbf{A}$  has (modulo foundational problems, see section 3.2) a *topological hull* (= *Mac Neille completion*)  $T(\mathbf{A})$ , characterized uniquely (up to isomorphism) by the following equivalent conditions:

- (1)  $T(\mathbf{A})$  is the smallest topological extension of  $\mathbf{A}$ ,
- (2)  $T(\mathbf{A})$  is the largest initially and finally tight extension of  $\mathbf{A}$ ,
- (3)  $T(\mathbf{A})$  is an initially and finally tight topological extension of  $\mathbf{A}$ ,
- (4)  $T(\mathbf{A})$  is the injective hull of  $\mathbf{A}$  in **Const**.

Examples of topological hulls:  $T(\mathbf{Top}_0) = \mathbf{Top}$ ,  $T(\mathbf{Unif}_2) = \mathbf{Unif}$ ,  $T(\mathbf{Poset}) = \mathbf{Prost}$ ,  $T(\mathbf{Met})$  is the construct **PMet** of pseudometric spaces.

### 2.2. Cartesian closed topological hulls

Every construct  $\mathbf{A}$  with finite concrete products has (modulo foundational problems, see section 3.2) a *cartesian closed topological hull* (= *Antoine completion*)  $\mathbf{CT}(\mathbf{A})$ , characterized uniquely (up to isomorphism) by the following equivalent conditions:

- (1)  $\mathbf{CT}(\mathbf{A})$  is the smallest finally tight cartesian closed topological extension of  $\mathbf{A}$ ,
- (2)  $\mathbf{CT}(\mathbf{A})$  is a cartesian closed topological extension of  $\mathbf{A}$ , in which  $\mathbf{A}$  is finally dense and powers  $B^A$  of  $\mathbf{A}$ -objects  $A, B$  are initially dense,
- (3)  $\mathbf{CT}(\mathbf{A})$  is the injective hull of  $\mathbf{A}$  in **Const<sub>p</sub>**.

Examples of cartesian closed topological hulls:  $\mathbf{CT}(\mathbf{Top}) = \mathbf{EpiTop}$  [13, 32, 18],  $\mathbf{CT}(\mathbf{PrTop}) = \mathbf{PsTop}$  [18],  $\mathbf{CT}(\mathbf{Tych}) = c$ -embedded convergence spaces [18, 19],  $\mathbf{CT}(\mathbf{Unif}) = \mathbf{bornological}$  uniform spaces [7],  $\mathbf{CT}(\mathbf{Prox}) = \mathbf{coreflective}$  hull of **Prox** in **Unif** [33],  $\mathbf{CT}(\mathbf{HComp}) = \mathbf{coreflective}$  hull of **HComp** in **Unif** [9],  $\mathbf{CT}(\mathbf{Poset}) = \mathbf{Prost}$ .

Recent references: [9, 10, 11, 12, 14, 28, 37].

### 2.3. Hereditary topological hulls

Every construct  $\mathbf{A}$  with subspaces has (modulo foundational problems, see section 3.2) a *hereditary topological hull*  $\text{HT}(\mathbf{A})$ , characterized uniquely (up to isomorphism) by the following equivalent conditions:

- (1)  $\text{HT}(\mathbf{A})$  is the smallest finally tight hereditary topological extension of  $\mathbf{A}$ ,
- (2)  $\text{HT}(\mathbf{A})$  is the injective hull of  $\mathbf{A}$  in  $\mathbf{Const}_h$ .

Examples of hereditary topological hulls:  $\text{HT}(\mathbf{Top}) = \mathbf{PrTop}$ ,  $\text{HT}(\mathbf{Poset}) = \mathbf{Rere}$ ,  $\text{HT}(\mathbf{Met})$  is the construct  $\mathbf{Dist}$  of distance spaces (obtained by dropping the triangle-inequality in the definition of a pseudometric space).

Details will appear in the author's "Hereditary topological constructs", Proc. Sixth Prague Topol. Symp. 1986.

### 2.4. Concrete quasitopos hulls

Every construct  $\mathbf{A}$  with finite concrete products and subspaces has (modulo foundational problems, see Section 3.2) a *concrete quasitopos hull* (= *Wyler-completion*)  $\text{CQ}(\mathbf{A})$ , characterized uniquely (up to isomorphism) by the following equivalent conditions:

- (1)  $\text{CQ}(\mathbf{A})$  is the smallest finally tight concrete quasitopos extension of  $\mathbf{A}$ ,
- (2)  $\text{CQ}(\mathbf{A})$  is a concrete quasitopos extension of  $\mathbf{A}$ , in which  $\mathbf{A}$  is finally dense and objects of the form  $(B^A)^+$ , with  $A$  and  $B$  being  $\mathbf{A}$ -objects, are initially dense,
- (3)  $\text{CQ}(\mathbf{A})$  is the injective hull of  $\mathbf{A}$  in  $\mathbf{Const}_{ph}$ .

Examples of concrete quasitopos hulls:  $\text{CQ}(\mathbf{Top}) = \mathbf{PsTop}$  [38],  $\text{CQ}(\mathbf{Poset}) = \mathbf{Rere}$ ,  $\text{CQ}(\mathbf{Met}) = \mathbf{Dist}$ .  $\text{CQ}(\mathbf{Unif})$  is the construct of submetrizable bornological merotopic spaces [8].  $\text{CQ}$  (the construct of finite simplicial complexes) =  $\mathbf{Simp}$ .  $\text{CQ}(\mathbf{STOP}) =$  construct of superspaces [40].

Recent references: [2, 3, 4].

## 3. Constructions

In section 3.1 constructions of the hulls will be provided. The foundational problems, raised by these constructions, will be discussed briefly in section 3.2.

### 3.1. Construction of the hulls

Each of the hulls under discussion is a finally tight extension. It can therefore be obtained as a full subconstruct of the *largest finally tight extension*. This can be described as follows: If  $\mathbf{A}$  is a construct, the objects of  $\text{Max}(\mathbf{A})$  are pairs  $(X, S)$ ,

consisting of a set  $X$  and a structured sink  $S$ , i.e., a class of pairs  $(A, a)$ , consisting of an  $\mathbf{A}$ -object  $A$  and a function  $a: UA \rightarrow X$ , subject to the following conditions:

- (1) If  $(A, a) \in S$  and  $b: B \rightarrow A$  is an  $\mathbf{A}$ -morphism, then  $(B, a \cdot b) \in S$ ,
- (2) if  $A$  is an  $\mathbf{A}$ -object and  $a: UA \rightarrow X$  is a constant map, then  $(A, a) \in S$ .

Morphisms in  $\text{Max}(\mathbf{A})$  from  $(X, S)$  to  $(Y, T)$  are functions  $f: X \rightarrow Y$ , satisfying the condition:

$$(M) \quad (A, a) \in S \text{ implies } (A, f \cdot a) \in T.$$

The construct  $\mathbf{A}$  can be regarded as a full finally dense subconstruct of  $\text{Max}(\mathbf{A})$  via the embedding  $E: \mathbf{A} \rightarrow \text{Max}(\mathbf{A})$ , defined by  $EA = (UA, \{(B, b) \mid b: B \rightarrow A \in \text{Mor } \mathbf{A}\})$ . Each of the four hulls of  $\mathbf{A}$  can be obtained as the full subconstruct of  $\text{Max}(\mathbf{A})$ , consisting of those objects  $(X, S)$  for which  $S$  can be expressed as an intersection  $S = \bigcap_{i \in I} S(i)$  of some particular  $S(i)$ 's. These will be described, for each of the 4 hulls, as follows:

(1) The topological hull: if  $i = (b, B)$  is a pair, consisting of an  $\mathbf{A}$ -object  $B$  and a map  $b: X \rightarrow UB$ , then

$$S(i) = \{(A, a) \mid b \cdot a: A \rightarrow B \in \text{Mor } \mathbf{A}\}.$$

(2) The cartesian closed topological hull: if  $i = (B, C, b)$  is a triple, consisting of  $\mathbf{A}$ -objects  $B$  and  $C$  and a map  $b: UB \times X \rightarrow UC$ , such that for each  $x \in X$  the restriction  $b(-, x): B \rightarrow C$  is an  $\mathbf{A}$ -morphism, then

$$S(i) = \{(A, a) \mid b \circ (\text{id}_B \times a): B \times A \rightarrow C \in \text{Mor } \mathbf{A}\}.$$

(3) The hereditary topological hull: if  $i = (Y, B, b)$  is a triple, consisting of a subset  $Y$  of  $X$ , an  $\mathbf{A}$ -object  $B$  and a map  $b: Y \rightarrow UB$ , then

$$S(i) = \{(A, a) \mid b \circ a_Y: A_Y \rightarrow B \in \text{Mor } \mathbf{A}\},$$

where  $A_Y$  is the subspace of  $A$  with underlying set  $a^{-1}[Y]$  and  $a_Y: UA_Y \rightarrow Y$  is the corresponding restriction of  $a: UA \rightarrow X$ .

(4) The concrete quasitopos hull: if  $i = (Y, B, C, b)$  is a quadruple consisting of a subset  $Y$  of  $X$ ,  $\mathbf{A}$ -objects  $B$  and  $C$  and a map  $b: UB \times Y \rightarrow UC$ , such that for each  $y \in Y$  the restriction  $b(-, y): B \rightarrow C$  is an  $\mathbf{A}$ -morphism, then

$$S(i) = \{(A, a) \mid b \circ (\text{id}_B \times a_Y): B \times A_Y \rightarrow C \in \text{Mor } \mathbf{A}\}$$

where  $A_Y$  and  $a_Y$  are defined as in (3).

Recent references: [2, 9, 10, 38]

### 3.2. Foundational problems

As is well known, such concepts as the 'set of all sets' or the 'category of all categories' can easily lead to contradictions. There are various ways to avoid these.

Most familiar, perhaps, is the distinction between sets and (proper) classes resp. between classes and (proper) conglomerates, leading to such concepts as the ‘class of all sets’ and the ‘quasicategory of all categories’.

If  $\mathbf{A}$  is a construct and  $P(\mathbf{A})$  is one of the hulls of  $\mathbf{A}$ , discussed before, then  $P(\mathbf{A})$  usually fails to be a construct: in fact if  $\mathbf{A}$  is not small (the objects of  $\mathbf{A}$  form a proper class) then the objects of  $P(\mathbf{A})$  are proper classes too; hence the collection of  $P(\mathbf{A})$ -objects is a proper conglomerate; hence  $P(\mathbf{A})$  is a quasiconstruct only. If this quasiconstruct is isomorphic to some construct, it will be called a *legitimate quasiconstruct*. In this case, there are no real problems. But sometimes  $P(\mathbf{A})$  is ‘too big’ to be legitimate; then it is called *illegitimate*. Sufficient (and necessary) conditions for a construct  $\mathbf{A}$  to have a legitimate hull, are exhibited:

- (a) for the topological hull in [5],
- (b) for the cartesian closed topological hull in [6],
- (c) for the concrete quasitopos hull in [2].

It may be worth mentioning that

- (1) there are constructs whose topological hull is illegitimate [5],
- (2) there are topological constructs, whose cartesian closed topological hull is illegitimate [6],
- (3) Spanier’s quasispaces form an illegitimate construct [27].

There are other ways to deal with the above mentioned problem (e.g., by means of a chain of Grothendieck universes), but none, to the author’s knowledge, can ‘explain away’ the problem.

#### 4. Generalizations

The above results are (particularly interesting) special cases of far more general results. If the category of sets is replaced by a category  $\mathbf{X}$ , and instead of constructs *concrete categories* over  $\mathbf{X}$ , i.e., pairs  $(\mathbf{A}, U)$  consisting of a category  $\mathbf{A}$  and a forgetful functor  $U: \mathbf{A} \rightarrow \mathbf{X}$ , are considered, then the above constructions and most of the results remain valid. Naturally, in the cartesian closed case  $\mathbf{X}$  should be required to be cartesian closed, in the quasitopos case  $\mathbf{X}$  should be required to be a quasitopos, and in the hereditary case  $\mathbf{X}$  should be required to carry a suitable factorization structure for morphisms. The general results not only imply the above results (of interest primarily to topologists and analysts) but also familiar results such as the characterization of (a) complete lattices as injective objects in **Poset**, (b) Mac Neille completions of partially ordered sets as injective hulls [17], and (c) locales as injective objects in the category of semilattices [21, 29].

In the general situation condition (C), saying that all constant maps are morphisms, is unnatural, hence dropped. It could have been omitted here too. This has not been done since condition (C) is responsible for the ‘familiarity’ of the concepts and hulls under discussion, e.g. it guarantees [25] that cartesian closedness is equivalent to the existence of decently behaved function spaces (condition (1) in 1.2), a

condition topologists and analysts are primarily interested in; also it is responsible for the hulls of familiar constructs to be not too exotic.

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