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Generalization of Mirsky's theorem on diagonals and eigenvalues of matrices ☆

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ABSTRACT

Mirsky proved that, for the existence of a complex matrix with given eigenvalues and diagonal entries, the obvious necessary condition is also sufficient. We generalize this theorem to matrices over any field and provide a short proof. Moreover, we show that there is a unique companion-matrix-type solution for this problem.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a complex matrix A of order n and $d_1, \ldots, d_n$ are its diagonal elements, then the sum of the λ_i's is necessarily equal to the sum of the d_i's. Mirsky [1] proved that the converse holds. If the data are real numbers, he proved that A can be chosen to be real as well. For a recent short proof of Mirsky's results see [2].

Instead of specifying the eigenvalues of a matrix we shall specify its characteristic polynomial. We shall work over any field F. Let

$$f(t) = t^n + c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \cdots + c_0$$

be a monic polynomial over F.

Theorem 1.1. *Given a sequence $d_1, \ldots, d_n$ in F with $d_1 + \cdots + d_n = -c_{n-1}$, there exists a unique sequence $b_1, \ldots, b_{n-1}$ in F such that $f(t)$ is the characteristic polynomial of the matrix*

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$$A = [a_{ij}] = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 & b_1 \\ 1 & d_2 & 0 & & 0 & b_2 \\ 0 & 1 & d_3 & & 0 & b_3 \\ \vdots & & & & & \\ 0 & 0 & 0 & & d_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & 1 & d_n \end{bmatrix}.$$

Proof. For any $S \subseteq \{1, \ldots, n\}$, let A_S be the submatrix of A whose entries are the a_{ij} with $i, j \in S$, and let $M_S = \det A_S$. It suffices to prove that the system of $n-1$ equations

$$\sum_{S:|S|=n-k+1} M_S = (-1)^{n-k+1} c_{k-1}, \quad k = 1, \ldots, n-1$$

in $n-1$ unknowns $b_1, \ldots, b_{n-1}$ has a unique solution in F. (By $|S|$ we denote the cardinality of S.) This is true because of the following claim: for each k we have

$$\sum_{S:|S|=n-k+1} M_S = (-1)^{n-k} b_k + g_k,$$

where g_k is a polynomial in the unknowns $b_{k+1}, \ldots, b_{n-1}$ only.

To prove this claim, it suffices to show that if b_k occurs in M_S and $|S| \leq n-k+1$ then $S = \{k, \ldots, n\}$. By the hypothesis b_k occurs in M_S, and so $\{k, n\} \subseteq S$ and there must exist a permutation π of S such that $\pi n = k$ and

$$\prod_{i \in S} a_{\pi i, i} \neq 0.$$

Hence, $\pi i \in \{i, i+1\}$ for $i \in S \setminus \{n\}$. As $k \in S$ and $\pi n = k$, we have $\pi k = k+1$. If $k < n-1$ then $\pi(k+1) \neq k+1$ and so $\pi(k+1) = k+2$, etc. By repeating this argument, we conclude that $\{k, \ldots, n\} \subseteq S$. As $|S| \leq n-k+1$, it follows that $S = \{k, \ldots, n\}$. This completes the proof of our claim and the theorem. □

If $d_n = -c_{n-1}$ and all other $d_i = 0$, then A becomes the well known Frobenius companion matrix of $f(t)$:

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & & 0 & -c_1 \\ 0 & 1 & 0 & & 0 & -c_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 0 & -c_{n-2} \\ 0 & 0 & 0 & & 1 & -c_{n-1} \end{bmatrix}.$$

The next theorem, which has much simpler proof, provides explicit formulae for the unknown elements $b_1, \ldots, b_{n-1}$ and establishes the existence assertion of Theorem 1.1, but not the uniqueness.

For $k = 0, 1, \ldots, n-1$, denote by $h_r(d_1, \ldots, d_k)$ the sum of all monomials in $d_1, \ldots, d_k$ of degree r. (In particular $h_0(d_1, \ldots, d_k) = 1$.) Let I_n be the identity matrix of order n.

Theorem 1.2. *If* $d_n = -c_{n-1} - d_1 - \cdots - d_{n-1}$ *and*

$$b_k = -\sum_{i=k-1}^{n} c_i h_{i-k+1}(d_1, \ldots, d_k), \quad k = 1, \ldots, n-1, \tag{1.1}$$

where $c_n = 1$, *then* $\det(tI_n - A) = f(t)$.

Proof. Let $T = [t_{ij}]$ be the upper triangular matrix with entries $t_{ij} = h_{j-i}(d_1, \ldots, d_i)$, $1 \le i \le j \le n$. As all $t_{ii} = 1$, T is invertible. It suffices to verify that $AT = TC$, which is straightforward. □

For example, if $n = 4$ then the above formulae read

$$\begin{aligned}
b_1 &= -c_0 - c_1 d_1 - c_2 d_1^2 - c_3 d_1^3 - d_1^4, \\
b_2 &= -c_1 - c_2(d_1 + d_2) - c_3 \left(d_1^2 + d_1 d_2 + d_2^2\right) - \left(d_1^3 + d_1^2 d_2 + d_1 d_2^2 + d_2^3\right), \\
b_3 &= -c_2 - c_3(d_1 + d_2 + d_3) - \left(d_1^2 + d_2^2 + d_3^2 + d_1 d_2 + d_1 d_3 + d_2 d_3\right).
\end{aligned}$$

References

[1] L. Mirsky, Matrices with prescribed characteristic roots and diagonal elements, J. London Math. Soc. 33 (1958) 14–21.
[2] E.A. Carlen, E.H. Lieb, Short proofs of theorems of Mirsky and Horn on diagonals and eigenvalues of matrices, Electron. J. Linear Algebra 18 (2009) 438–441.