Some results on generators of ideals

Shiv Datt Kumar\textsuperscript{a}, Satya Mandal\textsuperscript{b,∗}

\textsuperscript{a}Department of Mathematics, Government Post Graduate College, Ramnagar, Nainital, India
\textsuperscript{b}Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

Received 31 October 2000; received in revised form 28 March 2001
Communicated by A.V. Geramita

Abstract

We prove some results on projective generation of ideals. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 13C10

1. Introduction

In [1], Mandal proved the following theorem:

**Theorem 1.1.** Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring $A$. Let $I$ be an ideal in $R$ that contains a monic polynomial and $\mu(I/I^2) = r \geq \dim(R/I) + 2$. Then $I$ is also generated by $r$ elements.

It is natural to ask whether we can extend generators of $I(0) = \{ f(0) : f \in I \}$ to generators of $I$. One of our results (Theorem 3.2) deals with this question in the semilocal case.

Besides, we also prove the Laurent polynomial analogue of Nori’s Homotopy Conjecture as follows:

**Theorem 1.2.** Let $R = A[X,X^{-1}]$ be a Laurent polynomial ring over a noetherian commutative ring $A$. Let $I$ be an ideal in $R$ that contains a doubly monic polynomial and
Let $R = \mathbb{A}[X, X^{-1}]$ be a Laurent polynomial ring over a noetherian commutative ring $\mathbb{A}$. Let $I$ be an ideal in $R$ that contains a doubly monic polynomial and $P$ be a projective $\mathbb{A}$-module of rank $r \geq \dim (R/I) + 2$. Write $I(1) = \{ f(1) : f \in I \}$ let $s : P \rightarrow I(1)$ and $\phi : P[X, X^{-1}] \rightarrow I/I^2$ be two surjective maps such that

$$\phi \equiv s \text{ (modulo } (X - 1, I(1)^2))\).$$

Then there is a surjective map $\psi : P[X, X^{-1}] \rightarrow I$, such that $\psi$ lifts $\phi$ and that $\psi(1) = s$.

This theorem is the Laurent polynomial analogue of the corresponding theorem [2] that was proved for ideals in polynomial rings.

In this paper, all rings we consider are noetherian and commutative. For a ring $\mathbb{A}$ and an $\mathbb{A}$-module $M$, the minimal number of generators of $M$ will be denoted by $\mu(M)$. Also recall, a Laurent polynomial $f$ in $\mathbb{A}[X, X^{-1}]$ is said to be a doubly monic if the coefficients of the highest and the lowest degree terms are units.

2. Preliminaries

**Lemma 2.1.** Let $R = \mathbb{A}[X, X^{-1}]$ be a Laurent polynomial ring over a noetherian commutative ring $\mathbb{A}$. Let $I$ be an ideal in $R$ and $P$ be a projective $\mathbb{A}$-module. Write $I(1) = \{ g(1) : g \in I \}$. Let $s : P[X, X^{-1}] \rightarrow I(1)$ and $\phi : P[X, X^{-1}] \rightarrow I/I^2$ be two surjective maps such that

$$\phi \equiv s \text{ (modulo } (X - 1, I(1)^2))\).$$

Then, there is a map $\phi_1 : P[X, X^{-1}] \rightarrow I$ into $I$ that is a lift of $\phi$ and that $\phi_1(1) = s$.

**Proof.** Let $F : P[X, X^{-1}] \rightarrow I$ be any lift of $\phi$. Since $\text{Hom}(P[X, X^{-1}], \mathbb{A}[X, X^{-1}]) = \text{Hom}(P, \mathbb{A}) \otimes \mathbb{A}[X, X^{-1}]$ we have $F = (F_0 + F_1 X + \cdots + F_m)X^k$ where, $F_i \in \text{Hom}(P, \mathbb{A})$ and $k$ is an integer. Since, $F(1) \equiv s \text{ (modulo } I(1)^2\)), we have $F(1) - s \in I(1)^2 \text{Hom}(P, \mathbb{A})$. Therefore, $F(1) - s = f_1(1)g_1(1)\lambda_1 + \cdots + f_r(1)g_r(1)\lambda_r$ for some $f_1, \ldots, f_r, g_1, \ldots, g_r$ in $I$ and $\lambda_1, \ldots, \lambda_r$ in $\text{Hom}(P, \mathbb{A})$. Write

$$\phi_1 = F - (f_1g_1\lambda_1 + \cdots + f_rg_r\lambda_r).$$

Then $\phi_1(1) = F(1) - (F(1) - s) = s$ and $\lambda$ is a lift of $\phi$. \qed

3. Main theorems

**Theorem 3.1.** Let $R = \mathbb{A}[X, X^{-1}]$ be a Laurent polynomial ring over a noetherian commutative ring $\mathbb{A}$. Let $I$ be an ideal in $R$ that contains a doubly monic polynomial and...
$P$ be projective $A$-module of rank $r \geq \dim(R/I) + 2$. Write $I(1) = \{f(1) : f \in I\}$ let

$$s : P \to I(1) \quad \text{and} \quad \phi : P[X,X^{-1}] \to I/I^2$$

be two surjective maps such that

$$\phi \equiv s(\text{modulo}(X - 1,I(1)^2)).$$

Then there is a surjective map $\psi : P[X,X^{-1}] \to I$ such that $\psi$ lifts $\phi$ and that $\psi(1) = s$.

**Proof.** Write $J = I \cap A[X]$. Then we have $I(1) = \{f(1) : f \in J\} = J(1)$. Also since $I$ contains a doubly monic polynomial, $J$ contains a special monic polynomial (i.e. a monic polynomial with constant term 1).

We can find a lift $\phi_1 : P[X,X^{-1}] \to I$ of $\phi$ such that $\phi_1(1) = s$. Since $\text{Hom}(P[X,X^{-1}], \ A[X,X^{-1}]) = \text{Hom}(P,A) \otimes \ A[X,X^{-1}]$, we have $\phi_1 = \phi_2/X^k$ where $\phi_2 \in \text{Hom}(P[X],A[X])$ and $k$ is a positive integer. It follows that $\phi_2$ maps into $J$ and $\phi_2(1) = s$.

Let $F = o\phi_2$ where $o : J \to J/J^2$ is then natural map. Then, $F \equiv s(\text{modulo}(X - 1,I(1)^2))$. Let $g$ be a special monic in $J$. Since $F_X = \phi$ and since $(J/J^2)_g = 0$ we have $F$ is a surjective map. So, by the theorem of Mandal [2] there is a surjective $A[X]$-linear map $\psi : P[X] \to J$ that lifts $F$ and $F(1) = s$. Now if $\chi = \psi_k$ then $\chi : P[X,X^{-1}] \to I$ is a surjective $A[X,X^{-1}]$-linear map that lifts $\phi$ and $\chi(1) = s$. This completes the proof of the theorem. □

**Lemma 3.1.** Let $A$ be semilocal ring and $I$ be an ideal in $A$. Then $\mu(I) = \mu(I/I^2)$.

**Proof.** Let $I = (f_1, \ldots, f_r) + I^2$. Let $m_1, \ldots, m_k$ be the maximal ideals that do not contain $I$. We can assume that $f_1$ is in $m_1, \ldots, m_l$ and not in $m_{l+1}, \ldots, m_k$. Choose

$$g \in I^2 \cap (m_1 \cap \ldots \cap m_k) \setminus (m_1 \cup \ldots \cup m_l).$$

Write $F_1 = f_1 + g$. Then $F_1$ does not belong to $m_1, \ldots, m_k$. It follows $I = (F_1, f_2, \ldots, f_r)$. □

**Theorem 3.2.** Let $A$ be a semilocal commutative noetherian ring and $I$ be an ideal in $R = A[X]$ that contains a monic polynomial and $\mu(I/I^2) = r$. Let $I_0 = \{f(0) : f \in I\} \subseteq \text{rad}(A)$ be a complete intersection ideal of height $r$ or $I_0 = A$. Let $I_0 = (a_1, \ldots, a_r)$ be generated by $a_1, \ldots, a_r$. Then $I = (f_1, f_2, \ldots, f_r)$ where $f_1(0) = a_1, f_2(0) = a_2, \ldots, f_r(0) = a_r$.

**Proof.** Let $I = (f_1, f_2, \ldots, f_n) + I^2$. We can assume that $f_1$ is a monic polynomial. Now $I' = I/(f_1)$ is an ideal in $R' = A[X]/(f_1)$. Since $R'$ is a semilocal ring and $\mu(I'/I'^2) = r - 1$, it follows from the lemma that $I'$ is generated by $r - 1$ elements. So, after modifying the generators of $I$ we have $I = (f_1, \ldots, f_r)$ for some $f_2, \ldots, f_r$. □
Let $f_1(0) = b_1, \ldots, f_r(0) = b_r$. So, we have $I_0 = (b_1, b_2, \ldots, b_r)$. So, we have

$$
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_r
\end{pmatrix} =
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_r
\end{pmatrix}.
$$

Let $A = (a_{ij})$ denote the above matrix. First, assume that $I_0 \subseteq \text{rad}(A)$. Let “bar” denote “modulo $I_0$”. Since $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r$ and also $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_r$ are both free generators of $I_0/I_0^2$, it follows that $\det(A)$ is unit and hence $\det(A)$ is also a unit. So, $A$ is an invertible matrix in this case. When $I_0 = A$, since $A$ is semilocal, we can choose $A$ as an invertible matrix.

Now let

$$
\begin{pmatrix}
F_1 \\
F_2 \\
\vdots \\
F_r
\end{pmatrix} =
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr}
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_r
\end{pmatrix}.
$$

So, we have $I = (F_1, F_2, \ldots, F_r)$ and $F_1(0) = a_1, F_2(0) = a_2, \ldots, F_r(0) = a_r$. □

References