Computers and Mathematics with Applications 59 (2010) 1718–1726

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# A note on the finite element method for the space-fractional advection diffusion equation[☆]

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ARTICLE INFO

*Keywords:*
Space-fractional partial differential equation
Caputo derivative
Finite element method

ABSTRACT

In this paper, a note on the finite element method for the space-fractional advection diffusion equation with non-homogeneous initial-boundary condition is given, where the fractional derivative is in the sense of Caputo. The error estimate is derived, and the numerical results presented support the theoretical results.



## 1. Introduction

Fractional calculus, a mathematical topic developed in the 17th century, did not attract much attention until recent decades. It was recently found that many systems in various fields can be modeled with the help of fractional derivatives, such as in viscoelastic mechanics, the power-law phenomenon in fluids and complex networks, allometric scaling laws in biology and ecology, colored noise, electrode–electrolyte polarization, dielectric polarization, boundary layer effects in ducts, electromagnetic waves, quantitative finance, quantum evolution of complex systems, and fractional kinetics ([1,2] and many references cited therein).

In this paper, we mainly study one kind of typical fractional partial differential equations by using the finite element method, which reads in the following form:

$$\begin{cases} u_t + au_x + bD_x^{\alpha}u + cu = f, \quad \alpha \in (1,2], x \in \Omega, t \in (0,T], \\ u|_{t=0} = \varphi(x), \quad x \in \Omega, \\ u|_{\partial\Omega} = g, \quad t \in (0,T], \end{cases} \tag{1}$$

where $\Omega$ is a domain with boundary $\partial\Omega$, time $T > 0$, the anomalous diffusion item $D_x^{\alpha}u$ is an $\alpha$-th order fractional derivative of $u$ with respect to the space variable $x$ in the Caputo sense (which will be introduced later on), $u_x$ is the advection item, and $a, b, c, f, g$ are functions of $x, t$ which satisfy the conditions requested by the theorem of error estimates.

Considering Eq. (1), if $\alpha \in Z^+$, it is just the classical partial differential equation; in particular, if $\alpha = 2$ and $f = 0$, it often represents the typical Fokker–Planck equation which is commonly used to describe the Brownian motion of particles. If $\alpha = 2$, $c \in R$, it is also the traditional form of the advection dispersion equation. And if $\alpha$ is not an integer, it always relates to the anomalous diffusion phenomenon.

In general, three kinds of fractional derivatives are used: the Grwünwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative. Their definitions are introduced below.

☆ This work was partially supported by the Natural Science Foundation of China under grant no. 10872119, Shanghai Leading Academic Discipline Project under grant no. S30104. and Huainan Normal University under grant no. 2007LKQ07.

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doi:10.1016/j.camwa.2009.08.071

**Definition 1.** The $\alpha$-th order Riemann–Liouville integral of function $u(x)$ is defined as follows:

$$I^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{u(s)}{(x-s)^{1-\alpha}}\mathrm{d}s,$$

where $\alpha>0$.

**Definition 2.** The fractional left derivative of Grwünwald–Letnikov type is given as follows:

$${}_GD^{\alpha}u(x)=\lim_{\substack{n\to+\infty\\ nh=x}}h^{-\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}f(x-kh).$$

This limit expression is not convenient to use. However, if $x\in C^{n+1}(\Omega)$, the above limit definition is equal to

$${}_GD^{\alpha}u(x)=\sum_{k=0}^{n}\frac{u^{(k)}(0)x^{k-\alpha}}{\Gamma(k+1-\alpha)}+\frac{1}{\Gamma(n+1-\alpha)}\int_0^x\frac{u^{(n+1)}(t)}{(x-t)^{-n+\alpha}}\mathrm{d}t,$$

where $n-1\le\alpha<n\in Z^+$. If $\alpha>1$, then the Laplace transform of the Grwünwald–Letnikov fractional derivative does not exist in the classical sense [3]. From a pure mathematics point of view such a class of functions is narrow.

**Definition 3.** The $\alpha$-th order Riemann–Liouville derivative of function $u(x)$ is defined as follows:

$$\mathbf{D}_x^{\alpha}u(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}I^{n-\alpha}u(x),\quad n-1<\alpha<n\in Z^+.$$

In this definition, the condition of $u(x)$ is weakened, and it is convenient to use and easy to connect with the typical derivative. But from the definition we can see that, if $u$ is a non-zero constant in $\Omega$, then $\mathbf{D}_x^{\alpha}u(x)\neq0$, which is different from the integer derivative. When we investigate the equations involving $\mathbf{D}_x^{\alpha}u(x)$, we need to know the values $\mathbf{D}_x^{\alpha-i}u(x)|_{x=0}$, $i=1,2,\ldots,n$. Although attempts have been made to give some interpretations of them, for example, see [4], not all these fractional derivatives have clear practical and physical meanings for physical measurements to be obtained. In order to overcome these shortcomings, a practical fractional derivative was introduced, which is as follows. It is convenient to model the fractional phenomenon by using this kind of definition since its initial value problem is the same as the classical initial value problem.

**Definition 4.** The $\alpha$-th order Caputo derivative of function $u(x)$ is defined as

$$D_x^{\alpha}u(x)=I^{n-\alpha}\frac{\mathrm{d}^nu(x)}{\mathrm{d}x^n},\quad n-1<\alpha<n\in Z^+.$$

From this definition we can see that, if $u$ is a constant in $\Omega$, then $D_x^{\alpha}u(x)=0$; this property is the same as that of the integer order derivative. It is known that, if

$$u(0)=u'(0)=\cdots=u^{(n-1)}(0)=0,$$

then

$$D_x^{\alpha}u(x)=\mathbf{D}_x^{\alpha}u(x).$$

Here, we do not concern ourselves with the existence and uniqueness of a solution to the considered equation (work on the existence and uniqueness of $u$ satisfying (1) can be found in [5]), but focus on the finite element method for solving the equation. So we always assume that there exists a sufficiently regular solution $u(x,t)$.

The rest of this paper is constructed as follows. In Section 2 the preliminary knowledge regarding fractional derivative space is introduced. The error estimates of the finite element method for solving Eq. (1) are studied in Section 3. And in Section 4, numerical examples are taken to confirm the theoretical results derived in Section 3.

## 2. Fractional derivative space

The following notations are used. The $L^p(\Omega)$ inner product is denoted by $(\cdot,\cdot)$, and the $L^p(\Omega)$ norm by $\|\cdot\|_{L^p}$, with the special cases $L^2(\Omega)$ and $L^{\infty}(\Omega)$ norms being denoted as $\|\cdot\|$ and $\|\cdot\|_{\infty}$. We denote the norm associated with the Sobolev space $W^{k,p}(\Omega)$ by $\|\cdot\|_{W^{k,p}}$, with a special case $W^{k,2}(\Omega)$ being rewritten as $H^k(\Omega)$ with norm $\|\cdot\|_{H^k}$ or $\|\cdot\|_k$ and semi-norm $|\cdot|_{H^k}$ or $|\cdot|_k$. For the definition of fractional order Sobolev spaces $W^{s,p}(\Omega)$, $s\in R^+\setminus Z^+$, we use the interpolation between two Banach spaces; see [6].

**Definition 5.** Let $0<\beta<1$, and $J^{\alpha}(\Omega)$ be a fractional dimensional space, defined below:

$$J^{\alpha}(\Omega)=\{u\in H^k|D^{\beta}u,D^{\beta*}u\in H^k\},\quad\alpha=\beta+k,$$

endowed with a semi-norm

$$|u|_{J^{\alpha}(\Omega)}=\|D^{\alpha}u\|_{L^2(\Omega)},$$

and a norm

$$\|u\|_{J^\alpha(\Omega)} = (\|u\|^2_{L^2(\Omega)} + |u|^2_{J^\alpha(\Omega)})^{\frac{1}{2}},$$

and let $J_0^\alpha(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J^\alpha(\Omega)}$ or $\|\cdot\|_\alpha$. Here $D^{\beta *}$ means the right fractional derivative in the Caputo sense.

**Definition 6.** For $u(\cdot, t)$ being defined on the entire time interval $(0, T]$, we define the norms

$$\|u\|_{\infty,k} = \sup_{0<t<T} \|u(\cdot, t)\|_k,$$

$$\|u\|_{0,k} = \left(\int_0^T \|u(\cdot, t)\|_k^2 \mathrm{d}t\right)^{1/2},$$

$$\|u\|(t) = \|u(\cdot, t)\|.$$

We present some inequalities on norms for Sobolev spaces which will be used later on.

**Lemma 1.** *Let $\Omega \subset R^d$ be bounded, $\partial\Omega \in C^1$. Then for $u$ and $v$ such that the given norms are finite, we have*

$$\|uv\| \le C \cdot \begin{cases} \|u\|_s \|v\|_{d/2-s}, & 0 < s < d/2, \\ \|u\|_\infty \|v\|, \\ \|u\|_s \|v\|, & s > d/2. \end{cases}$$

**Lemma 2.** *Let $\{S_h\}$ denote a family of partitions of $\Omega$, with grid parameter $h$. Associated with $S_h$ we let $X_h(\Omega)$ represent the finite-dimensional vector space of continuous piecewise linear polynomials. We denote by $u_h$ the continuous piecewise linear approximation to $u$. Then the following approximation properties hold.*

(1) [7] *For $u \in J^\alpha(\Omega)$, there is a constant $C$ such that, for all $0 < \beta < \alpha$,*

$$\|u - u_h\|_\beta \le Ch^{\alpha-\beta}\|u\|_\alpha.$$

(2) [8] *For $u \in J^\alpha(\Omega)$, there exists a constant $C$ such that, for all $\alpha - 1 < k < \alpha$,*

$$|u - u_h|_k \le Ch^{\alpha-k}|u - u_h|_\alpha.$$

**Lemma 3** (*[9] Fractional Poincaré–Friedrichs*). *For $u \in H_0^\alpha(\Omega)$, we have*

$$\|u\|_{L^2(\Omega)} \le C|u|_{H_0^\alpha(\Omega)},$$

*and for $0 < s < \alpha, s \ne n - 1/2, n \in Z^+$,*

$$|u|_{H_0^s(\Omega)} \le C|u|_{H_0^\alpha(\Omega)}.$$

It immediately follows from Lemma 3 that

**Lemma 4.** *For $u \in J_0^\alpha(\Omega)$, we have*

$$\|u\|_{L^2(\Omega)} \le C\|u\|_{H_0^\alpha(\Omega)},$$

$$\|u\|_{L^2(\Omega)} \le C\|u\|_{J_0^\alpha(\Omega)},$$

*and for $0 < s < \alpha$,*

$$\|u\|_{H_0^s(\Omega)} \le C\|u\|_{H_0^\alpha(\Omega)}.$$

## 3. A priori error estimate

Now we discuss the variational solution of Eq. (1). We first introduce a lemma.

**Lemma 5.** *For $u \in J_0^\alpha(\Omega), 0 < \beta < \alpha$, one has*

$$\mathbf{D}_x^\alpha u = \mathbf{D}_x^\beta \mathbf{D}_x^{\alpha-\beta} u, \qquad D_x^\alpha u = D_x^\beta D_x^{\alpha-\beta} u.$$

If the boundary condition of problem (1) is inhomogeneous,

$$g(x, t) \ne 0, \quad x \in \partial\Omega,$$

where $g \in H^{\alpha/2-1/2}(\partial\Omega)$ is pre-assumed. According to the theorem of a trace operator [10], there exists a $u_g \in J^{\alpha-1}$ such that

$$\gamma_0 u_g = g, \quad x \in \partial\Omega.$$

Now, let $u = v + u_g$; then Eq. (1) can be changed into an equation with a homogeneous boundary condition:

$$\begin{cases} v_t + av_x + bD_x^\alpha v + cv = f(x) - \dfrac{\partial u_g}{\partial t} - a\dfrac{\partial u_g}{\partial x} - bD_x^\alpha u_g - cu_g, & x \in \Omega, t \in (0, T], \\ v(x, 0) = \varphi(x) - u_g(x, 0), & x \in \Omega, \\ v(x, t) = 0, & x \in \partial\Omega, t \in (0, T], \end{cases} \tag{2}$$

where $a, b, c, f, u_g$ satisfy suitable conditions for the error estimates.

Now we select a test function $w \in J_0^{\alpha/2}(\Omega)$, multiply the first equation of Eq. (2) by $w$, and integrate over $\Omega$; then we have

$$(v_t, w) + (av_x, w) + (bD_x^\alpha v, w) + (cv, w) = \langle f, w\rangle - \left(\frac{\partial u_g}{\partial t}, w\right) - \left(a\frac{\partial u_g}{\partial x}, w\right) - (bD_x^\alpha u_g, w) - (cu_g, w).$$

Using Lemma 5 yields

$$\begin{aligned} &(v_t, w) - (v, a_x w) - (v, aw_x) - (D_x^{\alpha-1} v, (bw)_x) + (cv, w) \\ &\quad = \langle f, w\rangle - \left(\frac{\partial u_g}{\partial t}, w\right) - \left(a\frac{\partial u_g}{\partial x}, w\right) - (bD_x^\alpha u_g, w) - (cu_g, w). \end{aligned}$$

So the problem (2) is equal to the following variational problem.

**Definition 7.** There exists a function $v \in J_0^{\alpha/2}(\Omega)$, which is subject to

$$A(v, w) = F(w) - B(u_g, w), \quad \forall w \in \forall J_0^{\alpha/2}(\Omega) \tag{3}$$

where

$$\begin{aligned} &A(v, w) = (v_t, w) - (v, a_x w) - (v, aw_x) - (D_x^{\alpha-1} v, (bw)_x) + (cv, w), \\ &F(w) = \langle f, w\rangle, \\ &B(u_g, w) = \left(\frac{\partial u_g}{\partial t}, w\right) + \left(\frac{\partial u_g}{\partial x}, w\right) + (D_x^\alpha u_g, bw) + (cu_g, w). \end{aligned}$$

Here we assume that the following conditions hold.

$$\begin{aligned} &(D^{2\alpha} v, w) \le C_0 \|v\|_\alpha \|w\|_\alpha, \\ &(D^{2\alpha} v, v) \ge C_1 \|v\|_\alpha^2. \end{aligned} \tag{4}$$

Let $T_h$ be a partition of $\Omega$, for $k \in N$, and $P_k(\Omega)$ denote the space of polynomials on $\Omega$ of degree no greater than $k$; then we define the finite element space $X_h$ as follows:

$$\begin{aligned} &X_h = \{v_h : v_h|_T \in p_k(T), \forall T \in T_h, v_h \in C^0(\Omega)\}, \\ &X_{0h} = \{v_h \in X_h : v_h(Q) = 0, \forall Q \in \partial\Omega\}. \end{aligned}$$

Let $\Delta t$ denote the step size of the time domain, and let $t_n = n\Delta t, n = 0, 1, 2, \ldots, N$. For convenience, we set $v^n = v(\cdot, t_n), n = 0, 1, 2, \ldots N$. Now we apply the difference method to approximate the time derivative:

$$\frac{v_h^n - v_h^{n-1}}{\Delta t} - (v_h^n, a_x w) - (v_h^n, aw_x) - (D_x^{\alpha-1} v_h^n, (bw)_x) + (cv_h^n, w) = \langle f^n, w\rangle - B(u_g^n, w). \tag{5}$$

**Lemma 6** ([11]). *Let $T_h, 0 < h \le 1$, denote a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subset R^d$. Let $(K', P, N)$ be a reference finite element such that $P \subset W^{l,p}(K') \cap W^{m,q}(K')$ is a finite-dimensional space of functions on $K'$, $N$ is a basis for $P'$, where $1 \le p \le \infty$, $1 \le p \le \infty$, and $0 \le m \le l$. For $K \in T_h$, let $(K, P_K, N_K)$ be the affine equivalent element, $V_h = v : v$ be measurable and $v|_K \in P_K, \forall K \in T_h$. Then there exists a constant $C = C(l, p, q)$ such that*

$$\left[\sum_{k\in T_h} \|v\|_{W^{l,p}(K)}^p\right]^{1/p} \le Ch^{m-l+\min(0,d/p-d/q)} \cdot \left[\sum_{k\in T_h} \|v\|_{W^{m,q}(K)}^q\right]^{1/q}. \tag{6}$$

**Lemma 7** ([12]). *Let $\Delta t, H$ and $a_n, b_n, c_n, \gamma_n$ (for integer $n \ge 0$) be non-negative numbers such that*

$$a_N + \Delta t \sum_{n=0}^N b_n \le \Delta t \sum_{n=0}^N \gamma_n a_n + \Delta t \sum_{n=0}^N c_n + H \tag{7}$$

*for $N \geq 0$. Suppose that $\Delta t\gamma_n < 1$, for all n, and set $\sigma_n = (1-\Delta t\gamma_n)^{-1}$. Then*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \leq \exp\left(\Delta t \sum_{n=0}^{N} \sigma_n \gamma_n\right)\left\{\Delta t \sum_{n=0}^{N} c_n + H\right\} \tag{8}$$

*for $N \geq 0$.*

The priori error estimates for the approximation are given in Theorem 1.

**Theorem 1.** *Assume that (3) has a solution u, satisfying $u_{tt} \in L^2(0,T,L^2(\Omega))$ with $\varphi(x) \in H^{k+1}(\Omega)$. If $\Delta t \leq ch$, then the finite element approximation is convergent to the solution (2) on the interval $(0,T]$, as $\triangle t, h \to 0$. The approximation $u_h$ satisfies the following error estimates:*

$$\|u-u_h\|_{0,\alpha/2} \leq C(h^{k+1}\|u_t\|_{0,k+1} + h^{k+1-\frac{\alpha}{2}}\|u\|_{0,k+1} + h^{k+\frac{1}{2}}\|u\|_{0,k+1} + \Delta t\|u_{tt}\|_{0,0} + \Delta t), \tag{9}$$

$$\|u-u_h\|_{\infty,0} \leq C(h^{k+1}\|u_t\|_{0,k+1} + h^{k+1-\frac{\alpha}{2}}\|u\|_{\infty,k+1} + h^{k+\frac{1}{2}}\|u\|_{\infty,k+1} + \Delta t\|u_{tt}\|_{0,0} + \Delta t). \tag{10}$$

**Proof of Theorem 1.** In order to estimate the error, we first discuss the error at $t = t_n$.

Defining $\varepsilon^n = u^n - u_h^n$, we have that the true solution $u$ at $t = t_n$ satisfies

$$\begin{aligned}&(u_t^n, w) + (cu^n, w) - (u^n, wa_x) - (u^n, aw_x) + (D_x^{\alpha}u^n, bw)\\ &\quad= \langle f^n, w\rangle - \left(\frac{\partial u_g^n}{\partial t}, w\right) - \left(\frac{\partial u_g^n}{\partial x}, aw\right) + (D_x^{\alpha-1}u_g^n, bw_x) + (D_x^{\alpha-1}u_g^n, wb_x),\end{aligned} \tag{11}$$

where $w \in X_{0h}$. Subtracting (5) from (11), we obtain the following equation for $\varepsilon^n$:

$$\begin{aligned}&\left(u_t^n - \frac{u_h^n - u_h^{n-1}}{\Delta t}, w\right) + (c\varepsilon^n, w) - (\varepsilon^n, wa_x) - (\varepsilon^n, aw_x) + (D_x^{\alpha}\varepsilon^n, bw)\\ &\quad= \left(u_t^n - \frac{u_h^n - u_h^{n-1}}{\Delta t} + \frac{u^n - u^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, w\right) - (\varepsilon^n, aw_x) + (c\varepsilon^n, w) - (\varepsilon^n, wa_x) + (D_x^{\alpha}\varepsilon^n, bw)\\ &\quad= \left(u_t^n - \frac{u^n - u^{n-1}}{\Delta t}, w\right) + \left(\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, w\right) + (c\varepsilon^n, w) - (\varepsilon^n, wa_x) - (\varepsilon^n, aw_x) + (D_x^{\alpha}\varepsilon^n, bw)\\ &\quad= 0.\end{aligned} \tag{12}$$

Assume that $a, b, c$ satisfy our discussion. Now let

$M = \max\{\|a^n\|_k, \|a_x^n\|_k, \|b^n\|_k, \|c^n\|_k\}$.

Let $\Lambda^n = u^n - V^n$, $E^n = V^n - u_h^n$, in which $V^n \in X_h$; then $\varepsilon^n = \Lambda^n + E^n$. Also, let $w = E^n$. We have

$$\begin{aligned}(\varepsilon^n, a^nE_x^n) &= (-D_x^{1/2}a^n(E^n + \Lambda^n), D_x^{1/2*}E^n)\\ &\leq (\|a^nE^n\|_{1/2} + \|a^n\Lambda^n\|_{1/2})\|E^n\|_{1/2}\\ &\leq \|a^n\| \cdot (\|E^n\|_{1/2} + \|\Lambda^n\|_{1/2})\|E^n\|_{1/2}\\ &\leq C_1M(\|E^n\|_{1/2}^2 + \|\Lambda^n\|_{1/2}^2)\\ &\leq C_1Mh^{\alpha/2-1/2}h^{\alpha/2-1/2}(\|E^n\|_{\alpha/2}^2 + \|\Lambda^n\|_{\alpha/2}^2)\\ &= C_1Mh^{\alpha-1}(\|E^n\|_{\alpha/2}^2 + \|\Lambda^n\|_{\alpha/2}^2).\end{aligned} \tag{13}$$

$(\varepsilon^n, E^na_x^n) = (E^n + \Lambda^n, E^na_x^n)$,

and it follows that

$$(E^n, E^na_x^n) \leq C_2M\|E^n\|^2, \tag{14}$$

$$(\Lambda^n, E^na_x^n) \leq \|a_x^n\| \cdot \|\Lambda^n\| \cdot \|E^n\| \leq C_3M(\|\Lambda^n\|^2 + \|E^n\|^2). \tag{15}$$

$$(D_x^{\alpha}\varepsilon^n, b^nE^n) = (D_x^{\alpha}E^n, b^nE^n) + (D_x^{\alpha}\Lambda^n, b^nE^n), \tag{16}$$

in which

$$(D_x^{\alpha}E^n, b^nE^n) \geq \frac{C_4}{M}\|E^n\|_{\alpha/2}^2. \tag{17}$$

$$(D_x^{\alpha}\Lambda^n, b^nE^n) \leq \frac{C_5}{M}(\|E^n\|_{\alpha/2}^2 + \|\Lambda^n\|_{\alpha/2}^2). \tag{18}$$

$(\varepsilon_t^n, E^n) = (E_t^n + \Lambda_t^n, E^n)$;

then one has

$$(E_t^n, E^n) = \left(\frac{E^{n+1} - E^n}{\Delta t}, E^n\right) \geq \frac{1}{2\Delta t}(\|E^{n+1}\|^2 - \|E^n\|^2), \tag{19}$$

$$(\Lambda_t^n, E^n) \leq \frac{1}{2}(\|E^n\|^2 + \|(\Lambda^{n+1} - \Lambda^n)/\Delta t\|^2). \tag{20}$$

$$\begin{aligned}\left(u_t^n - \frac{u^n - u^{n-1}}{\Delta t}, E^n\right) &\leq \left\|u_t^n - \frac{u^n - u^{n-1}}{\Delta t}\right\| \|E^n\| \\ &\leq \frac{1}{2}\left(\left\|u_t^n - \frac{u^n - u^{n-1}}{\Delta t}\right\|^2 + \|E^n\|^2\right).\end{aligned} \tag{21}$$

and

$$\begin{aligned}\left(u_t - \frac{u^n - u^{n-1}}{\Delta t}\right)^2 &= \left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u_{tt}(\cdot, t)(t_{n-1} - t)\right)^2 \\ &\leq \left(\frac{1}{\Delta t}\right)^2 \int_{t_{n-1}}^{t_n} u_{tt}^2(\cdot, t)\mathrm{d}t \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2\mathrm{d}t \\ &\leq \Delta t \int_{t_{n-1}}^{t_n} u_{tt}^2\mathrm{d}t.\end{aligned} \tag{22}$$

So

$$\left\|u_t^n - \frac{u^n - u^{n-1}}{\Delta t}\right\|^2 = \Delta t \int_{t_{n-1}}^{t_n}\int_{\Omega} u_{tt}^2\mathrm{d}t\mathrm{d}x \leq C_6\Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}\|^2\mathrm{d}t. \tag{23}$$

Now we consider the norms over the time domain $(0, T]$. We first multiply (12) by $\Delta t$ then sum from $n = 1$ to $N$, and apply (13)–(18) and (20)–(22); we have

$$\begin{aligned}&\frac{1}{2}(\|E^N\|^2 - \|E^0\|^2) + \sum_{n=1}^{N}\frac{\Delta t}{M}C_4\|E^n\|_{\alpha/2}^2 \leq \sum_{n=1}^{N}((C_2 + C_3)M + 1)\Delta t\|E^n\|^2 + (\Delta t)^2\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} C_6\|u_{tt}\|^2\mathrm{d}t \\ &\quad + \sum_{n=1}^{N}\frac{C_5}{M}\Delta t\|E^n\|_{\alpha/2}^2 + h^{\alpha-1}\sum_{n=1}^{N}C_1M\Delta t\|E^n\|_{\alpha/2}^2 + \sum_{n=1}^{N}C_3M\Delta t\|\Lambda^n\|^2 \\ &\quad + \sum_{n=1}^{N}\frac{C_5}{M}\Delta t\|\Lambda^n\|_{\alpha/2}^2 + h^{\alpha-1}\sum_{n=1}^{N}C_1M\Delta t\|\Lambda^n\|_{\alpha/2}^2 + \sum_{n=1}^{N}\Delta t\left\|\frac{\Lambda^{n+1} - \Lambda^n}{\Delta t}\right\|^2.\end{aligned}$$

Noting that $\|E^0\|^2 = 0$, and setting $(0 <)C_7 \leq (C_4 - C_5)/M - h^{\alpha-1}C_1M$, we have

$$\begin{aligned}&\frac{1}{2}\|E^N\|^2 + \sum_{n=1}^{N}C_7\Delta t\|E^n\|_{\alpha/2}^2 \\ &\quad \leq \frac{1}{2}\|E^N\|^2 + \sum_{n=1}^{N}(C_4 - C_5)/M\Delta t\|E^n\|_{\alpha/2}^2 - h^{\alpha-1}\sum_{n=1}^{N}C_1M\Delta t\|E^n\|_{\alpha/2}^2 \\ &\quad \leq \sum_{n=1}^{N}(M(C_2 + C_3) + 1)\Delta t\|E^n\|^2 + \sum_{n=1}^{N}C_3M\Delta t\|\Lambda^n\|^2 \\ &\qquad + \sum_{n=1}^{N}C_5/M\;\Delta t\|\Lambda^n\|_{\alpha/2}^2 + h^{\alpha-1}\sum_{n=1}^{N}C_1M\Delta t\|\Lambda^n\|_{\alpha/2}^2 \\ &\qquad + \sum_{n=1}^{N}\Delta t\left\|\frac{\Lambda^{n+1} - \Lambda^n}{\Delta t}\right\|^2 + C_6(\Delta t)^2\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2\mathrm{d}t \\ &\quad \leq \sum_{n=1}^{N}\frac{\Delta t}{M}C_8\|E^n\|^2 + C_9h^{2(k+1)}\|u_t\|_{0,k+1}^2 + C_{10}h^{2(k+1-\frac{\alpha}{2})}\|u\|_{0,k+1}^2\end{aligned}$$

$$+C_{11}h^{2k+1}\|u\|^2_{0,k+1}+C_{12}(\Delta t)^2\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2\mathrm{d}t,$$

in which the interpolation error

$$\sum_{n=1}^{N}\Delta t\|\Lambda^n\|^2_{\alpha/2}\le C_{10}h^{2(k+1-\frac{\alpha}{2})}\|u\|^2_{0,k+1}.$$

$$\begin{aligned}\sum_{n=1}^{N}\Delta t\left\|\frac{\Lambda^{n+1}-\Lambda^n}{\Delta t}\right\|^2 &= \sum_{n=1}^{N}\Delta t\left\|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}1\cdot\frac{\partial\Lambda}{\partial t}\right\|^2\\ &\le \sum_{n=1}^{N}\Delta t\frac{1}{\Delta t^2}\int_{\Omega}\left(\int_{t_{n-1}}^{t_n}1\mathrm{d}t\right)\left(\int_{t_{n-1}}^{t_n}\left(\frac{\partial\Lambda}{\partial t}\right)^2\mathrm{d}t\right)\mathrm{d}x\\ &\le C_9h^{2k+2}\|u_t\|^2_{0,k+1}.\end{aligned}\tag{24}$$

Using Lemma 7, and supposing that $a_N=\|E^N\|$, $b_N=\|E^N\|_{\frac{\alpha}{2}}$, $\gamma_N=C_8$, $H=C_9h^{2(k+1)}\|u_t\|^2_{0,k+1}+C_{10}h^{2(k+1-\frac{\alpha}{2})}\|u\|^2_{0,k+1}+C_{11}h^{2k+1}\|u\|^2_{0,k+1}+C_{12}\Delta t^2\|u_{tt}\|^2_{0,0}$, we have

$$\|E^N\|^2\le C_9h^{2(k+1)}\|u_t\|^2_{0,k+1}+C_{10}h^{2(k+1-\frac{\alpha}{2})}\|u\|^2_{0,k+1}+C_{11}h^{2k+1}\|u\|^2_{0,k+1}+C_{12}\Delta t^2\|u_{tt}\|^2_{0,0}.$$

Thus

$$\begin{aligned}\|\varepsilon^n\|^2_{0,\alpha/2} &= \sum_{n=1}^{N}\Delta t\|\varepsilon^n\|^2_{\alpha/2}+O(\Delta t^2)\\ &\le \sum_{n=1}^{N}\Delta t(\|E^n\|^2_{\alpha/2}+\|\Lambda^n\|^2_{\alpha/2})+O(\Delta t^2)\\ &\le C(T+1)\|E^N\|^2_{\alpha/2}+h^{2(k+1-\alpha/2)}\|u\|^2_{0,k+1}+O(\Delta t^2)\\ &\le C_{13}h^{2(k+1)}\|u_t\|^2_{0,k+1}+C_{14}h^{2(k+1-\frac{\alpha}{2})}\|u\|^2_{0,k+1}+C_{15}h^{2k+1}\|u\|^2_{0,k+1}+C_{16}(\Delta t)^2\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|u_{tt}\|^2\mathrm{d}t+O(\Delta t^2)\\ &\le C(h^{2k+2}\|u_t\|^2_{0,k+1}+h^{2(k+1-\frac{\alpha}{2})}\|u\|^2_{0,k+1}+h^{2k+1}\|u\|^2_{0,k+1}+(\Delta t)^2\|u_{tt}\|^2_{0,0}+O(\Delta t^2)).\end{aligned}$$

So, we have the result:

$$\|u-u_h\|_{0,\alpha/2}\le C(h^{k+1}\|u_t\|_{0,k+1}+h^{k+1-\frac{\alpha}{2}}\|u\|_{0,k+1}+h^{k+\frac{1}{2}}\|u\|_{0,k+1}+\Delta t\|u_{tt}\|_{0,0}+O(\Delta t)).$$

Hence (7) is derived. The estimate for $\|u\|_{\infty,0}$ can be derived similarly. □

## 4. Numerical examples

In this section, we present numerical results for the Galerkin approximations which support the theoretical analysis in Section 3.

Let $S_h$ denote a uniform partition on $[0,a]$, and $X_h$ the space of continuous piecewise linear functions on $S_h$, i.e. $k=1$. In order to implement the Galerkin finite element approximation, we adapt finite element discretization along the space axis, and use a finite difference scheme along the time axis. We associate the shape function of space $X_h$ with the standard basis of hat functions on the uniform grid of size $h=\frac{1}{n}$. For $\phi_i$, $i=0,\ldots,n$, there exists a closed-form expression for $D^{\alpha-1}\phi_i$, and we note that the fractional derivatives are non-local, with the support of $\phi_i$ being the interval $((i-1)/n,a]$. For this choice regarding $X_h$ and $X_{0h}$ the approximation property holds, and we have the predicted rates of convergence for $\Delta t=ch^2$ of

$$\|u-u_h\|_{0,0}\sim O(h^2),$$
$$\|u-u_h\|_{\infty,0}\sim O(h^2),$$

if $\varphi(x)$ is smooth enough.

**Example 1.** We consider $u(x,t)=\mathrm{e}^{-t}x^2(2-x)^2$ as the exact solution of the equation in [13]

$$\begin{cases}\dfrac{\partial u}{\partial t}=-u(x,t)+D_x^{1.7}u(x,t)+f(x,t), & 0\le x\le 2,\ 0\le t\le 1,\\ u(x,0)=x^2(2-x)^2, & 0\le x\le 2,\\ u(0,t)=u(2,t)=0, & 0\le t\le 1,\end{cases}\tag{25}$$

**Table 1**
Numerical error results for Example 1.

| $h$ | $\|u-u_h\|_{\infty,0}$ | Convergence rate | $\|u-u_h\|_{0,0}$ | Convergence rate |
|---|---|---|---|---|
| 1/10 | 3.3580E−003 | – | 9.3331E−004 | – |
| 1/20 | 8.2381E−004 | 2.0272 | 2.0872E−004 | 2.1607 |
| 1/40 | 2.0499E−004 | 2.0067 | 5.0666E−005 | 2.0425 |
| 1/80 | 5.1190E−005 | 2.0016 | 8.8898E−006 | 2.5107 |
| 1/160 | 1.2793E−005 | 2.0004 | 3.1371E−006 | 1.5027 |
| 1/320 | 3.1982E−006 | 2.0001 | 7.8391E−007 | 2.0006 |

**Table 2**
Numerical error results for Example 2.

| $h$ | $\|u-u_h\|_{\infty,0}$ | Convergence rate | $\|u-u_h\|_{0,0}$ | Convergence rate |
|---|---|---|---|---|
| 1/5 | 3.8322E−006 | – | 7.8445E−007 | – |
| 1/10 | 1.0061E−006 | 1.9294 | 1.76363E−007 | 2.1531 |
| 1/20 | 2.5030E−007 | 2.0070 | 4.2823E−008 | 2.0420 |
| 1/40 | 6.2662E−008 | 1.9980 | 1.0626E−008 | 2.0107 |
| 1/80 | 1.5705E−008 | 1.9963 | 2.6516E−009 | 2.0027 |
| 1/160 | 3.9638E−009 | 1.98630 | 6.6271E−0010 | 2.0004 |

where

$$f(x,t)=\frac{\mathrm{e}^{-t}}{\cos(0.85\pi)}\left[\frac{24(x^{2.3}+(2-x)^{2.3})}{\Gamma(3.3)}-\frac{24(x^{1.3}+(2-x)^{1.3})}{\Gamma(2.3)}-\frac{8(x^{0.3}+(2-x)^{0.3})}{\Gamma(1.3)}\right].$$

If we select $\Delta t=ch^2$ and note that $u^0$ is smooth enough, then we have

$$\|u-u_h\|_{0,0.85}\sim O(h^2).$$
$$\|u-u_h\|_{\infty,0}\sim O(h^2).$$

Table 1 includes numerical calculations over a regular partition of [0,2]. We can see from it that the smaller the size of the grid, the better the finite element approximation. And we can observe that the experimental rates of convergence agree very well with the theoretical rates of 2 for the numerical solution.

**Example 2.** We consider another space-fractional differential equation. $u(x,t)=\mathrm{e}^{-t}x^3$ is the exact solution of the following equation in [14]:

$$\begin{cases}\dfrac{\partial u}{\partial t}=\dfrac{1}{6}\Gamma(2.2)x^{2.8}D_x^{1.8}u(x,t)+f(x,t), & 0\le x\le 1, 0\le t\le 1,\\ u(x,0)=x^3, & 0\le x\le 1,\\ u(0,t)=0,\quad u(1,t)=\mathrm{e}^{-t}, & 0\le t\le 1,\end{cases}$$

where

$$f(x,t)=-(1+x)\mathrm{e}^{-t}x^3.$$

We select $\Delta t=ch^2$, and we have

$$\|u-u_h\|_{0,0.9}\sim O(h^2).$$
$$\|u-u_h\|_{\infty,0}\sim O(h^2).$$

Table 2 show that error results for different sizes of space grid. We can still observe that the experimental rates of convergence are near to the theoretical rates of 2 for the numerical solution.

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