Global solutions for a general strongly coupled prey–predator model

Huiling Li\textsuperscript{a}, Peter Y.H. Pang\textsuperscript{b,∗}, Mingxin Wang\textsuperscript{a,c}

\textsuperscript{a} Department of Mathematics, Southeast University, Nanjing 210018, People’s Republic of China
\textsuperscript{b} Department of Mathematics, National University of Singapore, 2 Science Drive 2, 117543, Republic of Singapore
\textsuperscript{c} Science Research Center, Harbin Institute of Technology, Harbin 150080, People’s Republic of China

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A B S T R A C T

This work investigates global solutions for a general strongly coupled prey–predator model that involves (self-)diffusion and cross-diffusion, where the cross-diffusion is of the form \( \frac{v}{(1+u^\ell)} \) with \( \ell \geq 1 \). Very few mathematical results are known for such models, especially in higher spatial dimensions.

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1. Introduction and main result

For decades, many authors have studied the following prey–predator model of Lotka–Volterra type:

\[
\begin{align*}
\frac{du}{dt} &= u[a - bu - f(u,v)], \quad t > 0, \\
\frac{dv}{dt} &= v[-s + g(u,v)], \quad t > 0,
\end{align*}
\]  

(1.1)

where \( u \) and \( v \) represent the densities of prey and predator respectively, and the parameters \( a, b \) and \( s \) are positive constants. The functions \( f(u,v) \) and \( g(u,v) \) are respectively the response function and the per capita predator production, and are nonnegative and continuous.

Taking into account the spatial dependence of the prey and predator distribution, one is led to partial differential (as opposed to the above ordinary differential) models. In particular, the movements of the species involved are modelled by various kinds of diffusions in the partial differential system. The natural tendency of each species to diffuse to areas of smaller population concentration gives rise to (self-)diffusion, while the movement of one species in response to behavior of another species, for example, pursuit evasion, is modelled by cross-diffusion. These diffusion processes may be quite intricate as different concentration levels of prey and predator cause different population movements.

A general partial differential prey–predator model is of the form (see, for example, [1], [2, Ch.10])

\[
\begin{align*}
\frac{du}{dt} - \text{div}[K_{11}(u,v)\nabla u + K_{12}(u,v)\nabla v] &= u[a - bu - f(u,v)], \quad x \in \Omega, t > 0, \\
\frac{dv}{dt} - \text{div}[K_{21}(u,v)\nabla u + K_{22}(u,v)\nabla v] &= v[-s + g(u,v)], \quad x \in \Omega, t > 0,
\end{align*}
\]  

(1.2)

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∗ Corresponding author.
E-mail address: matpyh@nus.edu.sg (P.Y.H. Pang).

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where $K_{11}, K_{22},$ and $K_{12}, K_{21},$ respectively, embody the (self-)diffusion and cross-diffusion processes. Biology considerations require that $K_i(u, v)$ satisfy
\begin{equation}
K_1(u, v), K_2(u, v) > 0, \quad K_{12}(u, v) \geq 0, \quad K_{21}(u, v) \leq 0.
\end{equation}

In this work, we shall consider the model
\begin{equation}
\begin{cases}
u_i - \Delta (d_i u + d_4 u^2) = u[i - h(u, v)], & x \in \Omega, \quad t > 0, \\
u_t - \Delta (d_4 v^2 + \frac{1}{1 + u^2}) = v[-s + g(u, v)], & x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0, & x \in \partial \Omega, \quad t > 0, \\
u(0, 0) = \nu_0(x), \quad v(0, 0) = v_0(x), & x \in \Omega,
\end{cases}
\end{equation}
where the diffusion coefficients $d_1, d_3 > 0$ and $d_4 \geq 0$ are constants, $\ell$ is a positive constant, $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary $\partial \Omega$, and $\nu$ represents the outward unit normal to $\partial \Omega$. The homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary. Furthermore, throughout this work, we assume that functions $f$ and $g$ satisfy:

(H) For all $u, v \geq 0, 0 \leq f(u, v), g(u, v) \leq \text{Ch}(u)$ for some positive constant $C$ and continuous function $h(u)$.

The model (1.4) is general enough to include the ratio-dependent model of [3], the non-monotonic functional response model of [4,5], and the Beddington–DeAngelis functional response model [6]. To our knowledge, except the paper [7], there are few results for global solutions for prey–predator models with cross-diffusion. The purpose of the present work is to establish the global existence of solutions to (1.4). The main result is as follows:

\textbf{Theorem 1.} Let $\ell \geq 1$. Suppose that the initial data $u_0(x), v_0(x) \in C^{2+\alpha}(\bar{\Omega})$ are nonnegative functions which are not identically zero and satisfy the homogeneous Neumann boundary condition. Then the system (1.4) has a unique nonnegative solution $(u, v)$, such that $u \in C^{2+\alpha}((0, T), W^4_p(\bar{\Omega}))$ and $v \in C^{1+\alpha}((0, T), W^4_p(\bar{\Omega}))$. Moreover, if $\ell = 1$ or $\ell \geq 2$, then $v \in C^{2+\alpha, (2+\alpha)/2}(\bar{\Omega} \times [0, \infty))$.

To end this section, we remark that, while there are few known results for prey– predator models as mentioned above, there has been relatively good success with competition models; see for example [8–10]. For related results on global existence, we also refer the reader to [11,12] and the references therein.

2. A priori estimates

Thanks to [13], we know that the system (1.4) has a unique nonnegative smooth solution which exists locally in time. More precisely,

\textbf{Proposition 1 ([13])}. Let $u_0, v_0 \in W^p_1(\Omega), \quad p > N$, be nonnegative functions. Then the system (1.4) has a unique nonnegative smooth solution $(u, v)$, such that $u, v \in C((0, T), W^4_p(\bar{\Omega})) \cap C^\infty((0, T), C^\infty(\bar{\Omega}))$, where $T$ is the maximal existence time. Moreover, if $T < \infty$, then
\begin{equation}
\lim_{t \to T^-} \max\{\|u(\cdot, t)\|_{W^4_p(\bar{\Omega})}, \|v(\cdot, t)\|_{W^4_p(\bar{\Omega})}\} = \infty.
\end{equation}

Define $Q_T = \Omega \times (0, T)$.

\textbf{Lemma 1}. Let $(u, v)$ be a nonnegative solution to the system (1.4); then there exist two positive constants $M = \max\{a/b, \|u_0\|_{C^\infty(\Omega)}\}$ and $C(T)$ such that
\begin{equation}
0 < u(x, t) \leq M, \quad v(x, t) > 0, \quad \forall (x, t) \in Q_T.
\end{equation}
and
\begin{equation}
\sup_{0 < t < T} \|v(\cdot, t)\|_{L^1(\Omega)} \leq C(T), \quad \|\nabla u\|_{L^2(Q_T)} \leq C(T).
\end{equation}
Furthermore,
\begin{equation}
\|u\|_{L^2(Q_T)} \leq C(T), \quad \text{i.e., } u \in V_2(Q_T).
\end{equation}

\textbf{Proof}. First, we note that the assertion (2.1) is a direct consequence of the maximum principle. Next, by the assumption (H) and (2.1), integrating the second equation of (1.4) in $\Omega$, we obtain
\begin{equation}
dt \int_{\Omega} v dx = \int_{\Omega} v(u - g(u, v)) dx \leq C \int_{\Omega} v h(u) dx \leq M \int_{\Omega} v dx.
\end{equation}
From the Gronwall inequality it follows that
\[
\sup_{0 < t < T} \|v(\cdot, t)\|_{L^1(\Omega)} \leq C(T).
\]

Similarly, multiplying the first equation of (1.4) by \(u\) and integrating the result on \(\Omega\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + d_1 \int_{\Omega} |\nabla u|^2 dx + 2d_2 \int_{\Omega} u|\nabla u|^2 dx = \int_{\Omega} u^2 (a - bu - f(u, v)) dx + a \int_{\Omega} u^2 dx.
\]

Integrating the above inequality and then using the Gronwall inequality, we get
\[
\sup_{0 < t < T} \int_{\Omega} u^2 dx + d_1 \int_{0}^{T} \int_{\Omega} |\nabla u|^2 dx dt + 2d_2 \int_{0}^{T} \int_{\Omega} u|\nabla u|^2 dx dt \leq C(T);
\]
that is,
\[
\|u\|_{V^2(\Omega_T)} \leq C(T).
\]

This completes the proof. \(\Box\)

Next, we shall establish \(V^2(\Omega_T)\) estimates and then \(L^\infty(\Omega_T)\) estimates for \(v\). From now on, \(C_1, C_2, \ldots\) and \(M_1, M_2, \ldots\) will denote positive constants that may depend on \(\Omega, N\), the constants in the system (1.4), \(T\) (but not on \(t\)), and the norms \(\|u_0\|_{W^2_0(\Omega)}, \|u_0\|_{L^\infty(\Omega)}\) and \(\|v_0\|_{L^2(\Omega)}\).

**Lemma 2** \(V^2(\Omega_T)\) Estimates for \(v\). Let \(T > 0\). Then there exists a constant \(C(T) > 0\) such that
\[
\|v\|_{V^2(\Omega_T)} \leq C(T).
\]

**Proof.** Rewrite the equation for \(u\) as
\[
u_t = \nabla \cdot \left((d_1 + 2d_2u)\nabla u\right) + u(u - bu - f(u, v)). \tag{2.4}
\]
Since \(d_1 + 2d_2u\) and \(u(u - bu - f(u, v))\) are bounded on \(\Omega_T\) by the assumption (H) and Lemma 1, by applying the Hölder continuity result (see, for example, [14]) to (2.4), we have
\[
u \in C^{\alpha, \alpha/2}(\Omega_T), \quad \alpha > 0. \tag{2.5}
\]
Let \(z = (d_1 + 2d_2u)\nabla u\); then \(z\) satisfies the equation
\[
z_t = (d_1 + 2d_2u)\Delta z + u(d_1 + 2d_2u)(a - bu - f(u, v)) \tag{2.6}
\]
Recall that \(d_1 + 2d_2u \in C^{\alpha, \alpha/2}(\Omega_T)\) by (2.5), \((d_1 + 2d_2u)u(u - bu - f(u, v)) \in L^\infty(\Omega_T)\) by Lemma 1. By the assumption (H), the parabolic regularity theorem [14] implies that
\[
z \in L^{\infty(\Omega_T)} \quad \text{for any } p > 1. \tag{2.7}
\]
Thus, the Sobolev inequality yields
\[
z \in C^{1+\beta, (1+\beta)/2}(\Omega_T) \quad \text{for any } \beta \in (0, 1). \tag{2.8}
\]
Taking into account that \(u = \frac{1}{2d_2}(-d_1 + \sqrt{d_1^2 + 4d_2z})\) (note that when \(d_2 = 0\), this becomes \(u = z/d_1\)), we find by (2.8) that
\[
u \in C^{1+\beta, (1+\beta)/2}(\Omega_T), \quad \beta \in (0, 1). \tag{2.9}
\]

Multiplying the second equation of (1.4) by \(v\) and integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 dx + \int_{\Omega} \left( d_1 + 2d_4v + \frac{1}{1 + u^2}\right) |\nabla v|^2 dx = \int_{\Omega} \frac{\ell u^{\ell-1}v}{(1 + u^2)} \nabla u \cdot \nabla v dx + \int_{\Omega} v(g(u, v) - s) dx. \tag{2.10}
\]
Integrating (2.10) from 0 to \(t\), and noting the assumption (H) and (2.1), we deduce that, for all \(t \in [0, T]\),
\[
\int_{\Omega} v^2 dx - \int_{\Omega} v_0^2(x) dx + 2d_3 \int_{\Omega} |\nabla v|^2 dx dt + \frac{16d_4}{9} \int_{\Omega} |\nabla (v^{3/2})|^2 dx dt + 2 \int_{\Omega} \frac{|\nabla (v^{1/2})|^2}{1 + u^2} dx dt \leq 2\ell \int_{\Omega} \frac{u^{\ell-1}v}{(1 + u^2)} \nabla u \cdot \nabla v dx dt + C_1 \int_{\Omega} v^2 dx. \tag{2.11}
\]
Consequently, by (2.9), Lemma 1 and \( \ell \geq 1 \), it follows that, for all \( t \in [0, T] \),

\[
\left| \int_{Q_t} \frac{u^{\ell-1} v}{1 + u^\ell} \nabla u \cdot \nabla v \, dx \, dt \right| \leq C_2 \int_{Q_t} |v| |\nabla v| \, dx \, dt \\
\leq \varepsilon C_2 \int_{Q_t} |\nabla v|^2 \, dx \, dt + \frac{C_2}{4\varepsilon} \int_{Q_t} v^2 \, dx \, dt. \tag{2.12}
\]

Choose \( \varepsilon \) so that \( \varepsilon C_2 < d_5 \); then from (2.11) and (2.12) we obtain

\[
\int_{Q_T} v^2 \, dx + \int_{Q_t} |\nabla v|^2 \, dx \, dt + \int_{Q_t} |\nabla (v^{3/2})|^2 \, dx \, dt \leq \int_{Q_T} v_0^2(x) \, dx + C_3 \int_{Q_T} v^2 \, dx, \quad \forall \, t \in [0, T]. \tag{2.13}
\]

An application of the Gronwall inequality to (2.13) yields

\[
\int_{Q_T} v^2 \, dx + \int_{Q_t} |\nabla v|^2 \, dx \, ds + \int_{Q_t} |\nabla (v^{3/2})|^2 \, dx \, dt \leq C_4.
\]

As a result,

\[
\sup_{0 \leq t \leq T} \int_{Q_T} v^2 \, dx + \int_{Q_t} |\nabla v|^2 \, dx \, ds + \int_{Q_t} |\nabla (v^{3/2})|^2 \, dx \, dt \leq C_4, \tag{2.14}
\]

which shows that \( v \in V_2(Q_T) \), thus completing the proof of the lemma. \( \square \)

**Lemma 3** (\( L^\infty \) Estimates for \( v \)). Let \( T > 0 \). Then \( \|v\|_{L^\infty(Q_T)} \leq M_2 \) for some constant \( M_2 > 0 \).

**Proof.** The second equation of (1.4) can be written as the linear equation

\[
v_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( a_i(x, t) \frac{\partial v}{\partial x_i} \right) + \sum_{i=1}^N \frac{\partial}{\partial x_i} (a_i \nu) - \bar{a} v,
\]

where

\[
a_{ij} = (a_3 + 2d_4 + \frac{1}{1 + u^\ell}) \delta_{ij}, \quad a_i = -\frac{\ell u^{\ell-1}}{1 + u^\ell} \frac{\partial u}{\partial x_i}, \quad \bar{a} = -s + g(u, v).
\]

Since \( \ell \geq 1 \), \( 0 < u \leq M \), \( \nabla u \in C^{\beta/2} \bar{Q}_T \) by (2.9), \( \|v\|_{L^\infty(Q_T)} \) is finite by Lemma 2, and \( g(u, v) \) is bounded by the assumption (H) and Lemma 1, the maximum principle of [14, Theorem 7.1, p.181] ensures that \( v \) is bounded in \( \bar{Q}_T \). \( \square \)

**3. Proof of Theorem 1**

On the basis of the above results, we may demonstrate a proof of Theorem 1 analogous to that in the paper [9]. Let \( [0, T] \) be a maximal existence interval of the solution \((u, v)\) to the system (1.4). Rewrite the second equation of (1.4) as

\[
v_t = \nabla \cdot \left\{ \left( d_3 + 2d_4 v + \frac{1}{1 + u^\ell} \right) \nabla v - \frac{\ell u^{\ell-1}}{1 + u^\ell} \nabla u \right\} + f^*(x, t), \tag{3.1}
\]

where \( f^*(x, t) = v(-s + g(u, v)) \in L^\infty(\bar{Q}_T) \) by the assumption (H), Lemmas 1 and 3. Since \( \ell \geq 1 \), from Lemma 1, Lemma 3 and (2.9), it follows that \( \frac{1}{1 + u^\ell} \) is finite, \( \frac{\ell u^{\ell-1}}{1 + u^\ell} \) and \( \nabla u \) are all bounded. Therefore, the result of [15] shows that

\[
v \in C^{\sigma, \sigma/2}(\bar{Q}_T) \quad \text{for some} \quad 0 < \sigma < 1.
\]

We now turn to the equation of \( u \) in the system (1.4) and rewrite it as

\[
u_t = (d_1 + 2d_2 u) \Delta u + g^*(x, t), \tag{3.3}
\]

where \( g^*(x, t) = 2d_2 |\nabla u|^2 + u \left( a - b u - f(u, v) \right) \in C^{\sigma, \sigma/2}(\bar{Q}_T) \) by (2.9), (3.3), the assumption (H) and Lemma 1. Applying the Schauder estimates to (3.3), we have

\[
u \in C^{2+\sigma^*, (2+\sigma^*)/2}(\bar{Q}_T), \quad \sigma^* = \min\{\alpha, \sigma\}.
\]

In order to demonstrate the higher regularity, it is convenient to introduce the function

\[
\phi(x, t) = \left( d_3 + d_4 v + \frac{1}{1 + u^\ell} \right) v,
\]
which satisfies
\[ \phi_t = \left( d_3 + 2d_4 v + \frac{1}{1+u^\ell} \right) \Delta \phi + h^*(x, t) \]
with
\[ h^*(x, t) = v \left( d_3 + 2d_4 v + \frac{1}{1+u^\ell} \right) (-s + g(u, v)) - \frac{\ell u^{\ell-1} v}{(1+u^\ell)^2} u_t. \]

Noting that \( \ell \geq 1 \) and \( 0 \leq u \leq M \) in \( \tilde{Q}_T \times [0, T] \), it follows from (3.2) and (3.4) that \( (d_3 + 2d_4 v + \frac{1}{1+u^\ell}) \in C^{\sigma^*, \sigma^*/2}(\tilde{Q}_T) \) and \( h^*(x, t) \in C^{\sigma^*, \sigma^*/2}(\tilde{Q}_T) \) (if necessary, \( \sigma^* \) can be chosen to be smaller). Thus, an application of the Schauder estimates to (3.5) gives
\[ \phi \in C^{2+\sigma^*, (2+\sigma^*)/2}(\tilde{Q}_T). \]

Recall that \( \phi = (d_3 + d_4 v + \frac{1}{1+u^\alpha}) v \). Then (3.4) and \( \ell \geq 1 \) imply that
\[ v \in C^{1+\sigma^*, (1+\sigma^*)/2}(\tilde{Q}_T). \]

Here, we should point out that it is not difficult to see that for the case \( \ell = 1 \) or the case \( \ell \geq 2 \),
\[ v \in C^{2+\sigma^*, (2+\sigma^*)/2}(\tilde{Q}_T). \]

When \( 1 < \ell < 2 \), repeating the above procedure by making use of (3.4) and (3.7) in place of (2.9) and (3.2), one can find that (3.4) and (3.7) are fulfilled with \( \sigma^* \) replaced by \( \alpha \), that is, \( u \in C^{2+\alpha, (2+\alpha)/2}(\tilde{Q}_T) \) and \( v \in C^{1+\alpha, (1+\alpha)/2}(\tilde{Q}_T) \). Consequently,
\[ \sup_{0 \leq t < T} \| u(\cdot, t) \|_{W^{1, p}_p(\Omega)} < \infty, \quad \text{and} \quad \sup_{0 \leq t < T} \| v(\cdot, t) \|_{W^{1, p}_p(\Omega)} < \infty \]
for \( p > N \). Then the result of [13] asserts that \( T = \infty \). Therefore, \( u \in C^{2+\alpha, (2+\alpha)/2}(\tilde{Q}_T \times [0, \infty)) \) and \( v \in C^{1+\alpha, (1+\alpha)/2}(\tilde{Q}_T \times [0, \infty)) \). This concludes the proof of Theorem 1. \( \square \)

References