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Poset resolutions and lattice-linear monomial ideals ☆

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ABSTRACT

We introduce the class of lattice-linear monomial ideals and use the lcm-lattice to give an explicit construction for their minimal free resolution. The class of lattice-linear ideals includes (among others) the class of monomial ideals with a linear free resolution and the class of Scarf monomial ideals. Our main tool is a new construction by Tchernev that produces from a map of posets $\eta : P \to \mathbb{N}^n$ a sequence of multigraded modules and maps.

1. Introduction

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring where $\Bbbk$ is a field, considered with its standard $\mathbb{Z}^n$-grading (multigrading) and let N be an ideal in R generated by monomials.

A well-studied combinatorial object associated to N is the lcm-lattice L_N. This atomic lattice is comprised of the least common multiples of minimal monomial generators of N where ordering in L_N is given by divisibility. Gasharov, Peeva and Welker in [6] express the Betti numbers of R/N using the homology groups of certain open intervals in L_N. They further show that the isomorphism class of L_N, together with the labeling of its elements, determines the structure of the minimal free resolution of R/N.

Motivated by these results, we introduce the class of lattice-linear monomial ideals. A lattice-linear ideal has the mapping structure of its minimal free resolution encoded in the covering relations of its lcm-lattice. In our main result, Theorem 3.3, we construct explicitly the minimal free resolution of any lattice-linear ideal from its lcm-lattice.

The class of lattice-linear ideals contains extensively studied subclasses, including the class of monomial ideals with a linear free resolution [7–11] and the class of Scarf ideals [1]. For each of these two subclasses, minimal free resolutions have been constructed using different techniques which are also distinct from the one described in this paper.

☆ With an appendix by Alexandre Tchernev.

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The key tool used to produce lattice-linear resolutions is a new construction described in Section 2 which is due to Tchernev. This construction takes a finite poset P as its input and produces as its output $\mathcal{D}(P)$, a sequence of $\Bbbk$-vector spaces and $\Bbbk$-linear maps. The sequence $\mathcal{D}(P)$ is formed from the homology of certain open intervals in the poset P and is motivated directly by the results of [6] on the lcm-lattice. In general, this sequence may not even be a complex of vector spaces, and if it is indeed a complex, it may not necessarily be exact. Appendix A provides some technical details of the construction, where in Proposition A.1, Tchernev shows that if the poset P is assumed to be ranked then $\mathcal{D}(P)$ is a complex of vector spaces.

We turn our attention to free resolutions of monomial ideals in Section 3. When there exists a map of posets $\eta : P \to \mathbb{N}^n$, we homogenize $\mathcal{D}(P)$ to produce a sequence of multigraded modules and multigraded morphisms $\mathcal{F}(\eta)$ which approximates a free resolution of the monomial ideal N whose generators have their degrees given by the images of the atoms of P. In the case when $\mathcal{F}(\eta)$ is an acyclic complex, we call it a poset resolution of R/N. The framework of poset resolutions is utilized to describe the minimal resolution of a lattice-linear ideals, although other posets may provide appropriate structure for supporting resolutions. Indeed, both the Taylor resolution [15] and the Eliahou–Kervaire resolution [5,4] may be realized as poset resolutions. In Section 4, we give brief descriptions of these examples of poset resolutions as well as the examples of lattice-linear ideals mentioned above.

The remainder of the paper is devoted to the proof of Theorem 3.3, which makes use of a certain reformulation of the sequence $\mathcal{D}(P)$. More precisely, in Section 5 we describe a variation on the construction of $\mathcal{D}(P)$. In order to obtain this reformulation we assume that A, the set of atoms of P, forms a crosscut which is indeed the case if P is a lattice or a geometric semilattice. Under the assumption that A forms a crosscut, the Crosscut Theorem applies so that the poset P is homotopy equivalent to the atomic crosscut complex $\Gamma(P, A)$. The reformulation of $\mathcal{D}(P)$ is carried out by replacing the homology of the open intervals used in the original construction with the homology of atomic crosscut complexes associated to each element of P through the isomorphism on homology induced by homotopy equivalence. More precisely, we describe a canonical isomorphism between the vector space given by the homology of an open interval and the vector space given by the homology of the crosscut complex of said open interval. Through this isomorphism, the differential of $\mathcal{D}(P)$ is able to be defined on either of the resulting vector spaces, facilitating the use of two equivalent methods for construction of $\mathcal{D}(P)$. This reformulation is established in Proposition 5.5 and Corollary 5.6, where we utilize the homotopy equivalence between P and the atom crosscut complex $\Gamma(P, A)$ described in the proof of the Crosscut Theorem given by Björner in [2].

The proof of Theorem 3.3 is given in Section 6, where the poset map used to construct the minimal free resolution of a lattice-linear ideal is the multidegree map, $\text{mdeg}: L_N \to \mathbb{N}^n$, which sends a monomial to its multidegree. As mentioned above, since L_N is a lattice, we use the crosscut complexes of open intervals to provide the framework for the underlying vector space structure of $\mathcal{F}(\text{mdeg})$.

2. Poset combinatorics and $\mathcal{D}(P)$

Let $(P, \leqslant)$ be a finite poset with minimum element $\hat{0}$. When the meet (greatest lower bound) and join (least upper bound) of a subset $\sigma \subseteq P$ exist, they are denoted as $\wedge\sigma$ and $\vee\sigma$, respectively. A totally ordered subset $\sigma \subseteq P$ which has the form $\alpha_0 < \cdots < \alpha_k$ is called a *chain of length k*, and for $\alpha \in P$, the rank of an element α is $\text{rk}(\alpha) = \sup\{\ell\colon\ \alpha_0 < \cdots < \alpha_\ell = \alpha\}$. A subset of P comprised entirely of elements which are pairwise incomparable is called an anti-chain. An element $\beta \in P$ is said to be *covered* by α (which we write as $\beta \lessdot \alpha$) when $\beta < \alpha$ and there exists no $\gamma \in P$ such that $\beta < \gamma < \alpha$. An open interval in P is the subposet of chains

$$(\beta, \alpha) = \{\gamma \in P \mid \alpha < \gamma < \beta\},$$

with closed and half-open intervals denoted similarly. Recall that the set of atoms of P is

$$A = \{a \in P\colon\ \hat{0} \lessdot a\}.$$

Using the convention that $\operatorname{rk}(\hat{0}) = 0$, we have $\operatorname{rk}(a) = 1$ for every $a \in A$. The poset P is said to be *ranked* when $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta) + 1$ for every $\beta \lessdot \alpha \in P$.

For the purpose of topological analysis of P, recall that the order complex of a poset is the abstract simplicial complex $\Delta(P)$ whose vertices are the elements of $P \setminus \{\hat{0}\}$ and for $k > 0$, the k-dimensional faces of $\Delta(P)$ are in one-to-one correspondence with the length k chains of $P \setminus \{\hat{0}\}$. As is standard, whenever discussing topological properties of P, we are implicitly referring to the topological properties of $\Delta(P)$. On the other hand, given a simplicial complex Ω, the face poset $\mathcal{P}(\Omega)$ is the set of faces of Ω with partial ordering given by inclusion. These combinatorial correspondences allow the first barycentric subdivision of a simplicial complex Ω to be realized as the abstract simplicial complex $\mathbf{sd}(\Omega) = \Delta(\mathcal{P}(\Omega))$.

For $\alpha \in P$ we use the notation $\Delta_\alpha := \Delta(\hat{0}, \alpha)$ for the order complex of the associated open interval which we analyze in the following way.

Definition 2.1. For $\lambda \in P$, set $\mathbf{D}_\lambda = \Delta(\hat{0}, \lambda]$, the order complex of a half-closed interval, so that

$$\Delta_\alpha = \bigcup_{\lambda \lessdot \alpha} \mathbf{D}_\lambda.$$

For a fixed $\lambda \lessdot \alpha$, let

$$\Delta_{\alpha,\lambda} = \mathbf{D}_\lambda \cap \left(\bigcup_{\substack{\beta \lessdot \alpha \\ \lambda \neq \beta}} \mathbf{D}_\beta \right).$$

Remark 2.2. For each $\lambda \neq \beta$ where both λ and β are covered by α we have $\mathbf{D}_\lambda \cap \mathbf{D}_\beta \subset \Delta_\lambda$. Indeed, suppose that $\{\mu_0, \mu_1, \dots, \mu_k\}$ is a face of $\mathbf{D}_\lambda \cap \mathbf{D}_\beta$ which is associated to the chain $\mu_0 < \mu_1 < \cdots < \mu_k$ of P. By assumption, $\mu_k \leqslant \lambda$ and $\mu_k \leqslant \beta$. Supposing that $\mu_k = \lambda$, it follows that either $\lambda = \beta$ or $\lambda \leqslant \beta \lessdot \alpha$, each of which is a contradiction to our assumptions. Therefore, $\mu_j < \lambda$ for every $0 \leqslant j \leqslant k$ which implies that $\mu_0 < \mu_1 < \cdots < \mu_k$ is a chain of the open interval $(\hat{0}, \lambda)$ so that $\{\mu_0, \mu_1, \dots, \mu_k\}$ is a face of Δ_λ. For a fixed $\lambda \lessdot \alpha$, taking the union of the complexes $\mathbf{D}_\lambda \cap \mathbf{D}_\beta$ over all $\beta \lessdot \alpha$ with $\beta \neq \lambda$ we have the inclusion $\Delta_{\alpha,\lambda} \subseteq \Delta_\lambda$.

Using the family of simplicial complexes Δ_α we now describe a sequence of vector spaces and vector space maps

$$\mathcal{D}(P) : \cdots \longrightarrow \mathcal{D}_i \xrightarrow{\varphi_i} \mathcal{D}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{D}_1 \xrightarrow{\varphi_1} \mathcal{D}_0$$

whose structure is determined by the reduced simplicial homology of the complexes Δ_α with coefficients in $\Bbbk$.

Remark 2.3. Inherent in our study of an abstract simplicial complex X is the existence of the empty face $\emptyset \in X$. We utilize the convention (see, for instance [2]) that the empty abstract simplicial complex $\{\emptyset\}$, which has no vertices, is a (-1)-dimensional simplex.

Definition 2.4.

1. Set $\mathcal{D}_0 = \widetilde{H}_{-1}(\{\emptyset\}, \Bbbk)$, a one-dimensional $\Bbbk$-vector space.
2. For $i \geqslant 1$, set $\mathcal{D}_{i,\alpha} = \widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk)$ and

$$\mathcal{D}_i = \bigoplus_{\alpha \in P \setminus \{\hat{0}\}} \mathcal{D}_{i,\alpha}.$$

Remark 2.5. The structure of the poset P clearly determines in a canonical way the vector spaces $\mathcal{D}_{i,\alpha}$.

Remark 2.6. When $i=1$ and $\alpha \in A$, we have $\Delta_\alpha = \mathbf{D}_{\hat{0}} = \{\emptyset\}$ and thus, $\mathcal{D}_{1,\alpha} = \widetilde{H}_{-1}(\{\emptyset\}, \Bbbk)$, a one-dimensional $\Bbbk$-vector space. Alternatively, if $i=1$ and $\alpha \notin A$ then $\Delta_\alpha = \bigcup_{\lambda \lessdot \alpha} \mathbf{D}_\lambda \neq \{\emptyset\}$ and hence $\mathcal{D}_{1,\alpha} = \widetilde{H}_{-1}(\Delta_\alpha, \Bbbk) = 0$. Therefore,

$$\mathcal{D}_1 = \bigoplus_{\alpha \in A} \mathcal{D}_{1,\alpha} = \bigoplus_{\alpha \in A} \widetilde{H}_{-1}\big(\{\emptyset\}, \Bbbk\big) \cong \bigoplus_{\alpha \in A} \Bbbk.$$

Next, given $\lambda \lessdot \alpha \in P$, we consider the Mayer–Vietoris sequence in reduced homology for the triple

$$\left(\mathbf{D}_\lambda, \bigcup_{\substack{\beta \lessdot \alpha \\ \lambda \neq \beta}} \mathbf{D}_\beta, \Delta_\alpha\right).$$

We write $\iota : \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, \Bbbk) \to \widetilde{H}_{i-3}(\Delta_\lambda, \Bbbk)$ for the map induced in homology by the inclusion map and

$$\delta_{i-2}^{\alpha,\lambda} : \widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk) \to \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, \Bbbk)$$

for the connecting homomorphism from the Mayer–Vietoris sequence. Recall this homomorphism takes the class $[c] \in \widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk)$ to the class $[d_{i-2}(c')] \in \widetilde{H}_{i-3}(\Delta_{\alpha,\lambda}, \Bbbk)$ where $c' + c'' = c \in \widetilde{C}_{i-2}(\Delta_\alpha, \Bbbk)$, and c' and c'' are any components of c that are supported by $\mathbf{D}_\lambda$ and $\bigcup_{\lambda \neq \mu \lessdot \alpha} \mathbf{D}_\mu$ respectively. Here, d is the usual simplicial boundary map.

We now proceed with the definition of the maps $\varphi_i : \mathcal{D}_i \to \mathcal{D}_{i-1}$ for the sequence $\mathcal{D}(P)$.

Definition 2.7.

1. Define $\varphi_1 : \mathcal{D}_1 \to \mathcal{D}_0$ componentwise by $\varphi_1|_{\mathcal{D}_{1,\alpha}} = \mathrm{id}_{\widetilde{H}_{-1}(\{\emptyset\},\Bbbk)}$.
2. For $i \geqslant 2$ define $\varphi_i : \mathcal{D}_i \to \mathcal{D}_{i-1}$ componentwise by

$$\varphi_i|_{\mathcal{D}_{i,\alpha}} = \sum_{\lambda \lessdot \alpha} \varphi_i^{\alpha,\lambda}$$

where

$$\varphi_i^{\alpha,\lambda} : \mathcal{D}_{i,\alpha} \to \mathcal{D}_{i-1,\lambda}$$

is the composition $\varphi_i^{\alpha,\lambda} = \iota \circ \delta_{i-2}^{\alpha,\lambda}$.

Remark 2.8. The maps of vector spaces φ_i are clearly canonical and determined by the structure of the homology of the open intervals in the poset P.

Remark 2.9. A priori, the sequence $\mathcal{D}(P)$ is not necessarily a complex of vector spaces, and even if it is a complex, it need not be exact. While necessary and sufficient conditions for the construction to produce an exact complex are not known, if the poset is ranked, $\mathcal{D}(P)$ is a complex of vector spaces. This result appears as Proposition A.1 of Appendix A.

3. Poset resolutions and lattice-linear monomial ideals

Let $R = \Bbbk[x_1, \dots, x_n]$, write $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ for the unique graded maximal ideal of R and $x^{\mathbf{a}} = x_1^{\mathbf{a}_1} \cdots x_n^{\mathbf{a}_n}$ for a monomial of R. We appeal to the standard $\mathbb{Z}^n$-grading (multigrading) of R and use the notation of Section 2 for ordering in the partially ordered set $\mathbb{Z}^n$. We use the multidegree map $\mathrm{mdeg}: R \to \mathbb{Z}^n$ for which $x^{\mathbf{a}} \mapsto \mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ and identify the monomials in R with the n-tuples of $\mathbb{N}^n \subset \mathbb{Z}^n$.

Suppose that $\eta : P \to \mathbb{N}^n$ is a map of partially ordered sets, and A is the set of atoms of P. Let N be the ideal in R minimally generated by the monomials

$$\{x^{\eta(a)}\colon a \in A\}.$$

The sequence of vector spaces $\mathcal{D}(P)$ constructed in Section 2 and associated to P may be homogenized using the map η to produce a sequence of free multigraded R-modules and multigraded R-module homomorphisms which approximates a free resolution of the multigraded module R/N.

The homogenization of $\mathcal{D}(P)$ is carried out by defining $\mathcal{F}_0(\eta) = R \otimes_{\Bbbk} \mathcal{D}_0$ and multigrading the result with $\mathrm{mdeg}(x^{\mathbf{a}} \otimes v) = \mathbf{a}$ for each $v \in \mathcal{D}_0$. Similarly, for $i \geqslant 1$ set

$$\mathcal{F}_i(\eta) = \bigoplus_{\hat{0} \neq \lambda \in P} \mathcal{F}_{i,\lambda}(\eta) = \bigoplus_{\hat{0} \neq \lambda \in P} R \otimes_{\Bbbk} \mathcal{D}_{i,\lambda}$$

where the multigrading is defined as $\mathrm{mdeg}(x^{\mathbf{a}} \otimes v) = \mathbf{a} + \eta(\lambda)$ for each $v \in \mathcal{D}_{i,\lambda}$.

The differential in this sequence of multigraded modules is defined componentwise in homological degree 1 as

$$\partial_1|_{\mathcal{F}_{1,\lambda}(\eta)} = x^{\eta(\lambda)} \otimes \varphi_1|_{\mathcal{D}_{1,\lambda}}$$

and for $i \geqslant 1$, the map $\partial_i : \mathcal{F}_i(\eta) \to \mathcal{F}_{i-1}(\eta)$ is defined as

$$\partial_i|_{\mathcal{F}_{i,\alpha}(\eta)} = \sum_{\lambda \lessdot \alpha} \partial_i^{\alpha,\lambda}$$

where $\partial_i^{\alpha,\lambda} : \mathcal{F}_{i,\alpha}(\eta) \to \mathcal{F}_{i-1,\lambda}(\eta)$ takes the form $\partial_i^{\alpha,\lambda} = x^{\eta(\alpha)-\eta(\lambda)} \otimes \varphi_i^{\alpha,\lambda}$ for $\lambda \lessdot \alpha$. We now have a sequence of multigraded modules and maps

$$\mathcal{F}(\eta) : \cdots \longrightarrow \mathcal{F}_t(\eta) \xrightarrow{\partial_t} \mathcal{F}_{t-1}(\eta) \longrightarrow \cdots \longrightarrow \mathcal{F}_1(\eta) \xrightarrow{\partial_1} \mathcal{F}_0(\eta),$$

and are in a position to make the following definition.

Definition 3.1. If $\mathcal{F}(\eta)$ is an acyclic complex of multigraded modules, then we say that it is a *poset resolution* of the ideal N.

We now turn to the class of monomial ideals that are the focus of this paper. Recall that the lcm-lattice associated to a monomial ideal N is the set L_N of least common multiples of minimal generators of N along with partial ordering given by divisibility. By definition, L_N has minimum element $\hat{0} = 1$ which is considered to be the least common multiple of the empty set. Recall that monomials in R are identified with their degree in $\mathbb{N}^n$ and it is standard to consider $L_N \subset \mathbb{N}^n$. As an immediate consequence of [3, Theorem 3.1a], if the ith multigraded Betti number $\beta_{i,\alpha}(R/N) \neq 0$, then $\alpha \in L_N$. In particular, this means that if B_i is any multihomogeneous basis of the free module $\mathcal{F}_i$ in the minimal free resolution $\mathcal{F}$ of R/N then $\mathrm{mdeg}(v) \in L_N$ for each $v \in B_i$.

Definition 3.2. Let $\mathcal{F}$ be a minimal multigraded free resolution of R/N. We say that N is *lattice-linear* if multigraded bases B_k of the free modules $\mathcal{F}_k$ can be fixed for all k so that for any $i \geqslant 1$ and any $v \in B_i$ the differential

$$\partial^{\mathcal{F}}(v) = \sum_{v' \in B_{i-1}} m_{v,v'} \cdot v'$$

has the property that if the coefficient $m_{v,v'} \neq 0$ then $\operatorname{mdeg}(v') \lessdot \operatorname{mdeg}(v) \in L_N$.

We now state our main result.

Theorem 3.3. *Define the map* $\operatorname{mdeg} : L_N \to \mathbb{N}^n$ *by sending a monomial* $m \in L_N$ *to its multidegree* $\operatorname{mdeg}(m) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. *The monomial ideal N is lattice-linear if and only if* $\mathcal{F}(\operatorname{mdeg})$ *is its minimal free resolution.*

We postpone the proof of Theorem 3.3 until Section 6, first giving examples of poset resolutions and lattice-linear ideals in Section 4 and reformulating $\mathcal{D}(P)$ in Section 5.

4. Examples of poset resolutions

Poset resolutions provide a general framework from which to view a variety of (minimal) resolutions which have been previously constructed using disparate techniques which are each distinct from the one presented herein.

For instance, the Taylor resolution [15] of an ideal minimally generated by the monomials $\{m_1, \ldots, m_r\}$ can be realized as a poset resolution where $P = \mathcal{B}_r$, the Boolean lattice. The map

$$\eta : \mathcal{B}_r \to \mathbb{N}^n$$

is defined on a lattice element $I \subseteq \{1, \ldots, r\}$ via $I \mapsto \operatorname{mdeg}(m_I)$, where $m_I = \operatorname{lcm}(m_i\colon\ i \in I)$. Although generally nonminimal, this reinterpretation provides a poset resolution of an arbitrary monomial ideal.

If one considers a more restricted class of monomial ideals, and an appropriate poset, minimal resolutions are also able to be constructed. Indeed, recall that a monomial ideal N is called *stable* if for every monomial $m \in N$, the monomial $m \cdot x_i/x_d \in N$ for each $1 \leqslant i < d$, where $d = \max\{k\colon\ x_k \text{ divides } m\}$. The class of stable monomial ideals, introduced in [5], is shown to admit a minimal poset resolution in [4] using a poset of Eliahou–Kervaire admissible symbols.

We now turn to the poset whose structure is the main motivation for this paper, the lcm-lattice L_N. Theorem 3.3 establishes a minimal poset resolution for the class of lattice-linear ideals which contains ideals with a linear free resolution, whose minimal free resolutions have been constructed in [10] using tools from Discrete Morse Theory. Our methods allow us to provide a considerably simpler and more transparent approach to constructing these minimal free resolutions.

Proposition 4.1. *Every monomial ideal which has a linear minimal free resolution is a lattice-linear monomial ideal.*

Proof. Suppose that N is an ideal with linear resolution and aiming for a contradiction, that N is not lattice-linear. Then there exists $i > 0$ and $e \in F_i$ of degree α such that in the expansion of $\partial^{\mathcal{F}}(e)$ the element $e' \in F_{i-1}$ has multidegree β which is not covered by $\alpha \in L_N$. Therefore, there exists $\gamma \in L_N$ such that $\beta < \gamma < \alpha$. However, since N has a linear resolution, $\deg(\alpha) = \deg(\beta) + 1$, and there can be no multidegree γ which fits this criteria for comparability in L_N. Hence, N is lattice-linear. $\square$

Remark 4.2. As is the case with ideals having linear resolutions, the notion of lattice-linearity is dependent upon the characteristic of the ground field $\Bbbk$. For example, the ideal minimally generated by the ten monomials

$$\langle x_1x_2x_3,\quad x_1x_3x_5,\quad x_1x_4x_5,\quad x_2x_3x_4,\quad x_2x_4x_5,$$
$$x_1x_2x_6,\quad x_1x_4x_6,\quad x_2x_5x_6,\quad x_3x_4x_6,\quad x_3x_5x_6\rangle$$

in $R = \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$ which is the Stanley–Reisner ideal corresponding to a triangulation of the projective plane is known to have a linear resolution if and only if $\mathrm{char}(\Bbbk) \neq 2$. It is natural to ask whether the lattice-linearity of the ideal is preserved when linearity is not, however this ideal is lattice-linear if and only if $\mathrm{char}(\Bbbk) \neq 2$.

For ideals whose resolution is not linear in any characteristic, it is natural to ask whether lattice-linearity is preserved when the characteristic of $\Bbbk$ is changed. This is not the case, for the ideal minimally generated by the ten monomials

$$\langle x_1x_2^2x_3^3,\quad x_1x_3^3x_5^2,\quad x_1x_4^3x_5^2,\quad x_2^2x_3^3x_4^3,\quad x_2^2x_4^3x_5^2,$$
$$x_1x_2^2x_6,\quad x_1x_4^3x_6,\quad x_2^2x_5^2x_6,\quad x_3^3x_4^3x_6,\quad x_3^3x_5^2x_6\rangle$$

in $R = \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$ is lattice-linear if and only if $\mathrm{char}(\Bbbk) \neq 2$.

Remark 4.3. For a squarefree monomial ideal N, the linear strand of the minimal free resolution of R/N has been completely described by Reiner and Welker in [13]. Their technique utilizes a Mayer–Vietoris sequence in reduced homology applied to the links of faces in the Alexander dual simplicial complex Δ^* associated to N. The connection between the technique of Reiner and Welker and the one described in this paper is not obvious and is worthy of further study.

Lattice-linearity is also an appropriate generalization of another well-studied class of monomial ideals. Recall from [1] the Scarf simplicial complex

$$\Delta_N = \bigl\{I \subseteq \{1, \dots, r\}\colon\ m_I \neq m_J \text{ for all } J \subseteq \{1, \dots, r\} \text{ other than } I\bigr\},$$

where as in the Taylor resolution, $m_I = \mathrm{lcm}(m_i\colon\ i \in I)$. In fact, any face $I \in \Delta_N$ is uniquely determined from m_I as $I = \{i\colon\ m_i < m_I\}$. The ideal N is called a *Scarf* ideal if its minimal free resolution is supported on the reduced simplicial chain complex of Δ_N. For instance, the so-called *generic* [1,11] ideals are Scarf. Note that when N is Scarf, the differential in its minimal free resolution takes the unique basis element e_I labeled by the monomial m_I to

$$\sum_{j=1}^{|I|} (-1)^{j+1} \frac{m_I}{m_{I\setminus\{i_j\}}} \cdot e_{I\setminus\{i_j\}}$$

where $I = \{i_1, \dots, i_{|I|}\}$.

Proposition 4.4. *Every Scarf ideal is a lattice-linear monomial ideal. In particular, for $\eta : \mathcal{P}(\Delta_N) \to \mathbb{N}^n$ where $I \mapsto \mathrm{mdeg}(m_I)$, the complex $\mathcal{F}(\eta)$ is the minimal free resolution of R/N.*

Proof. Suppose that N is a Scarf ideal, set L_N as the lcm-lattice of N and let $\mathcal{F}$ denote the minimal free resolution of R/N with differential $\partial^{\mathcal{F}}$. Fix a homological degree $p > 0$ and an $I \subseteq \{1, \dots, r\}$. For every $J \subset I$, the monomial $m_J < m_I$ in L_N. Supposing that N is not lattice-linear, there exists $J = \{i_1, \dots, \hat{i}_j, \dots, i_p\}$ so that the coefficient of e_J in the expansion of $\partial^{\mathcal{F}}(e_I)$ is nonzero, and yet m_J is not covered by m_I in the lattice L_N. Thus, there exists $m \in L_N$ so that $m_J < m < m_I$. Since N is Scarf, I and J are uniquely determined from m_I and m_J. By definition, $m = \mathrm{lcm}(m_{i_1}, \dots, m_{i_t})$ for some $\{i_1, \dots, i_t\}$ and it follows that $J \subset \{i_1, \dots, i_t\} \subseteq I = J \cup \{i_j\}$. This forces $\{i_1, \dots, i_t\} = I$ and therefore, $m = m_I$, and $m_J \lessdot m_I$. Hence N is lattice-linear. $\square$

5. A reformulation of $\mathcal{D}(P)$

The structure of the poset P affects directly the make-up of the sequence $\mathcal{D}(P)$, which itself may prove difficult to analyze. Specifically, the homology of the open intervals $(\hat{0},\alpha)$ and the precise action of the maps φ_i may be difficult to calculate. As such, we consider another family of simplicial complexes associated to P, which under appropriate assumptions can take the place of the simplicial complexes Δ_α in the construction of $\mathcal{D}(P)$. To do so, we recall the following well-studied combinatorial notions.

A set $C \subset P$ is called a *crosscut* if it satisfies the following three properties; (i) C is an anti-chain, (ii) for every finite chain σ in P there exists some element in C which is comparable to each element in σ and (iii) if $S \subseteq C$ is bounded in P, then either the join $\vee S$ or the meet $\wedge S$ exists in P. For a crosscut C, the *crosscut simplicial complex* $\Gamma(P,C)$ is defined as the collection of all subsets of C which are bounded in P. The connection between P and the crosscut complex $\Gamma(P,C)$ is fundamental.

Theorem 5.1. *(See [2, Crosscut Theorem].) The poset P and $\Gamma(P,C)$ are homotopy equivalent.*

To proceed, we assume that P is a finite poset and that **its set of atoms A forms a crosscut**, which is the case when P is a lattice or geometric semilattice. The assumption that set of atoms A forms a crosscut of P implies that $A_\alpha = \{a \in A\colon\ a < \alpha\}$ also is a crosscut for the subposet $(\hat{0},\alpha)$ where $\alpha \in P$. In this case, the Crosscut Theorem applies not only to P but to the open intervals of the form $(\hat{0},\alpha)$ which are in turn homotopy equivalent to $\Gamma((\hat{0},\alpha), A_\alpha)$, itself an atom crosscut complex.

The proof of Theorem 3.3, given in Section 6 utilizes this homotopy equivalence to analyze the structure of $\mathcal{D}(L_N)$, the sequence of vector spaces based on the lcm-lattice of a monomial ideal N.

Remarks 5.2.

1. Björner's proof of the Crosscut Theorem in [2] relies on an order-reversing map of posets. For a poset P and its atom crosscut A, this map $h : \mathcal{P}(\Delta(P)) \to \mathcal{P}(\Gamma(P,A))$ is defined on a poset element $\sigma \in \mathcal{P}(\Delta(P))$ as
$$\sigma \mapsto \big\{x \in A\colon\ \sigma \in \Delta(P_{\geqslant x})\big\}$$
where $P_{\geqslant x} = \{y \in P\colon\ \ y \geqslant x\}$. The map h induces a simplicial map $h : \Delta(\mathcal{P}(\Delta(P))) \to \Delta(\mathcal{P}(\Gamma(P,A)))$ which is the desired homotopy equivalence.
2. The simplicial map h between the complexes $\Delta(\mathcal{P}(\Delta(P))) = \mathbf{sd}(\Delta(P))$ and $\Delta(\mathcal{P}(\Gamma(P,A))) = \mathbf{sd}(\Gamma(P,A))$ induces in the usual way a map between the complexes $\widetilde{\mathcal{C}}(\mathbf{sd}(\Delta(P)))$ and $\widetilde{\mathcal{C}}(\mathbf{sd}(\Gamma(P,A)))$ which in turn induces a chain map
$$h_\sharp : \widetilde{\mathcal{C}}\big(\mathbf{sd}\big(\Delta(P)\big)\big) \to \widetilde{\mathcal{C}}\big(\mathbf{sd}\big(\Gamma(P,A)\big)\big).$$
A homotopy equivalence $f : \Delta(P) \to \Gamma(P,A)$ can therefore be defined on the level of chains as $f_\sharp = \mathbf{unsd}_\sharp \circ h_\sharp \circ \mathbf{sd}_\sharp$ where
$$\widetilde{\mathcal{C}}\big(\Delta(P)\big) \xrightarrow{\mathbf{sd}_\sharp} \widetilde{\mathcal{C}}\big(\mathbf{sd}\big(\Delta(P)\big)\big) \xrightarrow{h_\sharp} \widetilde{\mathcal{C}}\big(\mathbf{sd}\big(\Gamma(P,A)\big)\big) \xrightarrow{\mathbf{unsd}_\sharp} \widetilde{\mathcal{C}}\big(\Gamma(P,A)\big).$$
Here, $\mathbf{unsd}_\sharp$ is the chain map induced from the simplicial map $\mathbf{unsd}$, which is defined by fixing a total ordering $\prec$ on the set of atoms A and mapping a subset of atoms to its minimum element under this ordering. Invoking the Algebraic Subdivision Theorem [12, Theorem 17.2], $\mathbf{unsd}_\sharp$ is a chain map that is a homotopy inverse to the subdivision map $\mathbf{sd} : \Gamma(P,A) \to \mathbf{sd}(\Gamma(P,A))$.
To wit, recall that $\mathrm{St}(v,K)$, the *star* of a vertex v in a simplicial complex K, is the union of the interiors of the simplices in K that have v as a vertex. Given a vertex A' of $\mathbf{sd}(\Gamma(P,A))$ then viewing its minimum element a' as a vertex of $\Gamma(P,A)$ it follows that
$$\mathrm{St}\big(A', \mathbf{sd}\big(\Gamma(P,A)\big)\big) \subset \mathrm{St}\big(a', \Gamma(P,A)\big).$$

By [12, Lemma 15.1], it follows that **unsd** is a simplicial approximation to the identity, and therefore the Algebraic Subdivision Theorem applies.

3. Under barycentric subdivision, a face $\sigma = \{y_0, \ldots, y_k\} \in \Delta(P)$ with $y_0 < \cdots < y_k \in P$ has image

$$\mathbf{sd}_\sharp(\sigma) = \sum_{\rho \in \Sigma_{k+1}} \varepsilon_\rho \big\{\{y_{\rho(k)}\}, \{y_{\rho(k-1)}, y_{\rho(k)}\}, \ldots, \{y_{\rho(0)}, \ldots, y_{\rho(k)}\}\big\} \tag{5.3}$$

where Σ_{k+1} is the group of permutations on the set $\{0, 1, \ldots, k\}$ and ε_ρ denotes the sign of the permutation ρ. Applying h to the chain

$$\big\{\{y_{\rho(k)}\}, \{y_{\rho(k-1)}, y_{\rho(k)}\}, \ldots, \{y_{\rho(0)}, \ldots, y_{\rho(k)}\}\big\}$$

yields the face

$$\{A_{y_{\rho(k)}}, A_{y_{\max(\rho(k),\rho(k-1))}}, \ldots, A_{y_{\max(\rho(k),\ldots,\rho(0))}}\}.$$

Unless ρ is the identity of Σ_{k+1}, this face has dimension less than or equal to $k-1$. It follows that under the chain map $h_\sharp$, the sum in (5.3) has image

$$h_\sharp\big(\mathbf{sd}_\sharp(\sigma)\big) = \{A_{y_k}, A_{y_{k-1}}, \ldots, A_{y_0}\}. \tag{5.4}$$

Considering the simplicial map $\mathbf{unsd} : \mathbf{sd}(\Gamma(P, A)) \to \Gamma(P, A)$, if $a_{y_t} = a_{y_{t-1}}$ for some $1 \leqslant t \leqslant k$, then the face

$$\{a_{y_k}, \ldots, a_{y_0}\} = \mathbf{unsd}\big(\{A_{y_k}, A_{y_{k-1}}, \ldots, A_{y_0}\}\big)$$

has dimension less than or equal to k. Thus,

$$\mathbf{unsd}_\sharp\big(\{A_{y_k}, A_{y_{k-1}}, \ldots, A_{y_0}\}\big) = 0$$

except when $a_{y_k} \prec a_{y_{k-1}} \prec \cdots \prec a_{y_0}$. Therefore,

$$f_\sharp(\sigma) = \mathbf{unsd}_\sharp\big(h_\sharp\big(\mathbf{sd}_\sharp(\sigma)\big)\big) = \begin{cases} \{a_{y_k}, \ldots, a_{y_0}\} & \text{when } a_{y_k} \prec \cdots \prec a_{y_0}, \\ 0 & \text{otherwise.} \end{cases}$$

With appropriate simplical background established, we have the following.

Proposition 5.5. *Suppose that the set of atoms A forms a crosscut in the poset P. For $\alpha \in P$ and $\lambda \lessdot \alpha$, the canonical isomorphism on homology between $\widetilde{H}_{i-2}(\Delta_\alpha, \Bbbk)$ and $\widetilde{H}_{i-2}(\Gamma((\hat{0}, \alpha), A_\alpha), \Bbbk)$ induced by the homotopy equivalence between $(\hat{0}, \alpha)$ and $\Gamma((\hat{0}, \alpha), A_\alpha)$ allows the map $\varphi_i^{\alpha,\lambda} : \mathcal{D}_{i,\alpha} \to \mathcal{D}_{i-1,\lambda}$ to be interpreted as the map between the vector spaces*

$$\phi_i^{\alpha,\lambda} : \widetilde{H}_{i-2}\big(\Gamma\big((\hat{0}, \alpha), A_\alpha\big), \Bbbk\big) \to \widetilde{H}_{i-3}\big(\Gamma\big((\hat{0}, \lambda), A_\lambda\big), \Bbbk\big)$$

described in (5.8).

The conclusion of this proposition leads immediately to the following.

Corollary 5.6. *If the set of atoms A forms a crosscut in the poset P, then the sequence $\mathcal{D}(P)$ has two equivalent definitions; one which uses the family of order complexes $\{\Delta_\alpha\colon \alpha \in P\}$ and one which uses the family of crosscut complexes $\{\Gamma((\hat{0}, \alpha), A_\alpha)\colon \alpha \in P\}$.*

Proof of Proposition 5.5. We first describe for each $\alpha \in P$, a simplicial decomposition of the atom crosscut complex $\Gamma((\hat{0},\alpha),A_\alpha)$ similar to that which was performed on the order complex Δ_α. More precisely, for $\lambda \in P$, let $\mathbf{G}_\lambda$ be the full simplex on $\{a \in A\colon a \leqslant \lambda\}$, the set of atoms which are comparable to λ. For $\alpha \in P$ this allows the decomposition of the atom crosscut complex as

$$\Gamma\big((\hat{0},\alpha),A_\alpha\big) = \bigcup_{\lambda \lessdot \alpha} \mathbf{G}_\lambda.$$

Similar to Definition 2.1, for a fixed $\lambda \lessdot \alpha$ we set

$$\Gamma_{\alpha,\lambda} = \mathbf{G}_\lambda \cap \Bigg(\bigcup_{\substack{\beta \lessdot \alpha \\ \lambda \neq \beta}} \mathbf{G}_\beta\Bigg).$$

Assuming that the atoms of P form a crosscut implies that the simplicial inclusion $\Gamma_{\alpha,\lambda} \subseteq \Gamma((\hat{0},\lambda),A)$ holds. This allows the crosscut complexes of open intervals of P to be analyzed using an appropriate Mayer–Vietoris sequence.

Indeed, given $\lambda \lessdot \alpha$, we consider the Mayer–Vietoris sequence in reduced homology for the triple

$$\Bigg(\mathbf{G}_\lambda, \bigcup_{\lambda \neq \beta \lessdot \alpha} \mathbf{G}_\beta, \Gamma\big((\hat{0},\alpha),A_\alpha\big)\Bigg).$$

Mirroring the definition of φ_*, we set

$$\iota : \widetilde{H}_{i-3}(\Gamma_{\alpha,\lambda},\Bbbk) \to \widetilde{H}_{i-3}\big(\Gamma\big((\hat{0},\lambda),A_\lambda\big),\Bbbk\big)$$

to be the map induced in homology by the inclusion map and

$$\delta^{\alpha,\lambda}_{i-2} : \widetilde{H}_{i-2}\big(\Gamma\big((\hat{0},\alpha),A_\alpha\big),\Bbbk\big) \to \widetilde{H}_{i-3}(\Gamma_{\alpha,\lambda},\Bbbk)$$

to be the connecting homomorphism from the Mayer–Vietoris sequence. This homomorphism takes the class $[c] \in \widetilde{H}_{i-2}(\Gamma((\hat{0},\alpha),A_\alpha),\Bbbk)$ to the class $[d_{i-2}(c')] \in \widetilde{H}_{i-3}(\Gamma_{\alpha,\lambda},\Bbbk)$ where $c' + c'' = c \in \mathcal{C}_{i-2}(\Gamma((\hat{0},\alpha),A_\alpha),\Bbbk)$, and c' and c'' are components of c that are supported by $\mathbf{G}_\lambda$ and by $\bigcup_{\lambda\neq\beta\lessdot\alpha} \mathbf{G}_\mu$ respectively. Again, d is the usual simplicial boundary map.

Define

$$\phi_1 : \Bigg(\bigoplus_{\alpha \in A} \widetilde{H}_{-1}\big(\Gamma\big((\hat{0},\alpha),A_\alpha\big),\Bbbk\big)\Bigg) \to \widetilde{H}_{-1}\big(\{\emptyset\},\Bbbk\big)$$

componentwise as the identity map $\phi_1|_{\widetilde{H}_{-1}(\Gamma((\hat{0},\alpha),A_\alpha),\Bbbk)} = \mathrm{id}_{\widetilde{H}_{-1}(\{\emptyset\},\Bbbk)}$. For $i \geqslant 2$ define

$$\phi_i : \Bigg(\bigoplus_{\alpha \in P} \widetilde{H}_{i-2}\big(\Gamma\big((\hat{0},\alpha),A_\alpha\big),\Bbbk\big)\Bigg) \to \Bigg(\bigoplus_{\alpha \in P} \widetilde{H}_{i-3}\big(\Gamma\big((\hat{0},\alpha),A_\alpha\big),\Bbbk\big)\Bigg)$$

componentwise by

$$\phi_i|_{\widetilde{H}_{i-2}(\Gamma((\hat{0},\alpha),A_\alpha),\Bbbk)} = \sum_{\lambda \lessdot \alpha} \phi_i^{\alpha,\lambda} \tag{5.7}$$

where

$$\phi_i^{\alpha,\lambda} : \widetilde{H}_{i-2}\big(\Gamma\big((\hat{0},\alpha), A_\alpha\big), \Bbbk\big) \to \widetilde{H}_{i-3}\big(\Gamma\big((\hat{0},\lambda), A_\lambda\big), \Bbbk\big) \tag{5.8}$$

is the composition $\phi_i^{\alpha,\lambda} = \iota \circ \delta_{i-2}^{\alpha,\lambda}$.

Claim. *For every $\beta \lessdot \alpha \in P$ and every $i \geqslant 1$ the map f induces canonically a commutative diagram*

$$\begin{tikzcd}
\widetilde{H}_i(\Delta_\alpha, \Bbbk) \arrow[r, "\mathfrak{f}_i"] \arrow[d, "\varphi_i"'] & \widetilde{H}_i(\Gamma((\hat{0},\alpha), A_\alpha), \Bbbk) \arrow[d, "\phi_i"] \\
\widetilde{H}_{i-1}(\Delta_\beta, \Bbbk) \arrow[r, "\mathfrak{f}_{i-1}"] & \widetilde{H}_{i-1}(\Gamma((\hat{0},\beta), A_\beta), \Bbbk)
\end{tikzcd}$$

where $\mathfrak{f}_*$ is an isomorphism in homology.

Proof of claim. Remark 5.2 implies that for each α and each $\sigma \in \Delta(\hat{0},\alpha]$ we have $f_\sharp(\sigma) \in \widetilde{\mathcal{C}}(\mathbf{G}_\alpha)$ and therefore the horizontal maps in our diagram are well-defined homology isomorphisms induced from a homotopy equivalence.

To prove commutativity, fix $\beta \lessdot \alpha$, suppose that $[\pi]$ is a homology class in $\widetilde{H}_i(\Delta_\alpha)$ and write

$$\begin{aligned}
\pi &= \sum_{\dim(\sigma)=i} c_\sigma \cdot \sigma \\
&= \sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot \sigma + \sum_{y_i^\sigma \nleqslant \beta} c_\sigma \cdot \sigma,
\end{aligned} \tag{5.9}$$

for a representative of this class where $\sigma = \{y_0^\sigma, \dots, y_i^\sigma\}$ is oriented by considering the chain $y_0^\sigma < \cdots < y_i^\sigma \in P$ and $c_\sigma \in \Bbbk$ is a scalar. Applying the isomorphism $\mathfrak{f}$, we have

$$\begin{aligned}
\mathfrak{f}\big([\pi]\big) &= \big[f_\sharp(\pi)\big] \\
&= \Bigg[\sum_{\dim(\sigma)=i} c_\sigma \cdot f_\sharp(\sigma)\Bigg] \\
&= \Bigg[\sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot f_\sharp(\sigma) + \sum_{y_i^\sigma \nleqslant \beta} c_\sigma \cdot f_\sharp(\sigma)\Bigg].
\end{aligned}$$

Since the terms appearing in the first summand are faces in $\mathbf{G}_\beta$ and the terms appearing in the second summand are faces in

$$\bigcup_{\beta \neq \gamma \lessdot \alpha} \mathbf{G}_\gamma,$$

then applying the map ϕ_i to $[f_\sharp(\pi)]$ yields

$$\phi_i\big(\mathfrak{f}\big([\pi]\big)\big) = \Bigg[\sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot d_i \circ f_\sharp(\sigma)\Bigg].$$

On the other hand, taking $[\pi] \in \widetilde{H}_i(\Delta_\alpha)$, and first applying the map φ_i, we obtain the homology class

$$\varphi_i([\pi]) = \left[\sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot d_i(\sigma) \right].$$

Utilizing the isomorphism $\mathfrak{f}$, we now have

$$\begin{aligned} \mathfrak{f}(\varphi_i([\pi])) &= \left[\sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot f_\sharp \circ d_i(\sigma) \right] \\ &= \left[\sum_{y_i^\sigma \leqslant \beta} c_\sigma \cdot d_i \circ f_\sharp(\sigma) \right] \\ &= \phi_i([f_\sharp(\pi)]) \end{aligned}$$

so that commutativity is proved and the proposition follows. □

6. Proof of main theorem

We begin this section with the following general facts regarding notation. Any sequence $\mathcal{S}$ of morphisms of free multigraded modules can be decomposed as

$$\mathcal{S} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{S}_\alpha$$

where each $\mathcal{S}_\alpha$ is a sequence of maps of vector spaces, and is called the *multigraded strand of $\mathcal{S}$ in degree α*. We denote by $(\mathcal{S}_\alpha)_i$ the ith component of the sequence $\mathcal{S}_\alpha$. Using the multigrading, it is clear that

$$(\mathfrak{m}\mathcal{S})_\alpha = \sum_{\beta \lessdot \alpha} x^{\alpha-\beta} \mathcal{S}_\beta \subset \mathcal{S}_\alpha.$$

Further, we will identify $x^{\alpha-\beta}\mathcal{S}_\beta$ with $\mathcal{S}_\beta$ so that we may write $\mathcal{S}_\beta \subset \mathcal{S}_\alpha$ for $\beta \lessdot \alpha$. This allows us to consider $\mathcal{S}_\gamma \subset \mathcal{S}_\alpha$ for all $\gamma < \alpha$. In addition, we may now write

$$(\mathfrak{m}\mathcal{S})_\alpha = \sum_{\beta \lessdot \alpha} \mathcal{S}_\beta = \sum_{\gamma < \alpha} \mathcal{S}_\gamma \subset \mathcal{S}_\alpha.$$

Proof of Theorem 3.3. It is clear that if $\mathcal{F}(\eta)$ is the minimal free resolution then N is lattice-linear. It remains to show that lattice linearity implies that $\mathcal{F}(\eta)$ is a resolution of R/N. We remark that since L_N is a lattice, its set of atoms (the minimal generators of N) forms a crosscut, so that Theorem 5.5 applies. We therefore utilize the homotopy equivalence between Δ_α and $\Gamma((\hat{0}, \alpha), A_\alpha)$ and Corollary 5.6 to proceed with the proof using this alternate formulation of $\mathcal{D}(L_N)$ for our computation of $\mathcal{F}(\eta)$.

Suppose that N is a lattice-linear monomial ideal with minimal free resolution $\mathcal{F}$ and let B_i be a basis for $\mathcal{F}_i$ as in Definition 3.2. With this choice of basis, let $\mathcal{F}_{i,\alpha}$ be the free submodule of $\mathcal{F}_i$ spanned by the set

$$B_{i,\alpha} = \{v \in B_i \colon \operatorname{mdeg}(v) = \alpha\}.$$

Hence,

$$\mathcal{F}_i = \bigoplus_{\alpha \in L_N} \mathcal{F}_{i,\alpha}$$

and in particular

$$(\mathcal{F}_\alpha)_i = \bigoplus_{\beta \leqslant \alpha} V_{i,\beta}$$

where

$$V_{i,\beta} = \Bbbk\langle v \colon \ v \in B_{i,\beta}\rangle$$

and $x^{\alpha-\beta} V_{i,\beta}$ is identified with $V_{i,\beta}$.

Making use of $\mathcal{T}$, the Taylor resolution of R/N, consider the exact sequence

$$0 \longrightarrow \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta \longrightarrow \mathcal{T}_\alpha \longrightarrow \mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta \longrightarrow 0. \tag{6.1}$$

The exactness of the Taylor resolution implies that $\mathcal{T}_\alpha$ is an exact complex of vector spaces for $\hat{0} \neq \alpha \in L_N$. Indeed, $\mathcal{T}_\alpha$ is acyclic with $H_0(\mathcal{T}_\alpha) \cong (R/N)_\alpha$ and $(R/N)_\alpha = 0$ for $x^\alpha \in N$.

Passing from (6.1) to the long exact sequence in homology, the connecting homomorphism yields an isomorphism

$$H_i\biggl(\mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\biggr) \xrightarrow[\cong]{\mu_i >} H_{i-1}\biggl(\sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\biggr),$$

which takes the class

$$[\bar{v}] \in H_i\biggl(\mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\biggr)$$

to the class

$$\bigl[\partial^{\mathcal{T}}(v)\bigr] \in H_{i-1}\biggl(\sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\biggr)$$

whenever $\bar{v}$ is a cycle in $\mathcal{T}_\alpha / \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta$ represented by an element $v \in \mathcal{T}_\alpha$.

Since $\mathcal{F}$ is the minimal free resolution of R/N, we make the identifications

$$\begin{aligned}
\bigl(\mathcal{F}_\alpha/(\mathfrak{m}\mathcal{F})_\alpha\bigr)_i &= \biggl(\mathcal{F}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{F}_\beta\biggr)_i \\
&= (\mathcal{F}_\alpha)_i \Big/ \sum_{\beta \lessdot \alpha} (\mathcal{F}_\beta)_i \\
&= \bigoplus_{\beta \leqslant \alpha} V_{i,\beta} \Big/ \bigoplus_{\beta < \alpha} V_{i,\beta} \\
&= V_{i,\alpha}.
\end{aligned}$$

Fixing an embedding of $\mathcal{F}$ as a direct summand of $\mathcal{T}$, we have $\mathcal{T} = \mathcal{F} \bigoplus \mathcal{E}$ for some split exact complex of multigraded free modules $\mathcal{E}$, and in particular, $\mathcal{T}_\alpha = \mathcal{F}_\alpha \bigoplus \mathcal{E}_\alpha$ for every α. Since the induced map of complexes

$$\mathcal{F}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{F}_\beta \longrightarrow \mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta$$

is a split inclusion and an isomorphism in homology, we consider $V_{i,\alpha}$ as a subspace of the cycles $\mathrm{Z}_i(\mathcal{T}_\alpha / \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta)$ and obtain the canonical identification

$$\mathrm{Z}_i\left(\mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\right) = V_{i,\alpha} \oplus \mathrm{B}_i\left(\mathcal{T}_\alpha \Big/ \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta\right).$$

Recalling the definitions of $\Gamma((\hat{0}, \alpha), A_\alpha)$ and $\Gamma_{\alpha,\gamma}$ we see that

$$\sum_{\beta \lessdot \alpha} \mathcal{T}_\beta = \widetilde{\mathcal{C}}\big(\Gamma\big((\hat{0}, \alpha), A_\alpha\big), \Bbbk\big)$$

and

$$\sum_{\substack{\beta \lessdot \alpha \\ \gamma \neq \beta}} \mathcal{T}_{\gamma \wedge \beta} = \widetilde{\mathcal{C}}(\Gamma_{\alpha,\gamma}, \Bbbk).$$

Using these identifications, we have the following diagram for each $\gamma \lessdot \alpha$ and each $i \geqslant 2$:

$$\begin{tikzcd}
V_{i,\alpha} \arrow[r, "\partial^{\mathcal{F}} \circ i"] \arrow[d, "\mathrm{incl}"'] & (\sum_{\beta \lessdot \alpha} \mathcal{F}_\beta)_{i-1} \arrow[r, "\mathrm{proj}_\gamma"] & V_{i-1,\gamma} \arrow[d, "\mathrm{incl}"] \\
\mathrm{Z}_i(\mathcal{T}_\alpha / \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta) \arrow[d, "\mathrm{proj}"'] & & \mathrm{Z}_{i-1}(\mathcal{T}_\gamma / \sum_{\nu \lessdot \gamma} \mathcal{T}_\nu) \arrow[d, "\mathrm{proj}"] \\
H_i(\mathcal{T}_\alpha / \sum_{\beta \lessdot \alpha} \mathcal{T}_\beta) \arrow[d, "\mu_i"', "\cong"] & & H_{i-1}(\mathcal{T}_\gamma / \sum_{\nu \lessdot \gamma} \mathcal{T}_\nu) \arrow[d, "\cong"', "\mu_{i-1}"] \\
H_{i-1}(\sum_{\beta \lessdot \alpha} \mathcal{T}_\beta) \arrow[r, "\delta_{i-1}^{\alpha,\gamma}"] \arrow[d, equal] & H_{i-2}(\sum_{\substack{\beta \lessdot \alpha \\ \gamma \neq \beta}} \mathcal{T}_{\gamma \wedge \beta}) \arrow[r, "\iota"] \arrow[d, equal] & H_{i-2}(\sum_{\nu \lessdot \gamma} \mathcal{T}_\nu) \arrow[d, equal] \\
\widetilde{H}_{i-2}(\Gamma((\hat{0}, \alpha), A_\alpha), \Bbbk) \arrow[r, "\delta_{i-1}^{\alpha,\gamma}"] & \widetilde{H}_{i-3}(\Gamma_{\alpha,\gamma}, \Bbbk) \arrow[r, "\iota"] & \widetilde{H}_{i-3}(\Gamma((\hat{0}, \gamma), A_\gamma), \Bbbk)
\end{tikzcd} \tag{6.2}$$

We claim this diagram is commutative.

Let $v \in V_{i,\alpha}$ so that by the assumption of lattice-linearity,

$$\partial^{\mathcal{F}}(v) = \sum_{\beta \lessdot \alpha} v_\beta$$

where each $v_\beta \in V_{i-1,\beta}$. Canonically, $\mathrm{incl}(v) = v$. Under projection, the cycle v is sent to its corresponding class in homology, $[v]$. As mentioned above, the connecting map μ_i is an isomorphism, and

$$\mu_i\big([v]\big) = \big[\partial^{\mathcal{T}}(v)\big] = \big[\partial^{\mathcal{F}}(v)\big] = \Bigg[\sum_{\beta \lessdot \alpha} v_\beta\Bigg].$$

Applying $\delta_{i-1}^{\alpha,\gamma}$, which is the connecting Mayer–Vietoris map,

$$\delta_{i-1}^{\alpha,\gamma}\Bigg(\Bigg[\sum_{\beta \lessdot \alpha} v_\beta\Bigg]\Bigg) = \big[\partial^{\mathcal{T}}(v_\gamma)\big].$$

Lastly, ι is the homological inclusion map and thus, $\iota([\partial^{\mathcal{T}}(v_\gamma)]) = [\partial^{\mathcal{T}}(v_\gamma)]$.

Again taking $v \in V_{i,\alpha}$, we appeal to the differential of $\mathcal{F}$ and obtain

$$\big(\partial^{\mathcal{F}} \circ i\big)(v) = \partial^{\mathcal{F}}(v) = \sum_{\beta \lessdot \alpha} v_\beta.$$

Projecting onto $V_{i-1,\gamma}$, we have

$$\mathrm{proj}_\gamma\Bigg(\sum_{\beta \lessdot \alpha} v_\beta\Bigg) = v_\gamma.$$

The inclusion map now gives $\mathrm{incl}(v_\gamma) = v_\gamma$, and passing this cycle to homology yields $[v_\gamma]$. Through the isomorphism, $\mu_{i-1}([v_\gamma]) = [\partial^{\mathcal{T}}(v_\gamma)]$, completing the proof of the commutativity of the diagram.

We complete the proof of the theorem by establishing the connection between lattice linearity and the poset construction.

For each $i \geqslant 0$ define the isomorphism of free R-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}(\eta)_i$ on a basis element $v \in B_{i,\alpha} \subset B_i$ by applying the left column of (6.2), thus

$$\psi_i(v) = 1 \otimes \big[\mu_i \circ \mathrm{proj} \circ \mathrm{incl}(v)\big] = 1 \otimes \big[\partial^{\mathcal{F}}(v)\big] \in R \otimes \mathcal{G}_{i,\alpha} \subset \mathcal{F}(\eta)_i.$$

By the commutativity of (6.2), we have a commutative diagram,

$$\begin{tikzcd} (\mathcal{F})_i \arrow[r, "\partial^{\mathcal{F}}"] \arrow[d, "\psi_i"'] & (\mathcal{F})_{i-1} \arrow[d, "\psi_{i-1}"] \\ (\mathcal{F}(\eta))_i \arrow[r, "d_i"'] & (\mathcal{F}(\eta))_{i-1} \end{tikzcd} \tag{6.3}$$

for every $i \geqslant 2$. Furthermore, (6.3) commutes trivially for $i = 1$. It follows that the sequences $\mathcal{F}$ and $\mathcal{F}(\eta)$ are isomorphic, hence $\mathcal{F}(\eta)$ is the minimal free resolution of R/N. $\square$

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Appendix A. A sufficient condition for $\mathcal{D}(P)$ to be a complex[1]

We present with proofs two additional properties of the sequence $\mathcal{D}(P)$ as constructed in Section 2. These results provided the main motivation for our initial study of the sequence $\mathcal{D}(P)$, however they seem to not have any interesting direct consequences of their own. Thus rather than have them in a separate paper, it is more appropriate to include them as an appendix here.

The first main result is the following:

Proposition A.1. *If P is a ranked poset then $\mathcal{D}(P)$ is a complex of vector spaces.*

Before we proceed with the proof, we need to establish some additional notation. For the rest of this Appendix A the letter P always denotes a ranked poset. We set $\Delta_\alpha^{(0)} = \mathbf{D}_\alpha = \mathbf{D}_\alpha(P)$, and for $j \geqslant 1$, we write

$$\Delta_\alpha^{(j)} = \bigcup_{\substack{\gamma<\alpha \\ \mathrm{rk}(\gamma)=\mathrm{rk}(\alpha)-j}} \Delta_\gamma^{(0)}.$$

Since $\Delta_\alpha^{(0)}$ is contractible, we have (and will tacitly use throughout this Appendix A) the canonical identification

$$\widetilde{H}_i(\Delta_\alpha^{(0)}, \Delta_\alpha^{(1)}, \Bbbk) \xrightarrow[=]{\theta_{i,\alpha}} \widetilde{H}_{i-1}(\Delta_\alpha^{(1)}, \Bbbk) \tag{A.2}$$

for every i given by the connecting map in the long exact sequence in relative homology. In particular, the canonical decomposition of reduced chain complexes

$$\widetilde{\mathcal{C}}\big(\Delta_\alpha^{(j)}, \Delta_\alpha^{(j+1)}, \Bbbk\big) = \bigoplus_{\substack{\gamma<\alpha \\ \mathrm{rk}(\gamma)=\mathrm{rk}(\alpha)-j}} \widetilde{\mathcal{C}}\big(\Delta_\gamma^{(0)}, \Delta_\gamma^{(1)}, \Bbbk\big)$$

for each j gives rise to a decomposition on the level of reduced homology

$$\widetilde{H}_i\big(\Delta_\alpha^{(j)}, \Delta_\alpha^{(j+1)}\big) = \bigoplus_{\substack{\gamma<\alpha \\ \mathrm{rk}(\gamma)=\mathrm{rk}(\alpha)-j}} \widetilde{H}_i\big(\Delta_\gamma^{(0)}, \Delta_\gamma^{(1)}\big) = \bigoplus_{\substack{\gamma<\alpha \\ \mathrm{rk}(\gamma)=\mathrm{rk}(\alpha)-j}} \widetilde{H}_{i-1}\big(\Delta_\gamma^{(1)}\big)$$

which we refer to as *reindexing*.

We break the proof of Proposition A.1 into three lemmas.

Lemma A.3. *We have $\varphi_1 \circ \varphi_2 = 0$.*

Proof. Suppose $[w] \in \widetilde{H}_0(\Delta_\alpha^{(1)}, \Bbbk)$, with representative cycle w; so we have

$$w = \sum_{\lambda\in(\hat{0},\alpha)} c_\lambda \cdot \{\lambda\} \quad \text{with} \sum_{\lambda\in(\hat{0},\alpha)} c_\lambda = 0.$$

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Choosing a partition of $(\hat{0},\alpha)$ into a disjoint union

$$(\hat{0},\alpha)=\bigsqcup_{\beta\lessdot\alpha}P_\beta \tag{A.4}$$

of subsets P_β such that for every $\lambda\in P_\beta$ one has $\lambda\leqslant\beta$, we get

$$w=\sum_{\beta\lessdot\alpha}w_\beta \quad \text{with } w_\beta=\sum_{\lambda\in P_\beta}c_\lambda\cdot\{\lambda\}.$$

Therefore,

$$\varphi_2^{\alpha,\beta}\big([w]\big)=\big[d(w_\beta)\big]=\Bigg[\sum_{\lambda\in P_\beta}c_\lambda\cdot\{\emptyset\}\Bigg]$$

where d is the usual boundary map. Applying φ_1, we obtain

$$\varphi_1\circ\varphi_2\big([w]\big)=\sum_{\beta\lessdot\alpha}\varphi_1\circ\varphi_2^{\alpha,\beta}\big([w]\big)=\sum_{\beta\lessdot\alpha}\Bigg[\sum_{\lambda\in P_\beta}c_\lambda\cdot\{\emptyset\}\Bigg]=0,$$

which is the desired conclusion. □

The remaining two lemmas involve the maps μ and D from the canonical long exact sequence in reduced homology

$$\cdots\to\widetilde{H}_i\big(\Delta_\alpha^{(1)},\Delta_\alpha^{(3)},\Bbbk\big)\xrightarrow{\mu}\widetilde{H}_i\big(\Delta_\alpha^{(1)},\Delta_\alpha^{(2)},\Bbbk\big)\xrightarrow{D}\widetilde{H}_{i-1}\big(\Delta_\alpha^{(2)},\Delta_\alpha^{(3)},\Bbbk\big)\to\cdots. \tag{A.5}$$

Lemma A.6. *For each $i\geqslant 1$ the diagram*

$$\begin{tikzcd}
\widetilde{H}_i(\Delta_\alpha^{(1)},\Delta_\alpha^{(2)},\Bbbk) \arrow[r, "D"] \arrow[d, equal, "\text{reindex}"'] & \widetilde{H}_{i-1}(\Delta_\alpha^{(2)},\Delta_\alpha^{(3)},\Bbbk) \arrow[d, equal, "\text{reindex}"] \\
\bigoplus_{\beta\lessdot\alpha}\widetilde{H}_{i-1}(\Delta_\beta^{(1)},\Bbbk) \arrow[r, "\varphi_{i+1}"] & \bigoplus_{\gamma\lessdot\lessdot\alpha}\widetilde{H}_{i-2}(\Delta_\gamma^{(1)},\Bbbk)
\end{tikzcd}$$

is commutative. The notation $\gamma\lessdot\lessdot\alpha$ means $\gamma<\alpha$ and $\mathrm{rk}(\gamma)=\mathrm{rk}(\alpha)-2$.

Proof. Suppose that $[\overline{w}]$ is a representative for the homology class generated by the image $\overline{w}$ in $\widetilde{C}_i(\Delta_\alpha^{(1)},\Delta_\alpha^{(2)})$ of the relative cycle

$$w=\sum_{a_i^\sigma\lessdot\alpha}c_\sigma\cdot\sigma=\sum_{\beta\lessdot\alpha}w_\beta$$

of $(\Delta_\alpha^{(1)},\Delta_\alpha^{(2)})$, where $c_\sigma\in\Bbbk$, each face $\sigma=\{a_0^\sigma,\dots,a_i^\sigma\}$ is oriented by $a_0^\sigma<\cdots<a_i^\sigma$ and $w_\beta=\sum_{a_i^\sigma=\beta}c_\sigma\cdot\sigma$. Since w is a relative cycle, we must have

$$\sum_{a_i^\sigma=\beta}c_\sigma\cdot d(\hat{\sigma})=0$$

where $\hat{\sigma} = \{a_0^\sigma, \ldots, a_{i-1}^\sigma\}$ and d is the usual boundary map. Therefore, each w_β is a relative cycle for $(\Delta_\beta^{(0)}, \Delta_\beta^{(1)})$ and reindexing $[\overline{w}] = \sum_{\beta \lessdot \alpha}[\overline{w}_\beta]$ yields

$$\sum_{\beta \lessdot \alpha}\bigl[d(w_\beta)\bigr] = \sum_{\beta \lessdot \alpha}[v_\beta], \quad \text{where } v_\beta = d(w_\beta) = (-1)^i \sum_{a_i^\sigma = \beta} c_\sigma \cdot \hat{\sigma}$$

is a cycle in $\Delta_\beta^{(0)}$. Next, choose for each $\beta \lessdot \alpha$ a partition

$$(\hat{0}, \beta) = \bigsqcup_{\gamma \lessdot \beta} P_{\beta,\gamma}$$

of $(\hat{0}, \beta)$ such that $\lambda \leqslant \gamma$ for each $\lambda \in P_{\beta,\gamma}$ and write

$$v_\beta = \sum_{\gamma \lessdot \beta} w_{\beta,\gamma} \quad \text{where } w_{\beta,\gamma} = (-1)^i \sum_{\substack{a_i^\sigma = \beta \\ a_{i-1}^\sigma \in P_{\beta,\gamma}}} c_\sigma \cdot \hat{\sigma}.$$

Therefore for each $\delta < \alpha$ with $\mathrm{rk}(\delta) = \mathrm{rk}(\alpha) - 2$ the component of $\varphi_{i+1}(\sum_{\beta \lessdot \alpha}[v_\beta])$ in $\widetilde{H}_{i-2}(\Delta_\delta^{(1)}, \Bbbk)$ is given by

$$\sum_{\beta:\ \delta < \beta \lessdot \alpha} \varphi_{i+1}^{\beta,\delta}\bigl([v_\beta]\bigr) = \sum_{\beta:\ \delta \lessdot \beta \lessdot \alpha} \varphi_{i+1}^{\beta,\delta}\biggl(\biggl[\sum_{\gamma \lessdot \beta} w_{\beta,\gamma}\biggr]\biggr) = \sum_{\beta:\ \delta \lessdot \beta \lessdot \alpha}\bigl[d(w_{\beta,\delta})\bigr].$$

On the other hand, since v_β is a cycle, each $w_{\beta,\gamma}$ is a relative cycle for $(\Delta_\alpha^{(2)}, \Delta_\alpha^{(3)})$. As D is the connecting map in (A.6), we have

$$D\bigl([\overline{w}]\bigr) = \bigl[\overline{d(w)}\bigr] = \biggl[\sum_{\beta \lessdot \alpha} \bar{v}_\beta\biggr] = \sum_{\substack{\gamma < \alpha \\ \mathrm{rk}(\gamma) = \mathrm{rk}(\alpha) - 2}} \biggl[\sum_{\beta:\ \gamma \lessdot \beta \lessdot \alpha} \overline{w}_{\beta,\gamma}\biggr]$$

and reindexing yields

$$\sum_{\substack{\gamma < \alpha \\ \mathrm{rk}(\gamma) = \mathrm{rk}(\alpha) - 2}} \biggl[d\biggl(\sum_{\beta: \gamma \lessdot \beta \lessdot \alpha} w_{\beta,\gamma}\biggr)\biggr].$$

Therefore, its component in $\widetilde{H}_{i-2}(\Delta_\delta^{(1)}, \Bbbk)$ is equal to

$$\biggl[d\biggl(\sum_{\beta:\ \delta \lessdot \beta \lessdot \alpha} w_{\beta,\delta}\biggr)\biggr] = \sum_{\beta:\ \delta \lessdot \beta \lessdot \alpha}\bigl[d(w_{\beta,\delta})\bigr],$$

which proves the desired commutativity of our diagram. $\square$

Lemma A.7. *For each $i \geqslant 1$ the diagram*

$$\begin{tikzcd} \widetilde{H}_i(\Delta_\alpha^{(1)}, \Delta_\alpha^{(3)}, \Bbbk) \arrow[r, "\mu"] & \widetilde{H}_i(\Delta_\alpha^{(1)}, \Delta_\alpha^{(2)}, \Bbbk) \arrow[d, equal, "\text{reindex}"] \\ \widetilde{H}_i(\Delta_\alpha^{(1)}, \Bbbk) \arrow[u, "\pi"] \arrow[r, "\varphi_{i+2}"] & \bigoplus_{\beta \lessdot \alpha} \widetilde{H}_{i-1}(\Delta_\beta^{(1)}, \Bbbk) \end{tikzcd} \tag{A.8}$$

is commutative.

Proof. Suppose that $[v]$ is a homology class in $\widetilde{H}_i(\Delta_\alpha^{(1)}, \Bbbk)$ represented by the cycle

$$v = \sum_{a_i^\sigma < \alpha} c_\sigma \cdot \sigma.$$

Under projection

$$\pi\big([v]\big) = \big[\pi(v)\big] = \left[\sum_{a_i^\sigma < \alpha} c_\sigma \cdot \sigma'\right],$$

where $\sigma' = \pi(\sigma)$ is the image of σ under the standard projection $\widetilde{\mathcal{C}}(\Delta_\alpha^{(1)}) \to \widetilde{\mathcal{C}}(\Delta_\alpha^{(1)}, \Delta_\alpha^{(3)})$. Applying μ to $\pi([v])$ we have

$$\mu\big(\big[\pi(v)\big]\big) = \left[\sum_{a_i^\sigma < \alpha} c_\sigma \cdot \bar{\sigma}\right] = \sum_{\beta \lessdot \alpha} [\bar{w}_\beta],$$

where $\bar{}$ denotes images under the projection $\widetilde{\mathcal{C}}(\Delta_\alpha^{(1)}) \to \widetilde{\mathcal{C}}(\Delta_\alpha^{(1)}, \Delta_\alpha^{(2)})$ and

$$w_\beta = \sum_{a_i^\sigma \in P_\beta} c_\sigma \cdot \sigma$$

with P_β is as in (A.4). Note that since v is a cycle, each w_β is forced to be a relative cycle. Now reindexing yields $\sum_{\beta \lessdot \alpha} [d(w_\beta)]$ where clearly $[d(w_\beta)]$ is the component in $\widetilde{H}_{i-1}(\Delta_\beta^{(1)}, \Bbbk)$. On the other hand

$$\varphi_{i+2}\big([v]\big) = \sum_{\beta \lessdot \alpha} \varphi_{i+2}^{\alpha,\beta}\big([v]\big) = \sum_{\beta \lessdot \alpha} \big[d(w_\beta)\big]$$

and commutativity follows. □

Proof of Proposition A.1. We show $\varphi_{i-1} \circ \varphi_i = 0$. Lemma A.3 establishes this for $i = 2$, and the commutative diagrams of Lemmas A.6 and A.7 may be combined so that for $i \geqslant 3$ we have $\varphi_{i-1} \circ \varphi_i = (\text{reindex}) \circ D \circ \mu \circ \pi$. Since D and μ are consecutive maps in an exact sequence, $D \circ \mu = 0$. Therefore $\varphi_{i-1} \circ \varphi_i = 0$ for each i which completes the proof that $\mathcal{D}(P)$ is a complex. □

The general case can always be reduced to the case of a ranked poset because of our second main result:

Proposition A.9. *Let Q be any finite poset with minimum $\hat{0}$. Then there is a rank preserving canonical embedding $Q \subset P$, where P is a ranked poset with minimum $\hat{0}$ such that for every $i \geqslant 0$ and $\alpha \in P$ one has*

$$\mathcal{D}_{i,\alpha}(P) = \begin{cases} \mathcal{D}_{i,\alpha}(Q) & \text{if } \alpha \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathcal{D}_i(Q) = \mathcal{D}_i(P)$ for every $i \geqslant 0$.

Proof. The construction of the ranked poset P below is well known and is the standard way of obtaining a ranked poset from Q while preserving rank. More precisely, for all $\beta \lessdot \alpha$ in Q with $\mathrm{rk}_P(\alpha) - \mathrm{rk}_P(\beta) \geqslant 2$, let

$$P_{\alpha,\beta} = \left\{\gamma_i^{\alpha,\beta} \colon 1 \leqslant i \leqslant \mathrm{rk}_P(\alpha) - \mathrm{rk}_P(\beta) - 1\right\}$$

be a set of symbols, and let P be the disjoint union

$$P = Q \bigsqcup \left(\bigsqcup_{\beta \lessdot \alpha} P_{\alpha,\beta} \right).$$

The elements of P are ordered by describing their covering relations: the Hasse diagram of P is obtained from the Hasse diagram of Q by breaking up each edge $\beta \lessdot \alpha$ into n edges $\beta \lessdot \gamma_1^{\alpha,\beta} \lessdot \cdots \lessdot \gamma_{n-1}^{\alpha,\beta} \lessdot \alpha$ where $n = \mathrm{rk}_Q(\alpha) - \mathrm{rk}_Q(\beta)$. It is clear that in this way P is canonically determined by Q and $\mathrm{rk}_Q(\alpha) = \mathrm{rk}_P(\alpha)$ for each $\alpha \in Q$.

Since P can be obtained iteratively by breaking up one edge in two at a time, to prove the second claim of the proposition it is enough to assume that only one additional poset element γ is added to Q and $\beta \lessdot \gamma \lessdot \alpha$ in P for some $\beta \lessdot \alpha$ in Q.

We have

$$\Delta_\gamma(P) = \bigcup_{\rho \lessdot \gamma} \mathbf{D}_\rho(P) = \mathbf{D}_\beta(P) = \mathbf{D}_\beta(Q)$$

since γ uniquely covers β by construction. Thus, $\Delta_\gamma(P)$ is a cone with apex β and hence contractible. Therefore, $D_{i,\gamma}(P) = 0$ for each i. Next, let $\delta \in Q$. If $\delta \ngeqslant \alpha$ then $\Delta_\delta(P) = \Delta_\delta(Q)$ and so $D_{i,\delta}(P) = D_{i,\delta}(Q)$ for each i. Thus, it remains to consider the case $\delta \geqslant \alpha$. Let

$$\Omega = \mathrm{St}\big(\Delta_\delta(P), \{\alpha\}\big) = \mathbf{D}_\beta(P) * \Delta\big([\alpha,\delta)\big) = \mathbf{D}_\beta(Q) * \Delta\big([\alpha,\delta)\big).$$

Thus $\Delta_\delta(P) = \Delta_\delta(Q) \cup \Omega$ and $\Omega \cap \Delta_\delta(Q) = \mathbf{D}_\beta(Q) * \Delta([\alpha,\delta))$ is a cone with apex β, hence contractible. Since both Ω and $\Omega \cap \Delta_\delta(Q)$ are contractible we get the desired conclusion by considering the Mayer–Vietoris sequence in reduced homology on the triple $(\Delta_\delta(Q), \Omega, \Delta_\delta(P))$. $\square$

Remark A.10. Replacing the poset Q with the ranked poset P may result in different sequences $\mathcal{D}(Q)$ and $\mathcal{D}(P)$. Indeed, although the vector space components are identical by Proposition A.9, the maps in the sequences are different in general. In particular, the maps present in $\mathcal{D}(P)$ will in general have more trivial components.

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