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A Formula of S. Ramanujan

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IN MEMORY OF S. RAMANUJAN

In this paper, we discuss various equivalent formulations for the sum of an infinite series considered by S. Ramanujan. In the process, we also evaluate, in closed form, various classes of related infinite series.

1. INTRODUCTION

In Chapter 9, Volume 2 of his notebooks, Ramanujan (cf. [19, formula (11.3)] stated that

$$G(1) := \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)$$
$$= \frac{\pi}{4}\sum_{r=0}^{\infty}\frac{(-1)^r}{(4r+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}. \tag{1.1}$$

Recently, Berndt and Joshi (cf. [3, p. 31]), while editing this chapter, observed that this formula is incorrect; in fact, on taking the first three terms of the series defining $G(1)$, they found that $G(1) > 0.1529320988...$ whereas the extreme right of (1.1) is less than 0.1442780636.

In an attempt to obtain a corrected version of this formula, we are led to consider the following 12 classes of infinite series:

$$H_1(s) = \Sigma'\, r^{-s}k^{-1},$$
$$H_2(s) = \Sigma'\, r^{-s}(k+r)^{-1},$$
$$H_3(s) = \Sigma''\, r^{-s}k^{-1},$$

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$$H_4(s)=\Sigma' r^{-s}(-1)^{k-1}k^{-1},$$
$$H_5(s)=\Sigma' r^{-s}(-1)^{k-1}(r+k)^{-1},$$
$$H_6(s)=\Sigma'' r^{-s}(-1)^{k-1}k^{-1},$$
$$H_7(s)=\Sigma' (-1)^{r-1}r^{-s}k^{-1},$$
$$H_8(s)=\Sigma' (-1)^{r-1}r^{-s}(r+k)^{-1},$$
$$H_9(s)=\Sigma'' (-1)^{r-1}r^{-s}k^{-1},$$
$$H_{10}(s)=\Sigma' (-1)^{r-1}r^{-s}(-1)^{k-1}k^{-1},$$
$$H_{11}(s)=\Sigma' (-1)^{r-1}r^{-s}(-1)^{k-1}(r+k)^{-1},$$
$$H_{12}(s)=\Sigma'' (-1)^{r-1}r^{-s}(-1)^{k-1}k^{\ \ 1}$$

where Σ' ad Σ'' respectively denote

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r} \quad \text{and} \quad \sum_{r=1}^{\infty}\sum_{k=1}^{2r-1}$$

and s denotes a complex number with $\operatorname{Re} s>1$ in $H_1(s)$ through $H_6(s)$ and $\operatorname{Re} s>0$ in $H_7(s)$ through $H_{12}(s)$.

It may be noted that it is possible to shorten the above notation as follows: For $i, j \in \{0, 1\}$, let

$$H_1^{(i,j)}(s)=\Sigma' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}k^{-1},$$
$$H_2^{(i,j)}(s)=\Sigma' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}(r+k)^{-1},$$
$$H_3^{(i,j)}(s)=\Sigma'' (-1)^{i(r-1)}r^{-s}(-1)^{j(k-1)}k^{-1}.$$

Then we note that for $i, j \in \{0, 1\}$ and $k \in \{1, 2, 3\}$

$$H_{6i+3j+k}(s)=H_k^{(i,j)}(s).$$

Some of these infinite series have been studied, for positive integer values of s, by several authors and we refer to Euler [7, 8], Nielsen [18], Ramanujan [19], Rutledge and Douglass [20], Klamkin [13, 14], Williams [27], Briggs, Chowla, Kempner, and Mientka [4], Lehner and Newman [15], Gupta [10], Jordan [11], Kanemitsu [12], the author and Sivaramasarma [21, 22], Sivaramasarma [25], the author and Subbarao [23, 24], Bruckman [5], Georghiou and Philippou [9], Dixon and O'Cinneide [6], Matsuoka [17] and Apostol and Vu [2].

In Section 2 of this paper, we prove several relationships between $H_1(s)$ through $H_{12}(s)$. In Section 3, we discuss the evaluation of each of these infinite series for certain positive integer values of s. In particular, we give a new proof of the following identity due to Nielsen [18] which apparently goes back to Euler (cf. [18, footnotes on p. 47]): For integral $s \geqslant 2$,

$$2H_1(s) = (s+2)\,\zeta(s+1) - \sum_{i=1}^{s-2} \zeta(s-i)\,\zeta(i+1)$$

where $\zeta(s)$ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ in $\operatorname{Re} s > 1$ and its analytic continuation in $\operatorname{Re} s > 0$. This result and certain of its special cases were rediscovered by several authors (See Remark 3.1). We also evaluate, in closed form, $H_2(2b)$, $H_3(2b)$, $H_4(s)$, $H_6(2b)$, $H_{10}(2b)$, and $H_{11}(2b)$ where $s \geqslant 2$ and $b \geqslant 1$ are integers. We believe that all these results are new.

Our results of Section 3 give, in particular, the values of $H_1(2)$, $H_2(2)$, $H_3(2)$, $H_4(2)$, $H_6(2)$, $H_7(2)$, $H_{10}(2)$, and $H_{11}(2)$ explicitly in terms of $\zeta(2)$ and $\zeta(3)$. In Section 4, we complete this list by evaluating $H_5(2)$, $H_8(2)$, $H_9(2)$, and $H_{12}(2)$. These evaluations also involve Catalan's constant (see (4.13)).

In Section 5 we prove that

$$\begin{aligned} G(1) &= \tfrac{5}{64}\,\zeta(4) + \tfrac{1}{8}\,H_2(3) \\ &= -\tfrac{1}{64}\,\zeta(4) + \tfrac{1}{8}\,H_3(3) \\ &= \tfrac{1}{64}\,\zeta(4) + \tfrac{1}{8}\,H_6(3) \\ &= \tfrac{35}{64}\,\zeta(4) - \tfrac{1}{2}\,H_7(3) \\ &= \tfrac{5}{128}\,\zeta(4) + \tfrac{7}{32}\,\zeta(3)\log 2 - \tfrac{1}{8}\,H_{11}(3) \\ &= \tfrac{3}{16}\,\zeta(4) - \tfrac{1}{4}\,A_4 \end{aligned}$$

and

$$G(1) \simeq 0.16227.$$

We also note that

$$\begin{aligned} G(1) &= \tfrac{29}{64}\,\zeta(4) - \tfrac{7}{8}\,\zeta(3)\log 2 + \tfrac{1}{2}\,H_{10}(3) \\ &= -\tfrac{53}{64}\,\zeta(4) + \tfrac{7}{8}\,\zeta(3)\log 2 - \frac{\pi^2}{24}(\log 2)^2 + \frac{(\log 2)^4}{12} + Li_4(1/2). \end{aligned}$$

In the above A_4 is the constant defined by (cf. [20, p. 30])

$$A_4 = \sum_{r=2}^{\infty} \frac{(-1)^r}{r^2} \sum_{k=1}^{r-1} \frac{1}{k^2} \tag{1.2}$$

and $Li_4(z)$ is the polylogarithm of order 4 defined by [16]

$$Li_4(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^4}, \qquad |z| \leqslant 1.$$

Unfortunately, we could not evaluate, in closed form, any one of $H_2(3)$, $H_3(3)$, $H_6(3)$, $H_7(3)$, $H_{10}(3)$, $H_{11}(3)$, A_4, and $Li_4(\frac{1}{2})$.

2. Relationships between $H_1(s)$ through $H_{12}(s)$

We write $\sigma(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$ for $\operatorname{Re} s > 0$ so that $\sigma(s) = (1-2^{1-s})\zeta(s)$ for all s in $\operatorname{Re} s > 0$, $s \neq 1$ and $\sigma(1) = \log 2$.

The following Lemma is basic to our work.

Lemma 2.1. *For* $\operatorname{Re} s > 1$, *we have*

$$H_1(s) + H_2(s) = H_3(s) + \tfrac{1}{2}\zeta(s+1), \tag{2.1}$$

$$H_7(s) + H_8(s) = H_9(s) + \tfrac{1}{2}\sigma(s+1), \tag{2.2}$$

$$H_6(s) + H_{11}(s) = H_4(s) + \tfrac{1}{2}\zeta(s+1), \tag{2.3}$$

$$H_5(s) + H_{12}(s) = H_{10}(s) + \tfrac{1}{2}\sigma(s+1), \tag{2.4}$$

$$H_1(s) - H_7(s) = 2^{1-s} H_3(s) + 2^{-s}\zeta(s+1), \tag{2.5}$$

$$H_6(s) = H_2(s) + \tfrac{1}{2}\zeta(s+1), \tag{2.6}$$

$$H_{12}(s) = H_8(s) + \tfrac{1}{2}\sigma(s+1), \tag{2.7}$$

$$H_5(s) + H_8(s) = H_{10}(s), \tag{2.8}$$

$$H_2(s) + H_{11}(s) = H_4(s). \tag{2.9}$$

Proof. The results (2.1) through (2.5) follow directly from the definitions and the results (2.6) through (2.9) follow from the easily proved identity

$$\sum_{k=1}^{2r-1} \frac{(-1)^{k-1}}{k} = \sum_{k=1}^{r} \frac{1}{r+k} + \frac{1}{2r}.$$

3. Evaluation of $H_1(s)$ through $H_{12}(s)$ for Certain Values of s

Throughout this section, $a \geqslant 2$ and $b \geqslant 1$ denote integers and $\bar{\zeta}(s) = (2^s - 1)\zeta(s)$ for $\operatorname{Re} s > 1$.

Theorem 3.1. $2H_1(a) = (a+2)\zeta(a+1) - \sum_{i=1}^{a-2} \zeta(a-i)\zeta(i+1)$.

Remark 3.1. Theorem 3.1 has a long history and evidently Euler (cf. [7] and [8], p. 228) was the first to discover it in case $a=2$. Nielsen (cf. [18, pp. 37–51]) developed a systematic method of proving Theorem 3.1 and several related results. In case $a=2$, Theorem 3.1 was rediscovered by Ramanujan (cf. [19, Chap. 9, entry 9]; [3, p. 25]), Briggs *et al.* [4], Klamkin [13], and Bruckman [5]. Theorem 3.1 in case $a=3$ was rediscovered in a slightly different form, by Rutledge and Douglass [20] (who ascribe to F. T. Morley and whose paper is not available to the author) and it also appears as a problem proposed by Klamkin [14]. In 1953, Williams [27] and in 1983, Georghiou and Philippou [9] rediscovered Theorem 3.1 and in 1979, the author and Sivaramasarma [21] gave a new proof based on Apostol's extension of a transformation formula due to Lehner and Newman (cf. [1, p. 111]). Also the author and Sivaramasarma [21] obtained a result equivalent to Theorem 3.1. Matsuoka [17] and Apostol and Vu [2] independently discussed the nature of the analytic continuation of $H_1(s)$. In addition to this, Apostol and Vu's paper [2] contains several interesting and related results. Finally certain power series associated with the series defining $H_1(s)$ were investigated by Ramanujan (cf. [19, Chap. 9]) and for proofs of these assertions we refer to Berndt and Joshi (cf. [3, pp. 25–35]).

Proof of Theorem 3.1. We give yet another proof based on

LEMMA 3.1. *Let* $\sum_{r=1}^{\infty} f(r)$ *and* $\sum_{k=1}^{\infty} g(k)$ *be two absolutely convergent series of complex terms. Then*

$$\sum_{r=1}^{\infty} f(r) \sum_{k=1}^{r} g(k) + \sum_{r=1}^{\infty} g(r) \sum_{k=1}^{r} f(k) = \left(\sum_{r=1}^{\infty} f(r)\right)\left(\sum_{k=1}^{\infty} g(k)\right) + \sum_{r=1}^{\infty} f(r)\, g(r). \tag{3.1}$$

In particular, if for complex u, v *with* $\operatorname{Re} u > 1$, $\operatorname{Re} v > 1$

$$P(u, v) = \sum_{r=1}^{\infty} \frac{1}{r^u} \sum_{k=1}^{r} \frac{1}{k^v} \tag{3.2}$$

$$R(u, v) = \sum_{r,k=1}^{\infty} \frac{1}{r^u (r+k)^v}, \tag{3.3}$$

then

$$P(u, v) + P(v, u) = \zeta(u)\, \zeta(v) + \zeta(u+v) \tag{3.4}$$

$$R(u, v) + R(v, u) = \zeta(u)\, \zeta(v) - \zeta(u+v). \tag{3.5}$$

Proof. Since $\sum_{r=1}^{\infty} f(r)$ and $\sum_{k=1}^{\infty} g(k)$ converge absolutely, we have

$$\left(\sum_{r=1}^{\infty} f(r)\right)\left(\sum_{k=1}^{\infty} g(k)\right) = \sum_{r,k=1}^{\infty} f(r)\, g(k) = \left(\sum_{\substack{r,k=1\\ k\leqslant r}}^{\infty} + \sum_{\substack{r,k=1\\ r\leqslant k}}^{\infty} - \sum_{\substack{r,k=1\\ r=k}}^{\infty}\right) f(r)\, g(k)$$

from which (3.1) follows.

On taking $f(r)=r^{-u}$ and $g(k)=k^{-v}$ in (3.1), we obtain (3.4). Further, since

$$P(u, v) + R(u, v) = \zeta(u)\, \zeta(v),$$

(3.4) and (3.5) are in fact equivalent.

Now to prove Theorem 3.1, we write $S_r = 1 + 1/2 + \cdots + 1/r$ so that

$$S_r = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{r+k}\right) = r \sum_{k=1}^{\infty} \frac{1}{k(r+k)}.$$

Hence we have

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{S_r}{r^a} &= \sum_{r=1}^{\infty} \frac{1}{r^{a-1}} \sum_{k=1}^{\infty} \frac{1}{k(r+k)} \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{r^{a-1}k(r+k)^2} \\
&= \sum_{r,k=1}^{\infty} \frac{1}{r^{a-2}k(r+k)^2} + R(a-1, 2) \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{r^{a-2}k(r+k)^3} + R(a-1, 2) \\
&= \sum_{r,k=1}^{\infty} \frac{1}{r^{a-3}k(r+k)^3} + R(a-2, 3) + R(a-1, 2).
\end{aligned}$$

Continuing this procedure, we have by (3.5)

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{S_r}{r^a} &= \sum_{r,k=1}^{\infty} \frac{1}{rk(r+k)^{a-1}} + \sum_{i=2}^{a-1} R(i, a+1-i) \\
&= \sum_{r,k=1}^{\infty} \frac{r+k}{rk(r+k)^a} \\
&\quad + \frac{1}{2} \sum_{i=2}^{a-1} \{R(i, a+1-i) + R(a+1-i, i)\} \\
&= 2 \sum_{r,k=1}^{\infty} \frac{1}{r(r+k)^a} + \frac{1}{2} \sum_{i=2}^{a-1} \{\zeta(i)\, \zeta(a+1-i) - \zeta(a+1)\}.
\end{aligned}$$

Since

$$\sum_{r,k=1}^{\infty} \frac{1}{r(r+k)^a} = \sum_{n=2}^{\infty} \frac{1}{n^a} \sum_{r<n} \frac{1}{r} = \sum_{n=2}^{\infty} \frac{1}{n^a} \left(\sum_{r \leqslant n} \frac{1}{r} - \frac{1}{n} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^a} \left(\sum_{r \leqslant n} \frac{1}{r} - \frac{1}{n} \right) = H_1(a) - \zeta(a+1),$$

we find

$$H_1(a) = 2\,\{H_1(a) - \zeta(a+1)\} + \tfrac{1}{2} \sum_{i=2}^{a-1} \{\zeta(i)\,\zeta(a+1-i) - \zeta(a+1)\}$$

from which Theorem 3.1 follows.

Remark 3.2. The reciprocity relation (3.2) is of frequent occurance in the literature (cf. [18, 21, 23, 2]) and apparently goes back to Euler.

THEOREM 3.2. $2H_1(2a-1) = \sum_{i=2}^{2a-2} (-1)^i \zeta(i)\,\zeta(2a-i)$.

Proof. We need the following extension due to Apostol (cf. [1, p. 111]; [21]) of a transformation formula due to Lehner and Newman [15].

LEMMA 3.2. *Let $f(x, y)$ be a complex valued function defined for positive integral x and y. Then*

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} f(r,k) = \sum_{r=1}^{n} f(r,r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \{f(k,r) + f(r,k) - f(k,r-k)\}.$$

Proof. For a lattice point theoretic proof of the lemma, we refer to the author and Sivaramasarma [21]. Here we give a simpler proof. We have

$$\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} = \sum_{1 \leqslant r,k \leqslant n} f(r,k) - \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \leqslant n}} f(r,k)$$

$$= \left(\sum_{1 \leqslant r<k \leqslant n} + \sum_{1 \leqslant k<r \leqslant n} + \sum_{\substack{1 \leqslant r,k \leqslant n \\ r=k}} \right) f(r,k) - \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \leqslant n}} f(r,k)$$

$$= \sum_{k=2}^{n} \sum_{r=1}^{k-1} f(r,k) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} f(r,k) + \sum_{r=1}^{n} f(r,r)$$

$$- \sum_{u=2}^{n} \sum_{r+k=u} f(r,k)$$

$$= \sum_{r=1}^{n} f(r,r) + \sum_{r=2}^{n} \sum_{k=1}^{r-1} (f(r,k) + f(k,r)) - \sum_{u=2}^{n} \sum_{k=1}^{u-1} f(u-k,k)$$

and the lemma follows.

To prove Theorem 3.2, we take $f(x, y) = 1/x^{2a-1}(x+y)$ in Lemma 3.2 and obtain

$$\begin{aligned}\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^{2a-1}(r+k)} &= \frac{1}{2} \sum_{r=1}^{n} \frac{1}{r^{2a}} \\ &\quad + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \left\{ \frac{1}{r^{2a-1}(r+k)} + \frac{1}{k^{2a-1}(r+k)} - \frac{1}{k^{2a-1}r} \right\} \\ &= \frac{1}{2} \sum_{r=1}^{n} \frac{1}{r^{2a}} \\ &\quad + \sum_{r=2}^{n} \sum_{k=1}^{r-1} \left\{ \frac{k^{2a-2} - r^{2a-2}}{r^{2a-1}k^{2a-1}(r+k)} \right\}. \end{aligned} \tag{3.6}$$

For $a \geqslant 3$, we have

$$\begin{aligned}\sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^{a-1}(r+k)} &\leqslant \sum_{\substack{1 \leqslant r,k \leqslant n \\ r+k \geqslant n+1}} \frac{1}{r^{2}(r+k)} \leqslant \frac{1}{n+1} \sum_{1 \leqslant r \leqslant n} \frac{1}{r^2} \sum_{n+1-r \leqslant k \leqslant n} 1 \\ &= \frac{1}{n+1} \sum_{1 \leqslant r \leqslant n} \frac{1}{r} \leqslant \frac{1+\log n}{n}.\end{aligned}$$

Hence from (3.6) we find, on letting n tend to infinity,

$$\begin{aligned}\frac{1}{2}\zeta(2a) &= \sum_{r=2}^{\infty} \sum_{k=1}^{r-1} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)} \\ &= \sum_{r=2}^{\infty} \sum_{k=1}^{r} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)} = \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{r^{2a-2} - k^{2a-2}}{r^{2a-1}k^{2a-2}(r+k)} \\ &= \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{r^{2a-3} - r^{2a-4}k + r^{2a-5}k^2 - \cdots + rk^{2a-4} - k^{2a-3}}{r^{2a-1}k^{2a-2}} \\ &= \sum_{r=1}^{\infty} \sum_{k=1}^{r} \left\{ \frac{1}{r^2k^{2a-2}} - \frac{1}{r^3k^{2a-3}} + - + \frac{1}{r^{2a-2}k^2} - \frac{1}{r^{2a-1}k} \right\}\end{aligned}$$

so that by (3.4),

$$\begin{aligned}\zeta(2a) + 2H_1(2a-1) = \sum_{r=1}^{\infty} \sum_{k=1}^{r} \{ &(r^{-2}k^{-2a+2} + r^{-2a+2}k^{-2}) \\ &- (r^{-3}k^{-2a+3} + r^{-2a+3}k^{-3}) \\ &+ \cdots + (r^{-2a+2}k^{-2} + r^{-2}k^{-2a+2}) \}\end{aligned}$$

$$
\begin{aligned}
&= \{P(2, 2a-2) + P(2a-2, 2)\} \\
&\quad - \{P(3, 2a-3) + P(2a-3, 3)\} \\
&\quad + \cdots + \{P(2a-2, 2) + P(2, 2a-2)\} \\
&= \zeta(2)\,\zeta(2a-2) - \zeta(3)\,\zeta(2a-3) \\
&\quad + \cdots + \zeta(2a-2)\,\zeta(2) + \zeta(2a)
\end{aligned}
$$

from which the theorem follows for $a \geqslant 3$. Theorem 3.1 covers the case $a = 2$.

THEOREM 3.3. $\sum_{i=1}^{a-1} \zeta(2i)\,\zeta(2a-2i) = (a+\frac{1}{2})\,\zeta(2a)$.

Proof. Follows from Theorems 3.1 and 3.2.

THEOREM 3.4. $2H_1(2a-1) = (a+\frac{1}{2})\,\zeta(2a) - \sum_{i=1}^{a-2} \zeta(i+1)\,\zeta(2a-2i-1)$.

Proof. Follows from Theorems 3.2 and 3.3.

Remark 3.3. Theorems 3.2 through 3.4 appear in Nielsen (cf. [18, pp. 37–51]). Theorem 3.3 was rediscovered by Tornheim (cf. [26, corollary, p. 308]), Williams (cf. [27, Theorem I]), and Georghiou and Philippou [9]. Recently, Sivaramasarma (cf. [25, Chap. III]) gave a new proof of Theorem 3.3. His work also contains various related results.

THEOREM 3.5. $2H_4(a) = 2\zeta(a)\,\sigma(1) - a\zeta(a+1) + 2\sigma(a+1) + \sum_{i=1}^{a} \sigma(i)\,\sigma(a-i+1)$.

Proof. Following Nielsen (cf. [18, p. 47]) we write

$$d_{u,v} = \sum_{r=1}^{\infty} \frac{1}{(r+1)^u} \sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} \qquad \text{for any } u > 1,\ v \geqslant 1, \tag{3.7}$$

$$\gamma_{u,v} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(r+1)^u} \sum_{k=1}^{r} \frac{1}{k^v} \qquad \text{for any } u \geqslant 1,\ v \geqslant 1. \tag{3.8}$$

Then it is known (cf. [18, p. 47, Eq. 8 and p. 50, Eq. 6]) that

$$d_{u,v} = \zeta(u)\,\sigma(v) - \sigma(u+v) + \gamma_{v,u}, \tag{3.9}$$

$$\sum_{i=1}^{b} \sigma(i)\,\sigma(b-i+1) = b\zeta(b+1) - 2\sigma(b+1) + 2\gamma_{1,b}. \tag{3.10}$$

However, by definition,

$$\begin{aligned} d_{u,1} &= \sum_{r=2}^{\infty} \frac{1}{r^u} \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{k} \\ &= \sum_{r=2}^{\infty} \frac{1}{r^u} \left(\sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} - \frac{(-1)^{r-1}}{r} \right) \\ &= \sum_{r=1}^{\infty} \frac{1}{r^u} \left(\sum_{k=1}^{r} \frac{(-1)^{k-1}}{k} - \frac{(-1)^{r-1}}{r} \right) \\ &= H_4(u) - \sigma(u+1). \end{aligned} \tag{3.11}$$

Hence by (3.9) and (3.11)

$$\begin{aligned} H_4(a) &= d_{a,1} + \sigma(a+1) \\ &= \zeta(a)\,\sigma(1) + \gamma_{1,a} \end{aligned}$$

and the theorem follows from (3.10).

THEOREM 3.6. *We have*

$$4H_2(2b) = (2^{2b+1} - 2b - 3)\,\zeta(2b+1) + 2 \sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.12}$$

$$4H_3(2b) = (2^{2b+1} + 2b - 1)\,\zeta(2b+1) - 2 \sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) + \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.13}$$

$$4H_6(2b) = (2^{2b+1} - 2b - 1)\,\zeta(2b+1) + 2 \sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}, \tag{3.14}$$

and

$$\begin{aligned} 4H_{11}(2b) &= 4\zeta(2b)\,\sigma(1) + (6 - 2^{2-2b} - 2b)\,\zeta(2b+1) \\ &\quad - \bar{\zeta}(2b+1) + \sum_{i=1}^{2b} \sigma(i)\,\sigma(2b-i+1) \\ &\quad - 2 \sum_{i=1}^{b-1} \{\zeta(2b-i)\,\zeta(i+1) - \zeta(2i)\,\bar{\zeta}(2b+1-2i)\}. \end{aligned} \tag{3.15}$$

Proof. It is known due to Jordan (cf. [11, Eq. (8a)]) that

$$4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\frac{1}{2k-1}=(2^{2b+1}-1)\,\zeta(2b+1)-2\sum_{i=1}^{b-1}\zeta(2i)\,\bar{\zeta}(2b+1-2i). \qquad (3.16)$$

But

$$4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\frac{1}{2k-1}=4\sum_{r=1}^{\infty}\frac{1}{r^{2b}}\sum_{k=1}^{r}\left(\frac{1}{k}+\frac{1}{r+k}-\frac{1}{2k}\right)=2H_1(2b)+4H_2(2b).$$

Thus (3.12) follows from (3.16) and the above.

To prove (3.13), we have by (2.1)

$$4H_3(2b)=4H_1(2b)+4H_2(2b)-2\zeta(2b+1).$$

Hence Theorem 3.1 and (3.12) yield (3.13).

Formula (3.14) follows from (2.6) and (3.12) while (3.15) follows from (2.9), Theorem 3.5, and (3.12). This completes the proof of the theorem.

Remark 3.4. We believe that Theorems 3.5 and 3.6 are new. (3.12) in case $b=1$ is due to the author and Sivaramasarma (cf. [21, Eq. (1.13)]) who also proved its equivalence with a result due to Gupta [10]. (3.14) in case $b=1$ is due to the author and Subbarao (cf. [24, Eq. (2.16)]) while (3.16) in case $b=1$ is stated by Ramanujan (cf. [19, p. 108]) and proved by the author and Sivaramasarma [22]. Apparently, Jordan [11] was first to prove the general result (3.16).

THEOREM 3.7.

$$2^{2b+1}H_7(2b)=(2^{2b+1}b-2b-1)\,\zeta(2b+1)+2\sum_{i=1}^{b-1}\{\zeta(2i)\,\bar{\zeta}(2b+1-2i)-(2^{2b}-1)\,\zeta(2b-i)\,\zeta(i+1)\}. \qquad (3.17)$$

and

$$4^bH_{10}(2b)=(3\cdot 2^{2b}-4)\,\zeta(2b)\,\sigma(1)+\left(b+\frac{1}{2}-b\cdot 2^{2b}\right)\zeta(2b+1)-\sum_{i=1}^{b-1}\{\zeta(2i)\,\bar{\zeta}(2b-i+1)+2\bar{\zeta}(2i)\,\zeta(2b+1-2i)+(1-2^{2b+1})\,\zeta(2b-i)\,\zeta(i+1)\}-2^{2b}\sum_{i=1}^{b}\sigma(i)\,\sigma(2b-i+1). \qquad (3.18)$$

Proof. By (2.5), we have

$$H_7(s) = H_1(s) - 2^{1-s} H_3(s) - 2^{-s}\zeta(s+1).$$

Hence (3.17) follows from Theorem 3.1 and (3.13).

Now to prove (3.18), we first recall the following result due to Jordan (cf. [11], eq. (9a), p. 683):

$$4^{b+1} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^{2b}} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2r-1}\right)$$
$$= \bar{\zeta}(2b+1) + 4\bar{\zeta}(2b)\,\sigma(1) - 2 \sum_{i=1}^{b-1} \bar{\zeta}(2i)\,\zeta(2b+1-2i). \quad (3.19)$$

Since

$$4^{b+1} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^{2b}} \left(1 + \frac{1}{3} + \cdots + \frac{1}{2r-1}\right)$$
$$= 4^{b+1} \sum_{r=1}^{\infty} \frac{1+(-1)^{r-1}}{2r^{2b}} \sum_{k=1}^{r} \frac{1+(-1)^{k-1}}{2k}$$
$$= 4^{b} \sum_{r=1}^{\infty} \sum_{k=1}^{r} \left\{\frac{1}{r^{2b}k} + \frac{(-1)^{r-1}}{r^{2b}k} + \frac{(-1)^{k-1}}{r^{2b}k} + \frac{(-1)^{r-1}(-1)^{k-1}}{r^{2b}k}\right\}$$
$$= 4^{b}\{H_1(2b) + H_7(2b) + H_4(2b) + H_{10}(2b)\},$$

(3.18) follows from (3.19), Theorems 3.1, 3.5, and (3.17). This completes the proof of the theorem.

4. The Case $s = 2$

Our results in Section 3, in particular, yield the following

Theorem 4.1.

$$H_1(2) = 2\zeta(3), \quad (4.1)$$
$$H_2(2) = \tfrac{3}{4}\zeta(3), \quad (4.2)$$
$$H_3(2) = \tfrac{9}{4}\zeta(3), \quad (4.3)$$
$$H_4(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \tfrac{1}{4}\zeta(3) \quad (4.4)$$
$$H_6(2) = \tfrac{5}{4}\zeta(3) \quad (4.5)$$
$$H_7(2) = \tfrac{5}{8}\zeta(3) \quad (4.6)$$
$$H_{10}(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \tfrac{5}{8}\zeta(3) \quad (4.7)$$
$$H_{11}(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \zeta(3). \quad (4.8)$$

We complete this list by proving

THEOREM 4.2.

$$H_5(2) = \tfrac{3}{2}\zeta(2)\,\sigma(1) - \pi G + \tfrac{23}{16}\zeta(3), \tag{4.9}$$

$$H_8(2) = \pi G - \tfrac{33}{16}\zeta(3), \tag{4.10}$$

$$H_9(2) = \pi G - \tfrac{29}{16}\zeta(3), \tag{4.11}$$

$$H_{12}(2) = \pi G - \tfrac{27}{16}\zeta(3), \tag{4.12}$$

where G denotes the Catalan's constant defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915965594177.... \tag{4.13}$$

Proof. First we prove (4.11). For this we recall a result enunciated by Ramanujan (cf. [19, Chap. 9, entry 21]) and proved by Berndt and Joshi (cf. [3, p. 46]): For $|x| \leqslant \pi/4$

$$\begin{aligned}&\frac{\tan^2 x}{2^2} - \left(1 + \frac{1}{3}\right)\frac{(\tan x)^4}{4^2} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{(\tan x)^6}{6^2} - \&\, c\Big) \\ &= \frac{x^2}{2}\log_e \operatorname{Tan} x + x\left(\frac{\sin 2x}{1^2} + \frac{\sin 6x}{3^2} + \&\, c\right) \\ &\quad + \frac{1}{2}\left(\frac{\cos 2x}{1^3} + \frac{\cos 6x}{3^3} + \frac{\cos 10x}{5^3} + \&\, c\right) \\ &\quad - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&\, c\right).\end{aligned} \tag{4.14}$$

For the sake of completeness, we indicate a proof of (4.14). We denote the left- (resp. right-) hand side of (4.14) by $f(x)$ (resp. $g(x)$) and observe that

$$f(0+) = g(0+) \qquad \text{and} \qquad f'(x) = g'(x) = x^2/\sin 2x$$

in view of the well known

$$(\tan^{-1} x)^2 = \sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^{2r}}{r} \sum_{k=1}^{r} \frac{1}{2k-1}$$

and

$$\cos x + \frac{1}{3}\cos 3x + \cdots = \frac{1}{2}\log\left(\cot\frac{x}{2}\right).$$

On taking $x=\pi/4$ in (4.14) we find

$$\frac{1}{2^2}\left(1-\frac{1+1/3}{2^2}+\frac{1+1/3+1/5}{3^2}-\&\,c\right)$$
$$=\frac{\pi}{4}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&\,c\right)-\frac{1}{2}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&\,c\right). \tag{4.15}$$

Hence on one hand we have

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sum_{k=1}^{r}\frac{1}{2k-1}=\pi G-\frac{7}{4}\zeta(3) \tag{4.16}$$

and on the other

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\sum_{k=1}^{r}\frac{1}{2k-1}$$
$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^2}\left\{\sum_{k=1}^{2r-1}\frac{1}{k}+\frac{1}{2r}-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2r}\right)\right\}$$
$$=H_9(2)+\tfrac{1}{2}\sigma(3)-\tfrac{1}{2}H_7(2)$$
$$=H_9(2)+\tfrac{1}{16}\zeta(3) \tag{4.17}$$

where in the last step we used (4.6). Now (4.11) follows from (4.16) and (4.17). Also (4.10) follows from (2.2), (4.6), and (4.11); finally (4.9) follows from (2.8), (4.7), and (4.10); (4.12) follows from (2.7) and (4.10). This completes the proof of the theorem.

5. The Case $s=3$ and Ramanujan's $G(1)$

In this section, we relate $G(1)$ explicitly to each of $H_2(3)$, $H_3(3)$, $H_6(3)$, $H_7(3)$, $H_{11}(3)$, and A_4, where A_4 is given by (1.2).

Theorem 5.1. $G(1)=\frac{3}{16}\zeta(4)-\frac{1}{4}A_4$.

Proof. We first note (cf. [20, Eq. (18)]) that

$$\sum_{r=2}^{\infty}\frac{(-1)^r}{r^3}\sum_{k=1}^{r-1}\frac{1}{k}=-\frac{1}{2}A_4+\frac{\pi^4}{576}. \tag{5.1}$$

In fact

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} &= \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^2 n^2} \\ &= \sum_{r=2}^{\infty} (-1)^r \sum_{k=1}^{r-1} \frac{1}{k^2(r-k)^2} \\ &= \sum_{r=2}^{\infty} (-1)^r \sum_{k=1}^{r-1} \left\{ \frac{2}{r^3 k} + \frac{2}{r^3(r-k)} + \frac{1}{r^2 k^2} + \frac{1}{r^2(r-k)^2} \right\} \\ &= 4 \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k} + 2 \sum_{r=2}^{\infty} \frac{(-1)^r}{r^2} \sum_{k=1}^{r-1} \frac{1}{k^2}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k} &= -\frac{1}{2} A_4 + \frac{1}{4} \left(\frac{1}{2} \zeta(2) \right)^2 \\ &= -\frac{1}{2} A_4 + \frac{\pi^4}{576}\end{aligned}$$

which is (5.1). We also need

$$H_7(3) = \tfrac{23}{32} \zeta(4) + \tfrac{1}{2} A_4. \tag{5.2}$$

To see this, we have by (5.1)

$$\begin{aligned}-\frac{1}{2} A_4 + \frac{\pi^4}{576} &= \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \sum_{k=1}^{r-1} \frac{1}{k} \\ &= \sum_{r=2}^{\infty} \frac{(-1)^r}{r^3} \left(\sum_{k=1}^{r} \frac{1}{k} - \frac{1}{r} \right) \\ &= -\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^3} \sum_{k=1}^{r} \frac{1}{k} + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^4} \\ &= -H_7(3) + \sigma(4).\end{aligned}$$

Since $\sigma(4) = \frac{7}{8} \zeta(4)$ and $\zeta(4) = \pi^4/90$, we obtain (5.2).

Now consider

$$\begin{aligned}
G(1) &= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right) \\
&= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2r}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2r}\right)\right\} \\
&= \sum_{r=1}^{\infty} \frac{1+(-1)^r}{2r^3} \sum_{k=1}^{r} \frac{1}{k} - \frac{1}{16} \sum_{r=1}^{\infty} \frac{1}{r^3} \sum_{k=1}^{r} \frac{1}{k} \\
&= \frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r^3} \sum_{k=1}^{r} \frac{1}{k} - \frac{1}{2} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r^3} \sum_{k=1}^{r} \frac{1}{k} - \frac{1}{16} \sum_{r=1}^{\infty} \frac{1}{r^3} \sum_{k=1}^{r} \frac{1}{k} \\
&= \left(\frac{1}{2}-\frac{1}{16}\right) H_1(3) - \frac{1}{2} H_7(3). \qquad (5.3)
\end{aligned}$$

But by Theorem 3.1 and the fact tht $\zeta^2(2)=\pi^4/36=\frac{5}{2}\zeta(4)$, we have

$$\begin{aligned}
H_1(3) &= \tfrac{1}{2}(5\zeta(4)-\zeta^2(2)) \\
&= \tfrac{5}{4}\zeta(4). \qquad (5.4)
\end{aligned}$$

Hence from (5.3) and (5.4)

$$G(1)=\tfrac{35}{64}\zeta(4)-\tfrac{1}{2}H_7(3). \qquad (5.5)$$

Thus by (5.2)

$$G(1)=\tfrac{35}{64}\zeta(4)-\tfrac{1}{2}\left(\tfrac{23}{32}\zeta(4)+\tfrac{1}{2}A_4\right)-\tfrac{1}{4}A_4+\tfrac{3}{16}\zeta(4)$$

and Theorem 5.1 follows.

Remark 5.1. It may be noted that (5.5) above relates $G(1)$ explicitly with $H_7(3)$.

THEOREM 5.2. $G(1) \simeq 0.16227\ldots$.

Proof. It is known due to Rutledge and Douglass [20], that the first five digits of A_4 are 0.16265. Hence by Theorem 5.1, we see that

$$\begin{aligned}
G(1) &\simeq \tfrac{3}{16}(1.08232)-\tfrac{1}{4}(0.16265) \\
&= 0.16227.
\end{aligned}$$

THEOREM 5.3. $G(1)=\frac{5}{64}\zeta(4)+\frac{1}{8}H_2(3)$.

Proof. We have

$$\begin{aligned}G(1) &= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left(1+\frac{1}{3}+\cdots+\frac{1}{2r-1}\right)\\ &= \sum_{r=1}^{\infty} \frac{1}{(2r)^3}\left\{1+\frac{1}{2}+\cdots+\frac{1}{2r}-\left(\frac{1}{2}+\cdots+\frac{1}{2r}\right)\right\}\\ &= \frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}+\frac{1}{8}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{r+k}-\frac{1}{16}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{k=1}^{r}\frac{1}{k}\\ &= \tfrac{1}{16}H_1(3)+\tfrac{1}{8}H_2(3). \end{aligned} \tag{5.6}$$

Hence the theorem follows from (5.4).

COROLLARY 5.1. $H_2(3)=\frac{7}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.3.

THEOREM 5.4. $G(1)=-\frac{1}{64}\zeta(4)+\frac{1}{8}H_3(3)$.

Proof. From (5.6), (2.1), and (5.4), we have

$$\begin{aligned}G(1) &= \tfrac{1}{16}H_1(3)+\tfrac{1}{8}H_2(3)=\tfrac{1}{8}(H_1(3)+H_2(3))-\tfrac{1}{16}H_1(3)\\ &= \tfrac{1}{8}(H_3(3)+\tfrac{1}{2}\zeta(4))-\tfrac{1}{16}\cdot\tfrac{5}{4}\zeta(4)\\ &= -\tfrac{1}{64}\zeta(4)+\tfrac{1}{8}H_3(3).\end{aligned}$$

COROLLARY 5.2. $H_3(3)=\frac{13}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.4.

THEOREM 5.5. $G(1)=\frac{1}{64}\zeta(4)+\frac{1}{8}H_6(3)$.

Proof. Follows from Theorem 5.3 and (2.6).

COROLLARY 5.3. $H_6(3)=\frac{11}{8}\zeta(4)-2A_4$.

Proof. Follows from Theorems 5.1 and 5.5.

THEOREM 5.6. $G(1)=\frac{7}{32}\zeta(3)\log 2+\frac{5}{128}\zeta(4)-\frac{1}{8}H_{11}(3)$.

Proof. Follows from Theorems 5.3, (2.9), and Theorem 3.5 with $s=3$.

COROLLARY 5.4. $H_{11}(3)=\frac{7}{4}\zeta(3)\log 2-\frac{19}{16}\zeta(4)+2A_4$.

Proof. Follows from Theorems 5.1 and 5.6.

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References

1. T. M. Apostol, "Modular Functions and Dirichlet Series in Number Theory," Springer–Verlag, Berlin, 1976.
2. T. M. Apostol and T. H. Vu, Dirichlet series related to the Riemann zeta function, *J. Number. Theory* **19** (1984), 85–102.
3. B. C. Berndt and P. T. Joshi, "Infinite Series Identities, Transformations and Evaluations," Chap. 9, *in* Ramanujan's second notebook, Contemporary Mathematics Vol. 23, Amer. Math. Soc., Providence, R. I., 1983.
4. W. E. Briggs, S. Chowla, A. J. Kempner, and W. E. Mientka, On some infinite series, *Scripta Math.* **21** (1955), 28–30.
5. P. S. Bruckman, Problem H-320, *Fibonacci Quart.* **20** (1982), 186–187.
6. M. J. Dixon and C. A. O'Cinneide, Certain integrals and series related to the zeta function, preprint.
7. L. Euler, Meditationes circa singulare serierum genus, *Novi. Comm. Acad. Sci. Petropolitanas* **20** (1775), 140–186.
8. L. Euler, "Opera Omnia," Ser. 1, Vol. 15, Teubner, Berlin, 1927.
9. C. Georghiou and A. N. Philippou, Harmonic sums and the zeta function, The *Fibonacci Quart.* **21** (1983), 29–36.
10. H. Gupta, An identity, *Res. Bull. Panjab Univ.* (*N. S.*) **15** (1964/65), 347–349.
11. P. F. Jordan, Infinite sums of psi functions, *Bull. Amer. Math. Soc.* **79** (1973), 681–683.
12. S. Kanemitsu, On some sums involving Farey fractions, *Math. J. Okayama Univ.* **20** (1978), 101–113; Corrigendum, *ibid.* **22** (1980), 223–224.
13. M. S. Klamkin, Problem 4431, Soln. by R. Steinberg, *Amer. Math. Monthly* **59** (1952), 471–472.
14. M. S. Klamkin, Problem 4564, Solns. by J. V. Whittaker and the proposer, *Amer. Math. Monthly* **62** (1955), 129–130.
15. J. Lehner and M. Newman, Sums involving Farey fractions, *Acta Arith.* **15** (1968/69), 181–187.
16. L. Lewin, "Polylogarithms and Associated Functions," North-Holland, New York, 1981.
17. Y. Matsuoka, On the values of a certain Dirichlet series at rational integers, *Tokyo J. Math.* **5** (1982), 399–403.
18. N. Nielsen, "Handbuch der Theorie der Gamma Funktion," Chelsea, New York, 1965.
19. S. Ramanujan, "Notebooks," Vol. 2, Tata Institute of Fundamental Research, Bombay, 1957.
20. G. Rutledge and R. D. Douglass, Evaluation of $\int_0^1 (\log u/u) \log^2(1+u)\, du$ and related integrals, *Amer. Math. Monthly* **41** (1934), 29–36.
21. R. Sitaramachandrarao and A. Sivaramasarma, Some identities involving the Riemann zeta function, *Indian J. Pure Appl. Math.* **10** (1979), 602–607.
22. R. Sitaramachandrarao and A. Sivaramasarma, Two identities due to Ramanujan, *Indian J. Pure Appl. Math.* **11** (1980), 1139–1140.
23. R. Sitaramachandrarao and M. V. Subbarao, Transformation formulae for multiple series, *Pacific J. Math.* **113** (1984), 471–479.

24. R. SITARAMACHANDRARAO AND M. V. SUBBARAO, On some infinite series of L. J. Mordell and their analogues, *Pacific J. Math.* **119** (1985), 245–255.
25. A. SIVARAMASARMA, Some problems in the theory of Farey series and the Euler totient function, Doctoral dissertation, Andhra University, Waltair, India, 1980.
26. L. TORNHEIM, Harmonic double series, *Amer. J. Math.* **72** (1950), 303–314.
27. G. T. WILLIAMS, A new method of evaluating $\zeta(2n)$, *Amer. Math. Monthly* **60** (1953), 19–25.