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Spaces of Lattice Diagram Polynomials in One Set of Variables

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For a partition μ of n, let M_μ be the space span of all partial derivatives of the alternate associated to μ in two n-sets of variables X and Y. The $n!$-theorem of [M. Haiman, *J. Amer. Math. Soc.* **14** (2001), 941–1006] states that the dimension of M_μ is $n!$. In [F. Bergeron, N. Bergeron, A. Garsia, M. Haiman, and G. Tesler, *Adv. Math.* **142** (1999), 244–334], we introduced slightly more general spaces $M_{\mu/ij}$ as a tool for an elementary proof of the $n!$-theorem. For this one needs a formula for the bigraded characters in terms of Macdonald polynomials that would follow from a representation theoretic proof of our *four-term recurrence*. Here we give an explicit basis for the Y-free component of the space $M_{\mu/ij}$, and using this description we prove the Y-free analog of the four-term recurrence. Our basis of $M_{\mu/ij}$ has the further nice feature that it is a natural generalization of the Artin basis for the space of harmonic (covariant) polynomials for the symmetric group.

1. INTRODUCTION

We are going to explicitly describe certain $\mathcal{S}_n$-modules of polynomials, in n variables $x_1, \ldots, x_n$, that are closely related to classical harmonic polynomials for the symmetric group $\mathcal{S}_n$. These last polynomials can be characterized by the fact that they satisfy the conditions

$$\sum_{i=1}^{n} \partial_{x_i}^k P(x_1, \ldots, x_n) = 0, \qquad k = 1, 2, 3, \ldots. \tag{1.1}$$

A classical result of Steinberg states that the set M_{1^n}, of all harmonic polynomials for the symmetric group, is

$$M_{1^n} := \mathscr{L}_\partial[\Delta_{1^n}],$$

where $\Delta_{1^n} = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant, and $\mathscr{L}_\partial[\Delta_{1^n}]$ denotes the linear span of all partial derivatives of Δ_{1^n}. It is certainly striking to notice that the dimension of M_{1^n} is $n!$, and there are a lot of other nice results related to M_{1^n} and its generalization to reflection groups. The spaces studied here are natural generalizations of both these spaces and spaces studied by DeConcini and Procesi in [8] and Garsia and Procesi [11].

The point of departure of this work consists of replacing Δ_{1^n} by natural generalizations of the Vandermonde determinant. To this end, let us define a general *lattice diagram* to be any finite subset of $\mathbb{N} \times \mathbb{N}$. The case corresponding to diagrams of partitions is of special interest. Recall that a *partition* μ of n is a sequence $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$ of decreasing positive integers such that $n = \mu_1 + \cdots + \mu_k$. We denote $\mu \vdash n$ this fact. For $\mu \vdash n$, we set

$$\mu! := \mu_1!\, \mu_2! \cdots \mu_k!\,.$$

The lattice diagram associated to a partition μ is defined to be the set

$$\{(i, j) : 0 \le i \le k-1,\ 0 \le j < \mu_{i+1}\},$$

and we use the symbol μ for both the partition and its diagram. Most definitions and conventions used in this text are those of [5]. For example, the diagram of the partition (4, 2, 1) is geometrically represented as

2, 0			
1, 0	1, 1		
0, 0	0, 1	0, 2	0, 3

and it consists of the lattice *cells*

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (2, 0)\}.$$

Thus, the *coordinates* (r, c) of a cell are such that $r+1$ (resp. $c+1$) is the *row number* (resp. *column number*) for the cell position, counting from bottom up (resp. left to right).

Given a lattice diagram $D = \{(r_1, c_1), (r_2, c_2), \ldots, (r_n, c_n)\}$ we define the *lattice determinant*

$$\Delta_D(X; Y) := \det \left\| x_i^{r_j} y_i^{c_j} \right\|_{i,\, j=1}^{n},$$

where $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$. The polynomial $\Delta_D(X; Y)$ is bihomogeneous of degree $|r| = r_1 + \cdots + r_n$ in X and degree $|c| = c_1 + \cdots + c_n$ in Y. To ensure that this definition associates a unique polynomial to D, we order lattice cells in increasing lexicographic order.

We will need a few more definitions regarding partitions and diagrams. For an n-cell diagram D, a *tableau* of shape D is an injective map $T: D \to \{1, 2, \ldots, n\}$. If $T(r, c) = m$, we say that $h_T(m) := r$ is the *height* of m in T. We say that T is *row increasing* if $T(i, j) < T(k, j)$ whenever $i < k$ (when this has a meaning). Similarly we define *column increasing* tableaux, and a *standard* tableau, as one that is both row and column increasing. We denote by $\mathbb{S}_D$ the set of all standard tableau of shape D.

With these definitions out of the way, we come to our object of interest, namely the modules

$$M_D := \mathcal{L}_\partial[\Delta_D],$$

where D is some lattice diagram. The case where $D = 1^n$ (1^n is the partition of n with all parts equal to 1) corresponds to the classical module of harmonic polynomials. This generalization was first considered by Garsia and Haiman [9], in the special case where $D = \mu$ is the lattice diagram of a partition. Since then, several other cases have been studied (see [5] and [6]). For any n-cell lattice diagram D, the space M_D affords the structure of an $\mathcal{S}_n$-module, through the action of the symmetric group on polynomials consisting of permuting variables. More precisely, a permutation $\sigma \in \mathcal{S}_n$ acts *diagonally* on a polynomial $P(X;Y)$, as follows

$$\sigma P(X;Y) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}).$$

Under this action, $\Delta_D = \Delta_D(X;Y)$ is clearly an alternant and the action commutes with partial derivatives; hence M_D is an invariant subspace of $\mathbb{Q}[X, Y]$.

Moreover, since Δ_D is bihomogeneous, M_D affords the following natural bigrading. Denoting by $\mathcal{H}_{r,s}[M_D]$ the subspace consisting of the bihomogeneous elements of degree r in X and degree s in Y, we have the direct sum decomposition

$$M_D = \bigoplus_{r=0}^{|p|} \bigoplus_{s=0}^{|q|} \mathcal{H}_{r,s}[M_D].$$

The *bigraded Hilbert series* of M_D is

$$F_D(q,t) = \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r q^s \dim \mathcal{H}_{r,s}[M_D],$$

and the bigraded character of M_D is encoded by the following symmetric function,

$$H_D(X;q,t) = \sum_{r=0}^{|p|} \sum_{s=0}^{|q|} t^r q^s \mathcal{F}\big(\mathrm{ch}\, \mathcal{H}_{r,s}[M_D]\big),$$

where $\mathrm{ch}\, \mathcal{H}_{r,s}[M_D]$ denotes the character of $\mathcal{H}_{r,s}[M_D]$ and $\mathcal{F}$ is the Frobenius correspondence which maps the irreducible character χ^λ to the Schur function s_λ. We say that this symmetric function is the *bigraded Frobenius characteristic* of M_D.

The $n!$-conjecture of Garsia and Haiman states that for any partition diagram μ, $H_\mu(X;q,t)$ is none other than a renormalized version of the Macdonald polynomial associated to μ. This has recently been shown by Haiman [12] using an algebraic geometry approach. It develops that a very natural and combinatorial recursive approach to the $n!$-conjecture involves diagrams obtained by removing a single cell from a partition diagram. It is conjectured in [5] that, for such diagrams, the space M_D is a direct sum of left regular representations of $\mathcal{S}_n$. More precisely, if μ is a partition of $n+1$, we denote by μ/ij the lattice diagram obtained by removing one of its cells (i, j) from the diagram μ. We refer to the cell (i, j) as the *hole* of μ/ij. The conjecture in question states that the number of copies of the left regular representations in $M_{\mu/ij}$ is equal to the cardinality of the (i, j)-*shadow*, that is, the cardinality of $\{(s, t) \in \mu : s \geq i,\ t \geq j\}$.

This, and more, is all encoded in the following four-term recurrence for the bigraded Frobenius characteristic $H_{\mu/ij}$ of $M_{\mu/ij}$.

CONJECTURE 1.1. ([5]). *For all $(i, j) \in \mu$, we have $H_{\mu/ij} = C_{\mu/ij}$, where $C_{\mu/ij}$ is defined by the following "four-term" recurrence*

$$C_{\mu/ij} = \frac{t^{\ell} - q^{a+1}}{t^{\ell} - q^{a}} C_{\mu/i,\, j+1} + \frac{t^{\ell+1} - q^{a}}{t^{\ell} - q^{a}} C_{\mu/i+1,\, j} - \frac{t^{\ell+1} - q^{a+1}}{t^{\ell} - q^{a}} C_{\mu/i+1,\, j+1}. \tag{1.2}$$

Here ℓ and a give the number of cells that are respectively north and east of (i, j) in μ. As boundary conditions, we set $C_{\mu/i,\, j+1}$, $C_{\mu/i,\, j+1}$ or $C_{\mu/i,\, j+1}$ equal to zero when the corresponding cells $(i, j+1)$, $(i+1, j)$, or $(i+1, j+1)$ fall outside of μ. Furthermore, if (i, j) is a corner of μ, then μ/ij is a partition diagram ν, and we set

$$C_{\mu/ij} = H_{\nu}.$$

For any lattice diagram D, we consider M_D^0 the Y-free component of M_D; this is to say that

$$M_D^0 = \bigoplus_{r=0}^{|p|} \mathcal{H}_{r,\, 0}[M_D]\,.$$

In this paper, we study the spaces $M^0_{\mu/ij}$ and show that the Y-free specialization of Conjecture 1.1 holds for these spaces. In particular this implies that

$$\dim M^0_{\mu/ij} = \frac{n!}{\mu!}\ |\{\, (r, c) \in \mu \mid i \leq r \leq \ell\}|, \tag{1.3}$$

where ℓ is the largest integer for which the corresponding row of μ has at least j cells. Moreover, we obtain a formula for $H^0_{\mu/ij}$, the graded Frobenius character of $M^0_{\mu/ij}$. Our proof relies on an explicit construction of bases for M^0_{μ} (Theorem 2.3) and $M^0_{\mu/ij}$ (Theorem 3.1) that gives a very natural generalization of the classical basis of Artin for M_{1^n}.

2. A BASIS FOR M^0_μ

In preparation for our description of the modules $M^0_{\mu/ij}$, we need to recall and reformulate some results about the modules M^0_μ. Although a recursive description of a basis for M^0_μ is given in [7], and a direct description is given in [1], we give here a new description directly in terms of standard tableaux of shape μ. One can immediately link this description to Tanisaki's construction [13] of the defining ideal of M^0_μ. Moreover, we will see that it clearly generalizes the "Artin" basis for $M^0_{1^n}$

$$\mathscr{B}_n := \{\partial_X^{\mathbf{a}}\, \Delta_n(X) \mid \mathbf{a} = (a_1, a_2, \ldots, a_n), \quad a_i < i\,\},$$

where we use the vectorial notation $\partial_X^{\mathbf{a}} := \partial_{x_1}^{a_1}\, \partial_{x_2}^{a_2} \cdots \partial_{x_n}^{a_n}$.

First, let T be a tableau of shape D (any diagram), and define

$$Y_T := \prod_{(r,c)\in D} y^c_{T(r,c)},$$

so that, for $1 \le k \le n$, the exponent of y_k is c, if (r, c) is the position of k in T. For example, if

$$T = \begin{array}{|c|c|ccc}\hline 3 & 6 \\ \hline 1 & 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} \\ \hline\end{array}$$

then $Y_T = y_2\, y_6\, y_4^2\, y_5^3\, y_7^4$. Clearly, the monomial Y_T encodes the columns in which the entries of T appear. More precisely, let the *column set* of a diagram T, of shape D, be

$$\Gamma(T) := \Big\{\big(k, \Gamma_k(T)\big) \mid 0 \le k \le \max_{(r,c)\in D} c\Big\},$$

where for a given k, the set $\Gamma_k(T)$ is the set of entries in column k; that is,

$$\Gamma_k(T) := \{T(j,k) \mid (j,k) \in D\}.$$

Then $Y_T = Y_R$, if and only if T and R have the same column sets (up to the addition of pairs of the form $(k, \{\})$). Observe that T and R need not be of the same shape.

We will denote ∂Y_T the differential operator obtained by replacing each y_k in Y_T by ∂_{y_k}. With the same conventions, define

$$X_T := \prod_{(r,c)\in D} x^r_{T(r,c)} \qquad \text{and} \qquad Z_T := X_T\, Y_T.$$

It is easy to see that $T \mapsto Z_T$ establishes is a bijection between injective tableaux of shape D and monomials of the form

$$x^{r_1}_{\sigma_1}\, y^{c_1}_{\sigma_1}\, x^{r_2}_{\sigma_2} y^{c_2}_{\sigma_2} \cdots x^{r_n}_{\sigma_n} y^{c_n}_{\sigma_n},$$

with σ in $\mathcal{S}_n$ and $D = \{(r_1, c_1), (r_2, c_2), \ldots, (r_n, c_n)\}$ ordered lexicographically. If we set

$$\gamma_D := \prod_{(r,c)\in D} c!,$$

then we easily get the following lemma.

LEMMA 2.1. *For two tableaux T and R, both of shape D, one has*

$$\partial Y_T Z_R = \gamma_D\, X_R$$

if R can be obtained from T by a column fixing permutation of its entries. Otherwise

$$\partial Y_T Z_R = 0.$$

Given a tableau T of shape D, we define the *Garnir polynomial*

$$\Delta_T(X) = \prod_{\gamma\in\Gamma(T)} \det\left\| x_m^{h_T(\ell)} \right\|_{m,\, \ell\in\gamma}.$$

Recall that $h_T(\ell)$ is the height of ℓ in T; this is to say that it is the first coordinate of the cell in which it appears in T. We have the following proposition.

PROPOSITION 2.2. *For any tableau T of shape D', we either have*

$$\partial Y_T \Delta_D(X, Y) = \pm\, \gamma_D\, \Delta_T(X),$$

or $\partial Y_T \Delta_D(X, Y)|_{Y=0} = 0$.

Clearly this last possibility can only occur if one of the columns of D has a different number of cells than the corresponding column of D'.

Proof. We only give an outline of the proof and restrict ourselves to the case when each column of D has the same number of cells as the corresponding column of D'. In that case, Y_T has the same total Y-degree as $\Delta_D(X, Y)$, so that $\partial Y_T \Delta_D(X, Y)$ is a polynomial in the X variables only. Since the terms of $\Delta_D(X, Y)$ are

$$\text{sign}(\sigma)\; x_{\sigma_1}^{r_1}\, y_{\sigma_1}^{c_1}\, x_{\sigma_2}^{r_2} y_{\sigma_2}^{c_2} \cdots x_{\sigma_n}^{r_n} y_{\sigma_n}^{c_n},$$

in view of Lemma 2.1, the terms of $\partial Y_T \Delta_D(X, Y)$ are forced to be of the form $\partial Y_T\, Z_R$ for tableaux R that have the same column set as T. On the other hand, since $\partial Y_T \Delta_\mu(X, Y)$ alternates in sign under the action of column fixing permutations, it has to be a multiple of $\Delta_T(X)$. Hence, both polynomials having the same degree, we must have the equality stated. ∎

We now associate to each entry j, of a standard tableau T, a nonnegative integer in the following manner. Let (r_j, c_j) be the position of j in T, and let k be the largest entry of T, such that $c_k = c_j + 1$ and $k < j$. We set

$$\alpha_T(j) := r_j - r_k.$$

If there is no such k, set $\alpha_T(j) := r_j + 1$. For the example given below, the value of $\alpha_T(k)$ appears in the right tableau in the same position as k appears in the left tableau.

5				
4	8			
3	6			
1	2	7	9	10

3				
2	2			
1	2			
1	1	1	1	1

Clearly, if T is the unique standard tableau corresponding to a column (below)

n
$\vdots$
2
1

then $\alpha_T(j) = j$.

As shown in [7, 10] the space $\mathscr{L}_\partial[\Delta_T(X) : T \in \mathbb{S}_\mu]$ (the span of all partial derivatives of Garnir polynomials for tableaux of shape μ) coincides with the space M_μ^0. Using this characterization, we will now construct a basis for M_μ^0. But first, let us introduce some further notation.

For μ partition of n, let $\pi(\mu)$ be the set of partitions of $n - 1$ that can be obtained from μ by removing one of its corner. For $\nu \in \pi(\mu)$, we denote μ/ν the corner by which ν differs from μ. Let us label $\nu_1, \ldots, \nu_k$, the partitions in the set $\pi(\mu)$, following the increasing order of the column number in which the corresponding corners, the μ/ν_i's, appear. In other words, if (a_i, b_i), $1 \le i \le k$, are the respective coordinates of the corner cells μ/ν_i, then $b_1 < b_2 < \cdots < b_k$. Any standard tableau T of shape μ is such that n sits in a corner (a_j, b_j) of μ. Moreover, the value of $\alpha_T(n)$ depends only on the position of this corner (and on the shape μ), since all other entries of T are smaller. Denote by $\bar{\alpha}_j$ the value of $\alpha_T(n)$ (depending only on the shape of T and the position (a_j, b_j), where n is in T). It is clear that

(2.1) $$\bar{\alpha}_j = a_j - a_{j+1}.$$

THEOREM 2.3. *For any partition μ of n, the set of polynomials*

$$\mathcal{B}_\mu := \{\partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_\mu,\ \mathbf{m} = (m_1, m_2, \dots, m_n) \text{ and } 0 \le m_i < \alpha_T(i)\}$$

is a basis of M_μ^0.

Proof. We first show recursively that $\mathcal{B}_\mu$ is independent, assuming that the statement holds for partitions with at most $n-1$ cells. As before, for $\nu_j \in \pi(\mu)$, let (a_j, b_j) be the corner μ/ν_j with $b_1 < b_2 < \cdots < b_k$ and define

$$\mathbf{B}_j := \{X^{\mathbf{m}} \mid T \in \mathbb{S}_\mu,\ 0 \le m_i < \alpha_T(i),\ T(a_j, b_j) = n\}.$$

In view of (2.1), for $X^{\mathbf{m}} \in \mathbf{B}_j$, the dominant monomial of $\partial_X^{\mathbf{m}} \Delta_T(X)$ (in reverse lexicographic order) is of the form

$$x_n^{a_j - m_n} X^{\mathbf{p}} \qquad (\text{where} \quad p_n = 0),$$

with $0 \le m_n < a_j - a_{j+1}$. For k fixed with $a_{j+1} < k \le a_j$, our induction hypothesis gives that the set

$$\mathcal{B}_{j,k} := \{\partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_\mu,\ 0 \le m_i < \alpha_T(i),\ T(a_j, b_j) = n,\ m_n = a_j - k\},$$

is independent, since (in reverse lexicographic order) we have the following expansion of $\partial_X^{\mathbf{m}} \Delta_T(X)$

$$\partial_X^{\mathbf{m}} \Delta_T(X) = \partial_n^k \partial_X^{\mathbf{p}} \Delta_{T'}(X) + \underbrace{\cdots}_{\text{lower terms}},$$

where T' is the restriction of T to ν_j. Clearly, the sets $\mathcal{B}_{j,k}$ are mutually independent, so

$$\mathcal{B}_\mu = \biguplus_{j,k} \mathcal{B}_{j,k} \tag{2.2}$$

is independent.

We will now show that the number of elements of $\mathcal{B}_\mu$ is

$$|\mathcal{B}_\mu| = \frac{n!}{\mu!}, \tag{2.3}$$

using a recursive argument, assuming that the statement holds for $\nu_j \in \pi(\mu)$. By induction, it is clear that

$$|\mathcal{B}_{j,k}| = \frac{(n-1)!}{\nu_j!}$$

so that (in view of (2.2))

$$|\mathcal{B}_\mu| = \sum_{j=1}^{k} \bar{\alpha}_j \frac{(n-1)!}{\nu_j!}.$$

The result follows from the easy observation that

$$n = \sum_{j=1}^{k} \bar{\alpha}_j \,(a_j + 1),$$

since

$$a_j + 1 = \frac{\mu!}{\nu_j!}$$

is the length of the row of μ in which sits the corner (a_j, b_j). A "geometric" argument, which can be found in [11], shows that the dimension of M_μ^0 is at most $\frac{n!}{\mu!}$. Thus $\mathcal{B}_\mu$ is a basis. ∎

It is shown in [11] that the graded Frobenius characters of the M_μ^0's are none other than the Hall–Littlewood symmetric functions.

3. A BASIS FOR $M_{\mu/ij}^0$

The central result of this paper is the following description of a basis for $M_{\mu/ij}^0$, with μ any fixed partition of $n+1$ and (i, j) any given cell of μ. The proof that it is a generating set is postponed until the next section.

For a standard tableau T, let $\mathbf{B}_T$ simply denote the set

$$\mathbf{B}_T := \{X^{\mathbf{m}} \mid 0 \le m_s \le \alpha_T(s)\}.$$

For ν_ℓ a partition of n obtained from μ by removing the corner cell (a_ℓ, b_ℓ), the basis of $M_\ell := M_{\nu_\ell}$, described in the previous section, is

$$\mathcal{B}_\ell = \{\partial_X^{\mathbf{m}} \Delta_T(X) \mid T \in \mathbb{S}_{\nu_\ell},\ X^{\mathbf{m}} \in \mathbf{B}_T\}.$$

If T is a standard tableau of shape ν_ℓ, and $0 \le u \le a_\ell$ an integer, we denote $T\uparrow_{uv}$ the tableau of shape μ/uv (with $v = b_\ell$), such that

$$T\uparrow_{uv}(r, c) = \begin{cases} T(r, c) & \text{if } c \ne v \text{ or } \ r < u, \\ T(r-1, c) & \text{if } c = v \text{ and } r > u. \end{cases}$$

Since (u, v) is not in μ/uv, $T\uparrow_{uv}$ need not be defined at (u, v). In other words, the tableau $T\uparrow_{uv}$ is obtained from T by "sliding" upward by 1 the cells in column v that are on or above row u. For u and v as above, we set

$$\mathcal{A}_{uv} := \{\partial_X^{\mathbf{m}} \Delta_{T\uparrow_{u,v}}(X) \mid X^{\mathbf{m}} \in \mathbf{B}_T,\ T \in \mathbb{S}_{\nu_\ell}\}. \tag{3.1}$$

Observe that $\mathcal{A}_{uv}$ implicitly depends on the choice of corner (a_ℓ, b_ℓ) of μ, since v is equal to b_ℓ. Moreover, $\mathcal{A}_{a_\ell, b_\ell}$ is the basis of $M_{\nu_\ell}^0$ described in Theorem 2.3; thus $\mathcal{A}_{a_\ell, b_\ell}$ is independent.

Let $(a_1, b_1), \ldots, (a_m, b_m)$ (with $b_1 < \cdots < b_m$) be the set of corners of μ that are in the "shadow" of (i, j). This is to say that $i \leq a_\ell$ and $j \leq b_\ell$, for all $1 \leq \ell \leq m$. Once again, we denote $\bar{\alpha}_\ell$ the value of $\alpha_T(n+1)$ for any standard tableau of shape μ having $n+1$ in position (a_ℓ, b_ℓ). Defining

$$\mathscr{B}_{\mu/ij} := \bigcup_{\ell=1}^{m} \bigcup_{u=i}^{\min(i+\bar{\alpha}_\ell-1,\, a_\ell)} \mathscr{A}_{u,\, b_\ell}, \tag{3.2}$$

we have

THEOREM 3.1. *For μ a partition of $n+1$ and $(i, j) \in \mu$, $\mathscr{B}_{\mu/ij}$ is a basis of $M^0_{\mu/ij}$.*

Proof. In the remainder of this section, we will prove that (3.2) is an independent set, using a downward recursive argument, and that the number of elements of $\mathscr{B}_{\mu/ij}$ is

$$d_{\mu/ij} := \frac{n!}{\mu!} \sum_{\substack{i'>i \\ \mu_{i'}>j}} \mu_{i'}. \tag{3.3}$$

To complete the proof of the theorem, we will show in Section 4 that the dimension of $M^0_{\mu/ij}$ is at most equal to $d_{\mu/ij}$, so that $M^0_{\mu/ij}$ has to coincide with the span of $\mathscr{B}_{\mu/ij}$. ∎

The cardinality of $\mathscr{A}_{u,\, b_\ell}$ is clearly

$$\frac{n!}{\mu!}\, \mu_{a_\ell+1-u+i} = \frac{n!}{\mu!}\, \mu_{a_\ell+1}$$

since this is the same as $n!/\nu_\ell!$. If (3.2) is a disjoint union (which would follow from it being independent) and since every i' indexing the summation in (3.3) occurs exactly once in $\{a_\ell + 1 - u + i \mid 1 \leq \ell \leq m,\ i \leq u \leq \min(i + \bar{\alpha}_\ell - 1, a_\ell)\}$, then we must have

$$|\mathscr{B}_{\mu/ij}| = d_{\mu/ij}. \tag{3.4}$$

Let

$$D_X := \partial_{x_1} + \partial_{x_2} + \cdots + \partial_{x_n}.$$

We recall the following special case of [5, Proposition (I.2)].

PROPOSITION 3.2. *If $(i+1, j)$ is in μ, then*

$$D_X\, \Delta_{\mu/ij}(X, Y) =_{\text{cte}} \Delta_{\mu/i+1,\, j}(X, Y).$$

Otherwise $D_X\, \Delta_{\mu/ij}(X, Y) = 0$, and this corresponds to the case where (i, j) is on the top border of μ. The symbol "$=_{\text{cte}}$" stands for equality up to a nonzero constant.

It is easy to adapt the proof of this fact to show that, for T a standard tableau of shape ν_ℓ (with $\mu/\nu_\ell = (a_\ell, b_\ell)$) and $0 \le u \le a_\ell$, we have

$$D_X \, \Delta_{T\uparrow_{u,v}}(X) =_{\text{cte}} \begin{cases} \Delta_{T\uparrow_{u+1,v}}(X) & \text{if } \; u < a_\ell, \\ 0 & \text{if } \; u = a_\ell, \end{cases} \tag{3.5}$$

where $v = b_\ell$. Lemma 3.3 follows from the definition in Eq. (3.1).

LEMMA 3.3. *Using the same convention as above, we have*

$$D_X \, \mathcal{A}_{u,b_\ell} =_{\text{cte}} \begin{cases} \mathcal{A}_{u+1,b_\ell} & \text{if } u < a_\ell, \\ \{0\} & \text{if } u = a_\ell. \end{cases} \tag{3.6}$$

Since these two sets have the same cardinality we deduce, from the linear independence of $\mathcal{A}_{a_\ell, b_\ell}$, that each $\mathcal{A}_{u, b_\ell}$ is independent. Applying D_X to the definition in Eq. (3.2) we readily check that

$$\mathcal{B}_{\mu/i+1,j} = D_X \, \mathcal{B}_{\mu/i,j} \, . \tag{3.7}$$

But we know that $D_X \, \mathcal{A}_{a_m, b_m} = \{0\}$, and it is clear that $\mathcal{A}_{a_m, b_m}$ is a subset of $\mathcal{B}_{\mu/ij}$. A dimension count, together with the recursive assumption, forces $\mathcal{B}_{\mu/i+1,j}$ to be independent, since

$$d_{\mu/i+1,j} + |\mathcal{A}_{a_m, b_m}| = \frac{n!}{\mu!} \sum_{\substack{i' > i+1 \\ \mu_{i'} > j}} \mu_{i'} + \frac{n!}{\mu!} \, \mu_{i+1} = d_{\mu/ij} \, .$$

This with the result of the next section will conclude the proof of Theorem 3.1.

4. UPPER BOUND FOR THE DIMENSION OF $M^0_{\mu/ij}$

We now give an upper bound for the dimension of $M^0_{\mu/ij}$. Given a polynomial $P(X, Y)$, we denote by $P(\partial)$ the operator obtained from P by replacing all the variables x_i and y_j by ∂_{x_i} and ∂_{y_j}, respectively. If $P(X, Y) = M$ is a single monomial M we will write $P(\partial) = \partial_M$.

THEOREM 4.1. *For μ a partition of $n+1$,*

$$\dim M^0_{\mu/ij} \le \frac{n!}{\mu!} \sum_{\substack{i' > i \\ \mu_{i'} > j}} \mu_{i'} \, .$$

Proof. In [5], the bigraded $\mathcal{S}_n$-modules $M_{\mu/ij}$ and $\mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(X;Y)]$ are shown to be equivalent; hence their Y-free components,

$$M^0_{\mu/ij} \qquad \text{and} \qquad \mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(X;Y)]^0,$$

are equivalent. For any injective tableau T of shape μ, with $c_{n+1} \geq j$, we have $Y_T = y^j_{n+1}Y^{\mathbf{a}}$ and

$$\pm\gamma_\mu\,\Delta_T(X) = \partial Y_T\Delta_\mu(X,Y).$$

If $r_{n+1} < i$ then $\partial^i_{x_{n+1}}\Delta_T(X) = 0$, so that $\mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(X;Y)]^0$ is equal to

$$\mathcal{L}_\partial\big[\partial^i_{x_{n+1}}\Delta_T(X)\;\big|\;T\colon\mu\to\{1,2,\ldots,n+1\},\;c_{n+1}\geq j,\;r_{n+1}\geq i\big]. \tag{4.1}$$

For $\zeta_1,\zeta_2,\ldots,\zeta_{\ell(\mu)}$ and $\omega_1,\omega_2,\ldots,\omega_{\ell(\mu')}$ two families of pairwise distinct scalars, we construct a set of points $[\rho_\mu]_{i,j}$ in $\mathbb{C}^{2(n+1)}$ as follows. For every injective tableau T of shape μ, we define the point $\rho_T = (\zeta_{r_1},\zeta_{r_2},\ldots,\zeta_{r_{n+1}},\omega_{c_1},\omega_{c_2},\ldots,\omega_{c_{n+1}})$ in $\mathbb{C}^{2(n+1)}$ and set

$$[\rho_\mu]_{i,j} = \big\{\rho_T\;\big|\;T\colon\mu\to\{1,2,\ldots,n+1\},\;c_{n+1}\geq j,\;r_{n+1}\geq i\big\}.$$

That is, $\rho_T \in [\rho_\mu]_{i,j}$ when $n+1$ lies in the shadow of (i,j) in T. Note that $\rho_T \in [\rho_\mu] = [\rho_\mu]_{0,0}$ contains $n!$ points in correspondence with every injective tableau of shape μ.

We denote by $[\rho_\mu]^0_{i,j} = \pi([\rho_\mu]_{i,j})$, where π is the projection on $\mathbb{C}^{n+1}$ that keeps only the first $n+1$ entries. We see that the set of tableaux with $n+1$ entries strictly increasing in rows and where $n+1$ lies in a row i' such that $i' = r_{n+1}+1 > i$ and $\mu_{i'} = c_{n+1}+1 > j$ gives all the points of $[\rho_\mu]^0_{i,j}$ exactly once. One then easily verifies that the cardinality of $[\rho_\mu]^0_{i,j}$ is precisely

$$d_{\mu/ij} = \frac{n!}{\mu!}\sum_{\substack{i'>i\\ \mu_{i'}>j}}\mu_{i'}.$$

Following [5, Sect. 4] we associate to this set $J_{[\rho_\mu]^0_{i,j}}$, its vanishing ideal and define $H_{[\rho_\mu]^0_{i,j}} = (J_{[\rho_\mu]^0_{i,j}})^\perp$. The dimension of $H_{[\rho_\mu]^0_{i,j}}$ is then $d_{\mu/ij}$ as well.

Given a polynomial P, let $h(P)$ denotes its homogeneous component of highest degree. For any polynomial P in $J_{[\rho_\mu]^0_{i,j}}$, let

$$Q(X,Y) = P(X)\prod_{i'=1}^{i}(x_{n+1}-\zeta_{i'})\prod_{j'=1}^{j}(y_{n+1}-\omega_{i'}).$$

For any $\rho_T \in [\rho_\mu]$, the two products in the definition of Q vanish at ρ_T unless $n+1$ lies in the shadow of (i,j) in T. But if this is the case then $P(X)$ vanishes at $\pi(\rho_T)$. This shows that $Q(X,Y)$ is in $J_{[\rho_\mu]}$, the vanishing ideal of $[\rho_\mu]_{0,0} = [\rho_\mu]$. Hence $h(Q) = h(P)x_{n+1}^i y_{n+1}^j$ is in $gr(J_{[\rho_\mu]})$, its graded version, and $h(Q)(\partial)\Delta_\mu(X, Y) = 0$. For any injective tableau T of shape μ such that $c_{n+1} \geq j$ and $r_{n+1} \geq i$ we have $Y_T = y_{n+1}^j Y^{\mathbf{a}}$ and

$$\begin{aligned} h(P)(\partial)\, \partial_{x_{n+1}}^i \Delta_T(X) &= \pm\gamma_\mu^{-1}\, h(P)(\partial)\, \partial_{x_{n+1}}^i\, \partial Y_T \Delta_\mu(X, Y) \\ &= \pm\gamma_\mu^{-1}\, h(P)(\partial)\, \partial_{x_{n+1}}^i\, \partial_{y_{n+1}}^j \partial_Y^{\mathbf{a}} \Delta_\mu(X, Y) \\ &= \pm\gamma_\mu^{-1} \partial_Y^{\mathbf{a}}\, h(Q)(\partial)\Delta_\mu(X, Y) = 0. \end{aligned}$$

Thus $h(P)$ is in $I_{\partial_{x_{n+1}}^i \Delta_T(X)}$. We obtain this way that $grJ_{[\rho_\mu]_{i,j}^0}$ is a subset of $I_{\partial_{x_{n+1}}^i \Delta_T(X)}$ for any T with the prescribed conditions. The space in (4.1) is thus contained in $H_{[\rho_\mu]_{i,j}^0}$, which proves the theorem. ∎

5. FOUR-TERM RECURRENCE

Specializing Conjecture 1.1 to its Y-free component corresponds to setting $q = 0$ in the four-term recurrence (1.2). We now show that this specialization of Conjecture 1.1 holds by giving an explicit interpretation of the resulting recurrence in terms of the basis we have constructed for $M_{\mu/ij}^0$.

THEOREM 5.1. *If $H_{\mu/ij}^0$ denotes the graded Frobenius characteristic of $M_{\mu/ij}^0$ then*:

- *if $a = 0$ and $\ell > 0$,* $\quad H_{\mu/ij}^0 = \dfrac{t^{\ell+1} - 1}{t^\ell - 1} H_{\mu/ij+1}^0$;
- *if $a > 0$,* $\quad H_{\mu/i,j}^0 = H_{\mu/i,j+1}^0 + t\,H_{\mu/i+1,j}^0 - t\,H_{\mu/i+1,j+1}^0$.
- *if $a = 0$ and $\ell = 0$, $\quad H_{\mu/ij}^0$ is the graded Frobenius characteristic of M_ν^0, where ν is the partition μ/ij.*

Here (as before) ℓ and a give the number of cells that are respectively north and east of (i, j) in μ. If any of the cells $(i+1, j)$, $(i, j+1)$ or $(i+1, j+1)$ falls out of μ, then the corresponding term is considered to be 0.

Proof. Each of these assertions can be shown using the basis we have constructed. The third assertion is just a direct observation. The first assertion corresponds to a case for which there is just one corner (a_m, b_m) in the shadow of (i, j), with $b_m = j$, and then (3.2) can be written as

$$\mathscr{B}_{\mu/ij} := \bigcup_{u=0}^{\ell} \mathscr{A}_{i+u,\,j}.$$

Since, as long as $(k+1, j)$ is in μ, D_X is an isomorphism of representations between the homogeneous $\mathcal{S}_n$-modules $\mathcal{A}_{k,j}$ and $\mathcal{A}_{k+1,j}$ that lowers the degree by 1, we must have

$$\mathcal{F}_t(\mathcal{A}_{k,j}) = t\,\mathcal{F}_t(\mathcal{A}_{k+1,j}),$$

where $\mathcal{F}_t$ stands for the graded Frobenius characteristic. We deduce that, in the first case,

$$\mathcal{B}_{\mu/ij} = (1 + t + \cdots + t^{\ell})\,H_{\nu}^{0},$$

with $\mu/\nu = (a_m, b_m)$. This is clearly equivalent to the statement of the first case.

For the second case there are a few subcases, all similarly dealt with. The most interesting one is when $j = b_1$ and $m > 1$ for which the basis can clearly be broken down as

$$\mathcal{B}_{\mu/ij} = \mathcal{B}_{\mu/i,\,j+1} \uplus \bigcup_{u=i}^{i+\bar{\alpha}_1-1} \mathcal{A}_{u,\,b_1}.$$

We only need to show that the graded Frobenius characteristic of the linear span of

$$\bigcup_{u=i}^{i+\bar{\alpha}_1-1} \mathcal{A}_{u,b_1}$$

is given by

$$t\,(H^0_{\mu/i+1,\,j} - H^0_{\mu/i+1,\,j+1}). \tag{5.1}$$

Now we clearly have $\mathcal{B}_{\mu/i+1,\,j+1} \subset \mathcal{B}_{\mu/i+1,\,j}$, with $D_X \bigcup_{u=i}^{i+\bar{\alpha}_1-1} \mathcal{A}_{u,\,b_1}$ being the complement of $\mathcal{B}_{\mu/i+1,j+1}$ in $\mathcal{B}_{\mu/i+1,j}$. Under the hypothesis of this subcase, the graded Frobenius characteristic of the span of $\bigcup_{u=i}^{i+\bar{\alpha}_1-1} \mathcal{A}_{u,\,b_1}$ is thus given by (5.1). All other subcases are simple to show. ∎

6. REMARKS

REMARK 6.1. In [7, Proposition 2.2], N. Bergeron and Garsia show that the spaces M_μ^0 are nested into each other according to their indexing partition. That is,

$$\mu \preceq \lambda \Longrightarrow M_\mu^0 \subseteq M_\lambda^0,$$

where $\preceq$ denotes the dominance order. Moreover they show that

$$M_\mu^0 \cap M_\lambda^0 = M_{\mu\wedge\lambda}^0.$$

Using our basis, it is easy to show that both these results extend to the situation studied in this paper. Namely,

PROPOSITION 6.2. *For two partition μ and λ of $n+1$, we have*

$$\mu \preceq \lambda \Longrightarrow M^0_{\mu/ij} \subseteq M^0_{\lambda/ij}$$

and

$$M^0_{\mu/ij} \cap M^0_{\lambda/ij} = M^0_{\mu\wedge\lambda/ij},$$

whenever (i, j) appears in both μ and λ.

REMARK 6.3. For μ a partition of n (denoted $\mu \vdash n$), Macdonald has given an explicit description of the coefficients appearing in the Pieri formula for the H_μ

$$h_k^\perp H_\mu(X; q, t) = \sum_{\substack{\nu\vdash n-k \\ \nu\subseteq\rho}} c^k_{\mu\nu}(q, t) H_\nu(X; q, t),$$

where $h_k^\perp$ is the operator dual to multiplication by h_k (complete homogeneous) with respect to the usual scalar product on symmetric functions for which the Schür functions are orthonormal. These coefficients $c^k_{\mu\nu}(q, t)$ are rational functions in q and t. Now, let ρ be the partition of m corresponding to the shadow of (i, j) in μ, with m equal to the number of cells in this shadow. F. Bergeron has conjectured in [4] that the following symmetric function

$$\sum_{\substack{\nu\vdash m-k \\ \nu\subseteq\rho}} c^k_{\mu\nu}(q, t) H_{\mu-\rho+\nu}(X; q, t), \tag{6.1}$$

where $\mu - \rho + \nu$ stands for the partition obtained from μ by replacing ρ (the shadow of (i, j)) by ν, is the bigraded Frobenius characteristic of the module $M^k_{\mu/ij}$ obtained as the union of all modules M_D, for D ranging in the set of diagrams obtained from μ by removing k cells in the shadow of (i, j). This would imply that the dimension of $M^k_{\mu/ij}$ is equal to $\binom{m}{k}(n-k)!$. J.-C. Aval, in [2], has shown that this value is an upper bound and has generalized the construction of this paper to obtain an explicit basis for the Y-free component of $M^k_{\mu/ij}$. One can show that the graded Frobenius characteristic of the resulting space is the symmetric function obtained by taking the limit as $q \to 0$ of (6.1).

REMARK 6.4. One can explicitly characterize the defining ideal of the space $M^0_{\mu/ij}$. This will be the subject of a forthcoming paper [3].

REFERENCES

1. J.-C. Aval, Monomial bases related to the n! conjecture, *Discrete Math.* **224** (2000), 15–35.
2. J.-C. Aval, On Certain Spaces of Lattice Determinants, *Discrete Math*, in press.
3. J.-C. Aval and N. Bergeron, Vanishing ideals of lattice diagram determinants, *J. Comb. Theory Ser. A*, to appear.
4. F. Bergeron, "Spaces of Lattice Diagram Polynomials," in preparation.
5. F. Bergeron, N. Bergeron, A. Garsia, M. Haiman, and G. Tesler, Lattice diagram polynomials and extended Pieri rules, *Adv. Math.* **142** (1999), 244–334.
6. F. Bergeron, A. Garsia, and G. Tesler, Multiple left regular representations associated with alternants of the symmetric group, *J. Comb. Theory Ser. A* **91**, (2000), 49–83.
7. N. Bergeron and A. Garsia, On certain spaces of harmonic polynomials, *in* "Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications" (Donald St. P. Richards, Ed.), Contemp. Math. AMS, Vol. 138, pp. 51–86, Amer. Math. Soc., Providence, 1992.
8. C. De Concini and C. Procesi, Symmetric functions, conjugacy classes and the flag variety, *Invent. Math.* **64** (1981), 203–230.
9. A. Garsia and M. Haiman, A graded representation model for Macdonald's polynomials, *Proc. Nat. Acad.* (1993), 3607–3610.
10. A. Garsia and M. Haiman, "Orbit Harmonics and Graded Representations," Les Éditions du Lacim, UQAM, Montréal, to appear.
11. A. Garsia and C. Procesi, On certain graded $\mathcal{S}_n$-modules and the q-Kostka polynomials, *Adv. Math.* **94** (1992), 82–138.
12. M. Haiman, Hilbert schemes, polygraphs, and the Macdonald positivity conjecture, *J. Amer. Math. Soc.* **14** (2001), 941–1006.
13. T. Tanisaki, Defining ideals of the closure of conjugacy classes and representations of the Weyl groups, *Tohoku J. Math*. **34** (1982), 575–585.