On a Class of Generalized Nilpotent Groups

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We explore the class $\mathcal{B}$ of generalized nilpotent groups in the universe $\mathcal{C}/\overline{\mathcal{L}}$ of all radical locally finite groups satisfying $\text{min-}p$ for every prime $p$. We obtain that this class is the natural generalization of the class of finite nilpotent groups from the finite universe to the universe $\mathcal{C}/\overline{\mathcal{L}}$. Moreover, the structure of $\mathcal{B}$-groups is determined explicitly. It is also shown that $\mathcal{B}$ is a subgroup-closed $\mathcal{C}/\overline{\mathcal{L}}$-formation and that in every $\mathcal{C}/\overline{\mathcal{L}}$-group the Fitting subgroup is the unique maximal normal $\mathcal{B}$-subgroup.

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1. INTRODUCTION AND STATEMENT OF RESULTS

It is well known that there are numerous properties of finite groups which are equivalent to nilpotence, for instance, subnormality of each subgroup, normality of all Sylow subgroups, centrality of every chief factor, and normality of all maximal subgroups. If the attention is restricted to locally finite-soluble groups, the first three properties are sufficient to ensure local nilpotency and the latter three are enjoyed by each locally nilpotent group.

It is also well known that, for finite groups $G$, the conditions $G' \leq \phi(G)$ and $G/\phi(G)$ nilpotent are both equivalent to nilpotence. Taking into account that the Frattini subgroup $\phi(G)$ of a group $G$ is defined as the intersection of $G$ with all its maximal subgroups, it is rather clear that the condition $G' \leq \phi(G)$ is a weak property for infinite groups, even for locally finite groups, because an infinite group can have insufficient maximal subgroups or even none at all.

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Our first main objective in this paper is to use the Frattini-like subgroup \( \mu(G) \) introduced by Tomkinson in 1975 (see [6]) to study a class of generalized nilpotent groups in the universe \( \mathcal{cF} \) of all radical locally finite groups with \( \text{min}_p \) for all primes \( p \).

Before beginning the presentation of results, the reader should be aware of the notation that is used. Notation that is not specifically cited here is consistent with that used in [2, 3, 4].

Let \( U \) be a subgroup of group \( G \) and consider the properly ascending chains

\[ U = U_0 < U_1 < \cdots < U_\alpha = G \]

from \( U \) to \( G \). We define \( m(U) \) to be the least upper bound of the types \( \alpha \) of all such chains.

Clearly, \( m(U) = 1 \) if and only if \( U \) is a maximal subgroup of \( G \).

A proper subgroup \( M \) of \( G \) is said to be a \textit{major subgroup} of \( G \) if \( m(U) = m(M) \) whenever \( M \leq U < G \).

The intersection of all major subgroups of \( G \) is denoted by \( \mu(G) \). When \( G \) is finitely generated, \( \mu(G) \) coincides with the Frattini subgroup \( \phi(G) \) of \( G \). In [6, Proposition 2.1], Tomkinson shows that every proper subgroup of a group \( G \) is contained in a major subgroup of \( G \). Hence \( \mu(G) \) is always a proper subgroup of \( G \).

Following [8], a group \( G \) is \textit{semiprimitive} if it is the semidirect product \( G = [D]M \) of a finite soluble group \( M \) with trivial core and a divisible abelian group \( D \), such that every proper \( M \)-invariant subgroup of \( D \) is finite.

Let \( G \) be a \( \mathcal{cF} \)-group and consider two normal subgroups \( H, K \) of \( G \) such that \( K \) is contained in \( H \). Then \( H/K \) is said to be a \( \delta \)-\textit{chief factor} of \( G \) if \( H/K \) is either a minimal normal subgroup of \( G/K \) or a divisibly irreducible \( \mathbb{Z}G \)-module, that is, \( H/K \) has not proper infinite \( G \)-invariant subgroups.

Let \( \mathcal{B} \) be the class of all \( \mathcal{cF} \)-groups in which every proper subgroup has a proper normal closure. This is a class of generalized nilpotent groups in the universe \( \mathcal{cF} \) because every nilpotent \( \mathcal{cF} \)-group is in \( \mathcal{B} \) and every finite \( \mathcal{B} \)-group is nilpotent. Moreover, this class contains the class of all \( \mathcal{cF} \)-groups for which every subgroup is descendant.

Now we can establish the first of our main results. It shows that \( \mathcal{B} \)-groups play the same role in the class \( \mathcal{cF} \) as finite nilpotent groups do in the class of all finite groups. Moreover, we obtain a complete characterization of the \( \mathcal{B} \)-groups \( G \), through the Frattini-like subgroup \( \mu(G) \), analogously to the finite one for nilpotent groups and the Frattini subgroup.

**Theorem 1.** Let \( G \) be a group in the class \( \mathcal{cF} \). The following statements are pairwise equivalent:

(i) \( G \) is a \( \mathcal{B} \)-group.
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(ii) $G/\mu(G)$ is a $\mathcal{B}$-group.

(iii) $G' \subseteq \mu(G)$.

(iv) Every major subgroup of $G$ is a normal subgroup of $G$.

(v) $G$ is a direct product of nilpotent Sylow subgroups.

(vi) $G$ is locally nilpotent and the radicable part of $G$ is central.

(vii) Every $\delta$-chief factor of $G$ is central.

Assume that a group $G$ has the minimum condition on subgroups. If $G$ is a $\mathcal{B}$-group, then $G$ is a direct product of nilpotent Sylow subgroups by Theorem 1. Since $G$ has min, only finitely many of these Sylow subgroups are nontrivial. Hence $G$ is actually a nilpotent group. Consequently, if $G$ has min, $G$ is a $\mathcal{B}$-group if and only if $G$ is nilpotent.

In particular, if the group $G$ is the split extension of a quasicyclic 2-group by its involution, then $G$ is a locally nilpotent group, in fact it is hypercentral, but it is not a $\mathcal{B}$-group.

Consider the set $\{p_i\}_{i \geq 1}$ of all prime numbers in their natural order. Let $G_i$ be the split extension of the cyclic group $\langle x_i \rangle$ of order $p_i^i$ by its automorphism $y_i$ of order $p_i^{-1}$ which maps $x_i$ to $x_i^{p_i^i}$. Then $G_i$ is nilpotent of class $i$. Let $G = \bigcup_{i \geq 1} G_i$; then $G$ is a $\mathcal{B}$-group which is not nilpotent (see [6]).

These examples show that the class $\mathcal{B}$ is intermediate between the classes of nilpotent $c\mathcal{L}$-groups and locally nilpotent $c\mathcal{L}$-groups.

It is well known that for finite groups $G$, the Frattini subgroup $\phi(G)$ is nilpotent. This result can be extended to $c\mathcal{L}$-groups using Tomkinson's subgroup.

**Theorem 2.** Let $G$ be a group in the class $c\mathcal{F}$. Then $\mu(G)$ is a $\mathcal{B}$-group with finite Sylow subgroups.

Our next result analyzes the behavior of $\mathcal{B}$ as a class of $c\mathcal{F}$-groups. Recall that a class $\mathcal{F}$ of $c\mathcal{F}$-groups is said to be a $c\mathcal{F}$-formation if it satisfies the following properties:

1. If $G \in \mathcal{F}$ and $N$ is a normal subgroup of $G$, then $G/N \in \mathcal{F}$.
2. If $\{N_i\}_{i \in I}$ is a collection of normal subgroups of $G \in c\mathcal{F}$ such that $G/N_i \in \mathcal{F}$ for every $i \in I$ and $\bigcap_{i \in I} N_i = 1$, then $G \in \mathcal{F}$.

**Theorem 3.** $\mathcal{B}$ is a subgroup-closed $c\mathcal{F}$-formation.

Let $G$ be the semidirect product of a quasicyclic 2-group $H$ by a cyclic group of order 2 acting on $H$ by inverting each of its elements. Then $G$ is a semiprimitive group which is the union of an ascending chain of finite $\mathcal{B}$-subgroups, but $G$ is not a $\mathcal{B}$-group. According to [1, Theorem A], $\mathcal{B}$ is not a saturated $c\mathcal{F}$-formation.
Theorem 3 allows us to show the existence of the $\mathcal{B}$-radical in every group $G$ belonging to the class $\mathcal{cL}$.

It is known that the product of two normal nilpotent subgroups is nilpotent—this is Fitting’s theorem [4, Theorem 5.2.8]. The corresponding statement holds for locally nilpotent groups and is of great importance. Moreover, in any group $G$, there is a unique maximal normal locally nilpotent subgroup (called the Hirsch–Plotkin radical) containing all normal locally nilpotent subgroups of $G$ (see [4, (12.1.3)]). We obtain analogous results for the class $\mathcal{B}$ by defining the corresponding radical subgroup associated to this class.

**Theorem 4.** Let $G$ be a $\mathcal{cL}$-group. Assume that $H$ and $K$ are two normal $\mathcal{B}$-subgroups of $G$. Then $HK$ is a $\mathcal{B}$-group.

**Theorem 5.** Every group $G \in \mathcal{cL}$ has a unique largest normal $\mathcal{B}$-subgroup, denoted by $\delta(G)$.

From Theorems 3 and 5, we have that the product of arbitrarily many normal $\mathcal{B}$-subgroups of a group $G \in \mathcal{cL}$ belongs to the class $\mathcal{B}$. In particular, the subgroup $\delta(G)$ is the product of all normal $\mathcal{B}$-subgroups of $G$.

Note that for every group $G \in \mathcal{cF}$, the Fitting subgroup $F(G)$ is contained in $\delta(G)$ by Theorem 5. Now if $N \in \mathcal{B}$ is a normal subgroup of $G$, then every Sylow subgroup of $N$ is a normal nilpotent subgroup of $G$. Consequently, $N$ is contained in the Fitting subgroup of $G$, $F(G)$. This implies that $\delta(G) = F(G)$ for every group $G \in \mathcal{cF}$. It is known that, in general, the Fitting subgroup in an infinite group gives little information about the structure of the group. However, in this case it plays an important role as it inherits the properties of the $\mathcal{B}$-radical.

We say that a subclass $\mathcal{F}$ of $\mathcal{cF}$ is a $\mathcal{cF}$-Fitting class if it satisfies the following properties:

1. If $G \in \mathcal{F}$ and $H$ is a normal subgroup of $G$, then $H \in \mathcal{F}$.
2. If $G = \langle H_i; i \in I \rangle \in \mathcal{cF}$ and, for each $i \in I$, the subgroup $H_i$ is a normal $\mathcal{F}$-subgroup of $G$, then $G \in \mathcal{F}$.

It is clear that $\mathcal{B}$ is an example of a $\mathcal{cF}$-Fitting class. Perhaps it is worth noting at this point a well-known result of Bryce and Cossey concerning saturated formations and Fitting classes. It asserts that, in the finite soluble universe, a subgroup-closed Fitting formation is saturated ([3, (XI.1.2)]). This result is no longer true in infinite groups, as the class $\mathcal{B}$ shows.

Recall that a group $G$ is called a Fitting group if $G = F(G)$, that is, if $G$ is a product of normal nilpotent subgroups. A group $G$ is said to be a Baer group if it is generated by its abelian subnormal subgroups, or equivalently, if every finitely generated subgroup of $G$ is subnormal. It is known that a Fitting group is a Baer group, but the converse is not true, even in the
locally finite universe. However we have:

**Theorem 6.** Let \( G \) be a \( c\ell \)-group. Then the following statements are pairwise equivalent:

(i) \( G \) is a \( \mathcal{B} \)-group.

(ii) \( G \) is a Fitting group.

(iii) \( G \) is a Baer group.

(iv) \( G \) is hypocentral with hypocentral length \( \leq \omega \).

Let \( G = M(\mathbb{Q}, GF(p)) \) be the McLain group determined by the set \( \mathbb{Q} \) of all rational numbers and the field \( GF(p) \) (see [4, (12.1.9)]). It is clear that \( G \) is not in \( c\ell \). Moreover, it is known that \( G \) is the product of its normal abelian subgroups and hence is a Fitting group. Let \( H \) be the subgroup generated by the set of all \( 1 + e_{\alpha \beta} \), such that \( \alpha \in \mathbb{Q}, \beta \in X, \) and \( \alpha < \beta \), where \( X = \{x^2: x \in \mathbb{Q}\} \cup \{-x^2: x \in \mathbb{Q}\} \). We have that \( H \) is a proper subgroup of \( G \) and \( G = \langle H^G \rangle \). Therefore, \( G \) is a locally finite Fitting group which is not in the class \( \mathcal{B} \).

It is well known that the Fitting subgroup of a finite group \( G \) is the intersection of the centralizers of all chief factors of \( G \) (see [3, (A.13.8)]). This result is also true for the Hirsch–Plotkin radical of a periodic locally soluble group (see [2, (1.3.5), (6.2.4)]). As one might expect, there is an important connection between \( \delta(G) \) and the centralizers of the \( \delta \)-chief factors, as the following theorem shows.

**Theorem 7.** Suppose that \( G \) is a \( c\ell \)-group. Then \( \delta(G) \) is the intersection of the centralizers of all \( \delta \)-chief factors of \( G \).

### 2. PROOFS OF THE MAIN THEOREMS

We begin with the following lemmas:

**Lemma 1.** Let \( G \) be either a finite primitive soluble group or a semiprimitive group. If \( G \) is a \( \mathcal{B} \)-group, then \( G \) is abelian.

**Proof.** Suppose first that \( G \) is a finite primitive soluble group and let \( M \) be a maximal subgroup of \( G \) with trivial core. Since \( G \) is a \( \mathcal{B} \)-group and \( M \) is a proper subgroup of \( G \), it follows that \( \langle M^G \rangle \) is also a proper subgroup of \( G \). Hence, since \( M \) is maximal in \( G \), we have that \( M = \langle M^G \rangle \). This implies that \( M \) is a normal subgroup of \( G \). Therefore, \( M = 1 \) is a maximal subgroup of \( G \) and thus \( G \) is a cyclic group of order \( p \) for some prime \( p \).

Now, suppose that \( G = [D]M \) is a semiprimitive group, where \( D \) is a faithful divisibly irreducible \( \mathbb{Z}M \)-module and \( M \) is a finite soluble group with trivial core. Denote \( T = \langle M^G \rangle \), the normal closure of \( M \) in \( G \). Since \( G \)
is a \(\mathcal{R}\)-group, it follows that \(T\) is a proper subgroup of \(G\). Hence \(T = (D \cap T)M\) and \(D \cap T\) is a proper subgroup of \(D\). Moreover, as \(D \cap T \triangleleft G\) and \(D\) is divisibly irreducible, it follows that \(D \cap T\) is finite. Notice that \(T\) is a finite subgroup of \(G\) because \(M\) is also finite. Consequently, \(M\) has only finitely many conjugates in \(G\) and \(|G : N_G(M)|\) is finite. Consider now \(N_G(M) = N_D(M)M\) and assume that \(N_D(M) = D\). Then \(M\) is normal in \(G\). Since \(\text{Core}_G(M) = 1\), we have that \(M = 1\) and \(G\) is abelian.

Therefore, we may assume that \(N_D(M)\) is a proper subgroup of \(D\). The fact that \(D\) is a normal subgroup of \(G\) implies that \(N_D(M)\) is \(M\)-invariant and hence it is finite. Since \(M\) is also finite, we have that \(N_G(M)\) is finite and hence \(G\) is a finite group. This contradicts the fact that \(G\) is a semiprimitive group and the lemma is proved.

**Lemma 2.** Let \(G\) be a \(c\mathcal{F}\)-group. If \(G\) is a Chernikov \(\mathcal{R}\)-group, then \(G\) is nilpotent.

**Proof.** Let \(M\) be a major subgroup of \(G\). Denote \(M_G = \text{Core}_G(M)\). By [1, Theorem 1], we have that either \(G/M_G\) is a finite primitive soluble group or \(G/M_G\) is a semiprimitive group. Consequently, \(G/M_G\) is abelian by Lemma 1. Since \(G/\mu(G)\) is isomorphic to a subgroup of the cartesian product \(\text{Cr}_{M \in \mathcal{F}(G)} G/M_G\), we have that \(G/\mu(G)\) is also an abelian group.

Let \(G = G^0A\), where \(A\) is a finite subgroup of \(G\) and \(G^0\) is the radicable part of \(G\). The \(A\mu(G)\) is a normal subgroup of \(G\) because \(G/\mu(G)\) is abelian. Moreover, as \(G\) is a Chernikov group, it follows that \(\mu(G)\) is finite by [7, (1.2)]. Therefore, \(A\mu(G)\) is also a finite subgroup of \(G\). Now, applying [5, (3.29.1)], we have that \(G^0 = [G^0, A\mu(G)]\). Since \(A\mu(G)\) is a normal subgroup of \(G\), it follows that \([G^0, A\mu(G)]\) is contained in \(A\mu(G)\) and hence \([G^0, A\mu(G)]\) is finite. As a consequence, \(G^0 = C_{G^0}(A\mu(G))\). Since \(G = G^0(A\mu(G))\) and \(G^0\) is an abelian group, it follows that \(G^0\) is contained in the center of \(G\). Therefore, \(G/Z(G)\) is a finite group. Moreover, \(G/Z(G)\) is a \(\mathcal{R}\)-group. The \(G/Z(G)\) is nilpotent and so \(G\) is nilpotent.

**Proof of Theorem 1.** (i) implies (ii). This is clear from the fact that the class \(\mathcal{R}\) is closed under taking epimorphic images.

(ii) implies (iii). Note that if \(M\) is a major subgroup of \(G\), then \(\mu(G) \leq M_G\) and hence \(G/M_G\) is isomorphic to a quotient of \(G/\mu(G)\). Since \(G/\mu(G)\) is a \(\mathcal{R}\)-group, we have that \(G/M_G\) is also a \(\mathcal{R}\)-group. Therefore, by Lemma 1, \(G/M_G\) is an abelian group. Consequently, \(G/\mu(G)\) is also abelian and then \(G^0 \leq \mu(G)\).

(iii) implies (iv). Since \(G/\mu(G)\) is an abelian group, it follows that \(M/\mu(G)\) is a normal subgroup of \(G/\mu(G)\) for every major subgroup \(M\) of \(G\). Consequently, every major subgroup of \(G\) is a normal subgroup of \(G\).
(iv) implies (i). Let $H$ be a proper subgroup of $G$. Then, by [6, (2.3)], $H$ is contained in a major subgroup $M$ of $G$. Since $M$ is a normal subgroup of $G$, we have that $(H^G) \leq M$. In particular, $(H^G)$ is a proper subgroup of $G$. Thus $G$ is a $\mathfrak{H}$-group.

(iii) implies (v). Since $G' \leq \mu(G)$, it follows that $G/\mu(G)$ is an abelian group. In particular, $G/\mu(G)$ is a locally nilpotent group and then, by [6, (5.2)], $G$ is locally nilpotent. Thus, if we denote by $G_p$ the unique Sylow $p$-subgroup of $G$ for each prime $p$, then $G = \text{Dr}_p G_p$. We prove that $G_p$ is a nilpotent group for each prime $p$. Since $G$ is a $\mathfrak{S}$-group, we have that $G_p$ is a Chernikov group for each prime $p$ by [2, (2.5.13)]. Moreover, $G_p$ is isomorphic to $G/G_p'$, where $G_p' = \text{Dr}_q G_q$. Since (i) is equivalent to (iii), we have that $G_p$ is a $\mathfrak{S}$-group and consequently, $G_p$ is also a $\mathfrak{S}$-group for each prime $p$.

(v) implies (iii). Let $M$ be a proper subgroup of $G$ and denote $M_G = \text{Core}_G(M)$. Applying [1, Theorem 1], if $M$ is a maximal subgroup of $G$, then $G/M_G$ is a finite primitive soluble group. By hypothesis, $G$ is locally nilpotent. In particular, $G/M_G$ is locally nilpotent and, therefore, $G/M_G$ is nilpotent. Consequently, $G/M_G$ is a cyclic group of order $p$ for some prime $p$ and thus $G/M_G$ is an abelian group.

On the other hand, assume that $M$ is a nonmaximal major subgroup of $G$. Then $G/M_G$ is a semiprimitive group. By [8, (2.3)], $F(G/M_G) = (G/M_G)^\gamma = C_{G/M_G}(F(G/M_G))$ and $F(G/M_G)$ is a $p'$-group for some prime $p$. Moreover, $G/M_G$ is a direct product of nilpotent Sylow subgroups. Consequently, $F(G/M_G) = G/M_G$ and $G/M_G$ is a quasicyclic $p'$-group because $G/M_G$ is divisibly irreducible. In both cases we have proved that if $M$ is a major subgroup of $G$, then $G/M_G$ is an abelian group. As a consequence, $G/\mu(G)$ is also abelian and thus $G'$ is contained in the subgroup $\mu(G)$.

(v) implies (vi). It is clear that $G$ is locally nilpotent. For each prime $p$, $G$ has a normal Sylow $p$-subgroup $G_p$ and $G = \text{Dr}_p G_p$. Moreover, $G_p$ is Chernikov for all primes $p$ by [2, (2.5.13)]. Let $G_p^0$ be the radicable part of $G$. Then $G^0 \cap G_p = G_p^0$ for all primes $p$. Therefore, $G^0 = \text{Dr}_p G_p^0$. Since $G_p$ is nilpotent, we have that $G_p^0 \leq Z(G_p)$ by [2, (1.5.12)]. Consequently, $G_p^0 \leq Z(G)$ for all primes $p$ and $G^0 \leq Z(G)$.

(vi) implies (v). Arguing as above, we have that $G = \text{Dr}_p G_p$ and $G^0 = \text{Dr}_p G_p^0$. Since $G_p^0 \leq Z(G_p)$ and $G_p$ is Chernikov, it follows that $G_p/Z(G_p)$ is nilpotent and so $G_p$ is nilpotent for all primes $p$. Hence (v) holds.

(vi) implies (vii). Let $G$ be a locally nilpotent group such that $G^0 \leq Z(G)$ and consider a $\delta$-chief factor $H/K$ of $G$. If $H/K$ is a minimal normal subgroup of $G/K$, then, by [4, (12.1.6)], $H/K$ is central.
Suppose now that $H/K$ is a divisibly irreducible $\mathbb{Z}G$-module. In particular, $H/K$ is a divisible subgroup of $G/K$ and $H/K$ is contained in $(G/K)^0$, the radicable part of $G/K$. On the other hand, if we denote $T/K = (G/K)^0$, it follows from [2, (2.5.14)] that the Sylow $p$-subgroups of $G/(G^0K)$, and hence the Sylow $p$-subgroups of $T/(G^0K)$, are finite for each prime $p$. Furthermore, $T/(G^0K)$ is a locally nilpotent group and hence it is the direct product of its Sylow subgroups. Hence, every Sylow subgroup of $T/(G^0K)$ is a divisible subgroup and so it is trivial. Therefore, $T/K = (G/K)^0 = (G^0K)/K$. Consequently, $H/K \leq (G^0K)/K \leq (Z(G)K)/K \leq Z(G/K)$ as required.

(vii) implies (vi). By hypothesis, every $\delta$-chief factor of $G$ is central. In particular, every chief factor of $G$ is central. It follows from [2, (6.2.4)] that $G$ is locally nilpotent. Moreover, the Sylow subgroups of $G$ are Chernikov groups. Hence, to prove that $G^0 \leq Z(G)$, we may assume that $G$ is a Chernikov $p$-group for some prime $p$. Then $G = G^0A$, where $A$ is a finite subgroup of $G$ and $G^0$ is the radicable part of $G$. Since $G^0$ is a divisible abelian $p$-group of finite rank, it follows from [7, (1.3)] that there is a finite normal subgroup $C$ of $G$ contained in $G^0$ such that $G^0/C$ is a divisible product of divisible irreducible $\mathbb{Z}G$-modules, say

$$G^0/C = (G_1/C) \times (G_2/C) \times \cdots \times (G_n/C).$$

Since $G_i/C$ is a $\delta$-chief factor of $G$ for all $i \in \{1, \ldots, n\}$, $G_i/C \leq Z(G/C)$ for all $i \in \{1, \ldots, n\}$, and $G^0/C \leq Z(G/C)$. Hence, the commutator $[G^0, G]$ is contained in $C$ and $[G^0, G]$ is a finite group. Furthermore, applying [5, (3.29.1)], we have that $G^0 = [G^0, A]G^0(A)$. But, since the commutator $[G^0, A]$ is a finite group, it is clear that $G^0 = C_{G^0}(A)$. Therefore, as $G^0$ is abelian and $G = G^0A$, we conclude that $G^0 \leq Z(G)$.

Proof of Theorem 2. By [6, (5.3)], we have that $\mu(G)$ is locally nilpotent. Next we see that every Sylow $p$-subgroup of $\mu(G)$ is nilpotent for each prime $p$. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of $\mu(G)$. By [2, (2.5.13)], we know that $G/O_p(G)$ is a Chernikov group and, by [7, (1.2)], $\mu(G/O_p(G))$ is finite. Since $\mu(G)O_p(G)/O_p(G)$ is contained in $\mu(G/O_p(G))$, it follows that $\mu(G)O_p(G)/O_p(G)$ is finite and so is $PO_p(G)/O_p(G)$. Therefore, $P$ is finite and nilpotent. By Theorem 1, we have that $\mu(G)$ is a $\mathcal{R}$-group.

Proof of Theorem 3. First we prove that every subgroup of a $\mathcal{R}$-group is also a $\mathcal{R}$-group. Let $G$ be a $\mathcal{R}$-group and let $H$ be a subgroup of $G$. Since $G$ is locally nilpotent, we have that $H$ is also locally nilpotent. Let $H_p$ be the Sylow $p$-subgroup of $H$ for each prime $p$. Then $H_p$ is contained in the unique Sylow $p$-subgroup $G_p$ of $G$. By Theorem 1, we have that $G_p$ is nilpotent. Then $H_p$ is also nilpotent for each prime $p$. According to Theorem 1, $H$ is a $\mathcal{R}$-group.
On the other hand, it is clear that $\mathcal{B}$ is closed under taking epimorphic images. Let $\{N_i\}_{i \in I}$ be a collection of normal subgroups of $G \in c\mathcal{F}$ such that $G/N_i \in \mathcal{B}$ for every $i \in I$ and $\bigcap_{i \in I} N_i = 1$. Since $G/N_i$ is locally nilpotent for all $i \in I$, we know that $G$ is locally nilpotent [2, (6.2.11)]. Let $G^0$ be the radicable part of $G$. Then $G^0N_i/N_i$ is contained in $(G/N_i)^0$, which is central in $G/N_i$ for all $i \in I$. Therefore, $[G, G^0] \leq N_i$ for all $i \in I$. This implies that $[G, G^0] = 1$ and $G^0$ is contained in $Z(G)$. Applying Theorem 1, we have that $G$ is a $\mathcal{B}$-group.

**Proof of Theorem 4.** We know from Theorem 1 that $H$ and $K$ are locally nilpotent with nilpotent Sylow subgroups. By [2, (12.1.2)], $HK$ is locally nilpotent. Let $p$ be a prime and let $H_p$ and $K_p$ be the Sylow $p$-subgroups of $H$ and $K$, respectively. Then it is clear that $H_p$ and $K_p$ are normal in $G$. Let $S_p$ be the Sylow $p$-subgroup of $HK$. We have that $S_p \cap H = H_p$, $S_p \cap K = K_p$, and $S_p = H_pK_p$. Therefore, $S_p$ is a nilpotent group by Fitting’s theorem.

**Lemma 3.** Let $G$ be a Chernikov $c\mathcal{F}$-group. Assume that $G = \bigcup_{i \in I} G_i$, where $G_i$ is a normal $\mathcal{B}$-subgroup of $G$ for each $i \in I$ and $\{G_i; i \in I\}$ is a totally ordered set by inclusion. Then $G$ is a $\mathcal{B}$-group.

**Proof.** Let $G^0$ be the radicable part of $G$ and let $A$ be a finite subgroup of $G$ such that $G = G^0A$. If $a \in A$, then there exists $\alpha_a \in I$ such that $a \in G_{\alpha_a}$. Since $A$ is finite, it follows that $\{G_{\alpha_a}; a \in A\}$ is finite and so we can choose a maximal element, $K$ say. From the fact that $\{G_i; i \in I\}$ is totally ordered, we can conclude that $G_{\alpha_a} \leq K$ for each $a \in A$. Therefore, $A$ is contained in $K$ and the result follows from Theorem 4.

**Proof of Theorem 5.** Consider the nonempty set $\mathcal{F} = \{B \triangleleft G; B \in \mathcal{B}\}$. Let $\mathcal{C} = \{B_i; i \in I\}$ be a chain in $\mathcal{F}$. We show that $X = \bigcup_{i \in I} B_i$ is an upper bound for $\mathcal{C}$ in $\mathcal{F}$. First, it is clear that $X$ is a normal subgroup of $G$. By [2, (2.5.13)], $X/O_p(X)$ is a Chernikov group for all primes $p$, and $X/O_p(X)$ is the union of the elements of the chain $\{B_iO_p(X)/O_p(X); i \in I\}$. By Lemma 3, $X/O_p(X)$ is a $\mathcal{B}$-group. Since $\bigcap_p O_p(X) = 1$ and $\mathcal{B}$ is a $c\mathcal{F}$-formation by Theorem 3, we conclude that $X$ belongs to $\mathcal{B}$. It is now clear that $X$ is a subgroup which is an upper bound for $\mathcal{C}$. We may now apply Zorn’s Lemma to produce a maximal element of $\mathcal{F}$, $\delta(G)$ say. Let $N$ be a normal $\mathcal{B}$-subgroup of $G$. By Theorem 4, $\delta(G)N$ is a normal $\mathcal{B}$-subgroup of $G$. The maximality of $\delta(G)$ in $\mathcal{F}$ implies that $\delta(G) = \delta(G)N$ and $N \leq \delta(G)$. Consequently, $\delta(G)$ is the unique largest normal $\mathcal{B}$-subgroup of $G$.

**Proof of Theorem 6.** It is clear that (i) implies (ii) and that (ii) implies (iii).

(iii) implies (i). Let $G$ be a Baer group and consider a major subgroup $M$ of $G$. If $M$ is maximal in $G$, it follows by [4, (12.1.5)] that
$M$ is normal in $G$. Then $G/M_G$ is a cyclic group of order $p$, for some prime $p$, and hence it is abelian. Therefore, $G'/M_G$. Suppose that $M$ is not maximal in $G$. Then $G/M_G$ is a semiprimitive Baer group. We shall prove that $G/M_G$ is abelian. To see this, we may assume that $M_G = 1$. Then $G = [D]M$, where $D$ is a divisibly irreducible abelian $p$-group for some prime $p$ and $M$ is a finite soluble group with trivial core. Denote $M_i = \Omega_i(D)M$, where $\Omega_i(D)$ is the subgroup generated by the elements of $D$ of order dividing $p^i$. Then $G = \langle M_i: i \in \mathbb{N} \rangle$ and $M_i$ is finite and hence subnormal in $G$ for all $i \in \mathbb{N}$ because $G$ is a Baer group. Therefore, there exists a proper normal subgroup $T_i$ of $G$ such that $M_i \leq T_i$. Since $D$ is divisibly irreducible and $T_i = M(T_i \cap D)$, it follows that $T_i$ is finite. Therefore, every subgroup of $T_i$ is subnormal in $T_i$. Consequently, $T_i$ is nilpotent and then $M_i$ is nilpotent for all $i \in \mathbb{N}$. We conclude that $M_i \leq \delta(G)$ for all $i \in \mathbb{N}$ and hence $G$ is a $\mathcal{B}$-group. Consequently, $G$ is abelian by Lemma 1 and then $G/M_G$ is abelian for all major subgroups $M$ of $G$. By Theorem 1, $G$ is a $\mathcal{B}$-group.

(i) implies (iv). Let $G$ be a $\mathcal{B}$-group and let $p$ be a prime. Since $G/O_p^p(G)$ is a Chernikov by [2, (2.5.13)], then, applying Lemma 2, $G/O_p^p(G)$ is nilpotent. Hence, there exists $n_p \in \mathbb{N}$ such that $\gamma_{n_p}(G) = [G, \gamma_{n_p}(G)] \leq O_p^p(G)$ for each prime $p$, where $\gamma_i(G)$ denotes the $i$th term of the lower central series of $G$. Therefore, $\gamma_p^\omega(G) = \cap_{\beta<\omega} \gamma_{n_p}(G) \leq \gamma_{n_p}(G) \leq O_p^p(G)$ for every prime $p$ and then $\gamma_p^\omega(G) = 1$.

(iv) implies (i). $G/O_p^p(G)$ is a Chernikov hypocentral group and then it is nilpotent for every prime $p$. In particular, since $\mathcal{B}$ is a formation, we have that $G$ is a $\mathcal{B}$-group.

Proof of Theorem 7. Let $H/K$ be a $\delta$-chief factor of $G$. We show that $\delta(G)$ is contained in $C_G(H/K)$. Since $\delta(G)K/K \leq \delta(G/K)$, we may assume that $K = 1$. If $H$ is a minimal normal subgroup of $G$, then it follows from [2, (1.3.5)] and [2, (6.2.4)] that the Hirsch–Plotkin radical of $G$, $\rho(G)$, is contained in $C_G(H)$. Consequently, since $\delta(G) \leq \rho(G)$, we have that $\delta(G)$ is also contained in $C_G(H)$ as required. Suppose now that $H$ is a divisibly irreducible $\mathbb{Z}$-module. In particular, $H$ is a divisible subgroup of $G$ and so $H \leq G^0$. On the other hand, $\delta(G)$ is a $\mathcal{B}$-group. Then, applying Theorem 1, we have that $G^0 = (\delta(G))^0 \leq Z(\delta(G))$. Therefore, $\delta(G) \leq C_G(H)$ and we may conclude that $\delta(G) \leq \cap \{C_G(H/K): H/K$ is a $\delta$-chief factor of $G\}$.

Next, we denote $T = \cap \{C_G(H/K): H/K$ is a $\delta$-chief factor of $G\}$. We prove that $T \leq \delta(G)$. Since $T$ is a normal subgroup of $G$, we need only to show that $T$ is in the class $\mathcal{B}$. As it has been said above, $\rho(G)$ is the intersection of the centralizers of all chief factors of $G$ and, consequently, $T$ is contained in $\rho(G)$. It follows that $T$ is a locally nilpotent group. We see that $T$ has nilpotent Sylow subgroups. Let $T_p$ be the Sylow $p$-subgroup of $T$. 


for the prime \( p \). According to [2, (2.5.13)], \( T_p \) is a Chernikov group and \( T^0_p \) has finite rank. We argue by induction on the rank of \( T^0_p \). If \( T^0_p = 1 \), then \( T_p \) is a finite \( p \)-group and, consequently, it is nilpotent. Then we may assume that \( T_p \) is nontrivial. Consider the nonempty set \( \mathcal{F} = \{ A \leq T^0_p : A \text{ is a nontrivial divisible } G \text{-invariant subgroup of } G \} \). Since \( G \) satisfies \( \text{min-p} \) and \( \mathcal{F} \) is a nonempty set of \( p \)-subgroups of \( G \), there exists a minimal element \( A \) in \( \mathcal{F} \). Then \( A \) is a divisibly irreducible \( \mathbb{Z} G \)-module and hence it is a \( \delta \)-chief factor of \( G \). Consequently, \( A \) is contained in the center of \( T \). On the other hand, we have that \( T/A = \bigcap \{C_{G/A}(H/A)/(K/A) : (H/A)/(K/A) \text{ is a } \delta \text{-chief factor of } G/A \} \). Moreover, the rank of \( T^0_p/A = (T_p/A)^0 \) is less than the rank of \( T^0_p \). By induction, \( T_p/A \) is nilpotent (note that \( T_p/A \) is the unique Sylow \( p \)-subgroup of \( T/A \)). Since \( A \leq Z(T_p) \), we have that \( T_p \) is nilpotent. Therefore, \( T \) is a \( \mathcal{B} \)-group, as we wanted to see.

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