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Algebraic Computations with Continued Fractions

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General algorithms, viewed as transducers, are introduced for computing rational expressions with continued fraction expansions. Moreover, expansions of some algebraic numbers, like $\sqrt[3]{2}$ or those related to primitive matrices are considered.

1. INTRODUCTION

For irrational number x let

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ddots}}}$$

be its (regular) continued fraction expansion with partial quotients c_k ($c_0 \in \mathbf{Z}$ and $c_k \in \mathbf{N} \backslash \{0\}$). The notation $x = [c_0; c_1, c_2, c_3, ...]$ is classical and we write $x = [c_0; c_1, ..., c_{k-1}, \overline{c_k \cdots c_{k+T-1}}]^\infty$ to express that $(c_n)_n$ is ultimately periodic with $c_k \cdots c_{k+T-1} = c_{k+mT} \cdots c_{k+(m+1)T-1}$ for all $m \in \mathbf{N}$. The convergents of x are rational numbers p_k/q_k which can be defined by the product formula:

$$\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

The right-hand side of (1) will be denoted by $\Pi_{c_0 \cdots c_k}$. For rational numbers x the above expansion is finite with two possibilities. The first one, called regular, is given by $x = [a_0; a_1, ..., a_n]$ with $a_n \geqslant 2$; the second one is $x = [a_0; a_1 ..., a_n - 1, 1]$. The integer n is said to be the depth of x.

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In this paper we construct several algorithms, described in terms of transducers, which perform algebraic computations with continued fractions. In Section 2, a general factorization theorem for matrices is proved. This result, which is not really new, is one of the main ingredients needed to construct these transducers. A first application, given in Section 3, is closely related both to the classical work of M. Hall Jr. [Ha] on continuous fraction expansion of $h(x) = (ax+b)/(cx+d)$, where a, b, c, d are integers with $ad - bc \neq 0$ and the results of Raney [Ra] which relate fractional linear transformations, continued fractions, and finite transducers. In fact, our method leads to an efficient formal algorithm which produces consecutive partial quotients of $h(x)$ by reading the consecutive ones of x at a positive rate bounded below independently of the entry x.

In Section 4, an algorithm is given for biquadratic fractional transformations $(axy + bx + cy + d)/(exy + fx + gy + h)$, previously studied by Gosper [Go]. The algorithm is strongly related to two group homomorphisms from $GL_2(\mathbf{R})$ to $GL_4(\mathbf{R})$. Section 5 is concerned with rational values $R(x) = P(x)/Q(x)$, where P and Q are polynomials defined on $\mathbf{Z}$. In that case, the algorithm arises from a group homomorphism $f_n \colon GL_2(\mathbf{R}) \to GL_{n+1}(\mathbf{R})$, where n is the projective degree of R. We also pay attention to the fascinating continued fraction expansion of $\sqrt[3]{2}$. In Section 6, we give further consequences of the factorization theorem. First we exhibit a transducer which produces the continued fraction expansion of some algebraic numbers in connection with primitive matrices with nonnegative integer entries. Finally we show how to obtain easily by transducer the usual continued fraction expansion of real numbers ξ of the form $\xi = a_0 + b_0/(a_1 + (b_1/a_2 + (\cdots)))$, where a_k and b_k are integers.

2. DEFINITIONS AND BASIC TOOLS

2.1. *Transducers*

Given a set $\mathscr{A}$, also called alphabet, we denote by $\mathscr{A}^*$ the monoid generated by $\mathscr{A}$. Elements in $\mathscr{A}^*$ are viewed as words $w = w_1 w_2 \cdots w_n$, $w_i \in \mathscr{A}$, with length $|w| = n$. The empty word has length 0 and is denoted by $\wedge$. The number of times a given letter a occurs in a word w will be denoted by $|w|_a$. Let $\mathscr{B}$, $\mathscr{C}$, be any other alphabets and let $\Phi = \{\phi_c ; c \in \mathscr{C}\}$ and $\Psi = \{\psi_c ; c \in \mathscr{C}\}$ be two families of maps $\phi_c \colon \mathscr{B} \to \mathscr{B}$ and $\psi_c \colon \mathscr{B} \to \mathscr{A}^*$. The quintuple $\mathscr{T} = (\mathscr{C}, \mathscr{B}, \mathscr{A}, \Phi, \Psi)$ is called an alphabetic transducer over the semi-automaton $(\mathscr{C}, \mathscr{B}, \Phi)$ with input alphabet $\mathscr{C}$ and output alphabet $\mathscr{A}$. Elements of $\mathscr{B}$ are called states of the transducer.

For each $c \in \mathscr{C}$ the map ϕ_c corresponds to an input instruction of the semi-automaton and the map ψ_c corresponds to the output instruction associated to c. For a given state b_0 (called initial state), the transducer $\mathscr{T}$ associates, to any input word $c_1 c_2 \cdots c_n \in \mathscr{C}^n$, a sequence of states $b_0, b_1 = \phi_{c_1}(b_0), ..., b_n = \phi_{c_n}(b_{n-1})$. Now the output of $\mathscr{T}$ associated to $c_1 c_2 \cdots c_n$ is the word

$$[\Psi, \Phi]_{c_1 \cdots c_n}(b_0) = \psi_{c_1}(b_0)\, \psi_{c_2}(b_1) \cdots \psi_{c_n}(b_{n-1})$$

which belongs to $\mathscr{A}^*$. Notice that the above definition of transducers differs slightly from the classical one, where the alphabets and the space of states are usually assumed to be finite and instructions are usually associated to a finite set of words, which often includes the empty word $\wedge$.

In the sequel, the set $\mathscr{A}$ will often be a language (i.e., a set of words) over the alphabet of natural numbers $\mathbf{N}$ (including zero). In that case, there is a natural embedding of $\mathscr{A}^*$ into the free monoid $\mathscr{N}_0$ generated by $\mathbf{N}$ so that $\mathscr{A}^*$ will be also considered as Language over $\mathbf{N}$. We introduce the morphism α: $\mathscr{N}_0 \to \mathbf{GL_2}(\mathbf{Z})$ defined by $\alpha(a) = \left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $a \in \mathbf{N}$. Clearly, α induces a morphism from $\mathscr{A}^*$ into $\mathbf{GL_2}(\mathbf{Z})$, still denoted by α. Let $\mathscr{N}_1 = (\mathbf{N} \setminus \{0\})^*$ and let $\mathscr{N}_2$ be the sub-monoid in $\mathbf{GL_2}(\mathbf{Z})$ generated by all matrices $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $a \in \mathbf{N} \setminus \{0\}$. Classically, the morphism α restricted to $\mathscr{N}_1$ realizes an isomorphism between $\mathscr{N}_1$ and $\mathscr{N}_2$.

2.2. *A Factorization Theorem*

Let $\mathscr{M}_{k, \mathbf{N}}$ be the set of all $2 \times k$ matrices of rank 2

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}$$

where α_i and β_i are nonnegative integers. Notice that if $A \in \mathscr{M}_{2, \mathbf{N}}$ and $B \in \mathscr{M}_{k, \mathbf{N}}$, then $AB \in \mathscr{M}_{k, \mathbf{N}}$.

Definition 1. Let $\mathscr{D}_k$, $\mathscr{D}'_k$, and $\mathscr{E}_k$ be subsets of $\mathscr{M}_{k, \mathbf{N}}$ respectively defined by

(i) $A \in \mathscr{D}_k$ if $\alpha_i \geqslant \beta_i$ for all integers i, $1 \leqslant i \leqslant k$;

(ii) $A \in \mathscr{D}'_k$ is $\alpha_i \leqslant \beta_i$ for all integers i, $1 \leqslant i \leqslant k$;

(iii) $A \in \mathscr{E}_k$ if there exists (i, j) with $0 \leqslant i < j \leqslant k$ and $(\alpha_i - \beta_i)(\alpha_j - \beta_j) < 0$.

Remark 1. The family $\{\mathscr{D}_k, \mathscr{D}'_k, \mathscr{E}_k\}$ forms a partition of $\mathscr{M}_{k, \mathbf{N}}$.

Remark 2. In the case $k = 2$, elements of $\mathscr{E}_2$ are called "row-balanced" matrices; this definition was introduced by Raney [Ra].

We give below some matrix properties. Their proofs are straightforward and details are left to the reader.

LEMMA 1. *Let A be in $\mathscr{D}_2$ (resp. in $\mathscr{D}'_2$) and let $B \in \mathscr{M}_{k,\mathbf{N}}$. Then AB is in $\mathscr{D}_k$ (resp. in $\mathscr{D}'_k$).*

LEMMA 2. *Let c and c' be integers $\geqslant 0$ and let B, B' be matrices in $\mathscr{M}_{k,\mathbf{N}}$ which are not in $\mathscr{D}'_k$. Then*

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} B = \begin{pmatrix} c' & 1 \\ 1 & 0 \end{pmatrix} B' \quad \Leftrightarrow \quad c = c', \quad B = B'.$$

Notice that for any integers $u \geqslant v > 0$, the classical Euclidean division of u by v is equivalent to the equality

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

with integers a, $u'(=v)$, v' such that $a > 0$, and $u' > v' \geqslant 0$. Using this formula and Lemma 2, we get

LEMMA 3. *Assume that*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Pi_{a_0 a_1 \cdots a_r} \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{2}$$

with integers u, v, u', v' such that $u' \geqslant v' \geqslant 0$ and $a_i \in \mathbf{N} \backslash \{0\}$ for $i = 0, ..., r$. Then $u \geqslant v \geqslant 0$ and

(i) *if $u = 0$, then $u = u' = v' = 0$;*

(ii) *if $u > 0$ and $v = 0$, then the product is empty (in other words, $\binom{u}{0} = \Pi_\wedge \binom{u}{0}$, where $\Pi_\wedge = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$);*

(iii) *assume that $v > 0$:*

— *if $u' > v' > 0$, then $u > v$ and the word $a_0 \cdots a_r$ in $(\mathbf{N} \backslash \{0\})^*$ given by the factorization (2) is unique;*

— *if $u' = v' \neq 0$, then $u' = \gcd(u, v)$,*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Pi_{a_0 a_1 \cdots a_r} \begin{pmatrix} u' \\ u' \end{pmatrix}$$

and the factorization (2) is unique;

— *if* $u' > v' = 0$ *then* $u' = \gcd(u, v)$ *and there are two factorizations, namely*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Pi_{a_0 a_1 \cdots a_r} \begin{pmatrix} u' \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \Pi_{a_0 a_1 \cdots (a_r - 1) \cdot 1} \begin{pmatrix} u' \\ 0 \end{pmatrix}.$$

Moreover, for all cases with $v > 0$ *and* $u' > 0$, *the equality*

$$\frac{u}{v} = \left[a_0; a_1, ..., a_r + \frac{v'}{u'} \right]$$

holds.

This lemma will be mainly used in the following form.

LEMMA 4. *Let*

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}$$

be a matrix in $\mathscr{M}_{k, \mathbf{N}}$. *The following statements are equivalent*:

(i) *There exists* $a_0 \in \mathbf{N}$, $a_1, ..., a_r \in \mathbf{N} \backslash \{0\}$ *and a matrix* B *in* $\mathscr{D}_k$ *such that*

$$A = \Pi_{a_0 a_1 \cdots a_r} B.$$

(ii) *For all* j, $0 \leqslant j \leqslant k$,

— *if* $\beta_j = 0$ *then* $\alpha_j = 0$;

— *if* $\beta_j \neq 0$ *then* $a_0, a_1, ..., a_r$ *are the* $r+1$ *first partial quotients of one of the two continued fraction expansions of* α_j / β_j.

We are ready to set and prove the following factorization theorem.

THEOREM 1. *Let*

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}$$

be a matrix of $\mathscr{M}_{k, \mathbf{N}}$ *such that* $A \notin \mathscr{E}_k$:

(i) *There exists a unique factorization*

$$A = \Pi_{c_0 c_1 \cdots c_n} A', \tag{F}$$

where $c_0 \in \mathbf{N}$, $c_1, c_2, ..., c_n \in \mathbf{N} \backslash \{0\}$ *and* $A' \in \mathscr{E}_k$.

(ii) *For all integers i such that* $1 \leqslant i \leqslant k$ *and* $\beta_i \neq 0$, *the depth of* α_i/β_i *is* $\geqslant n$, *the n first partial quotients of* α_i/β_i *are* $c_0, c_1, ..., c_{n-1}$ *and the next partial quotient in the regular continued fraction expansion of* α_i/β_i *is* $\geqslant c_n$.

Proof. We first prove the existence of factorization (F). If $M \in \mathscr{D}'_k$, then $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M'$ with $M' \in \mathscr{D}_k$. Therefore, we may assume that the matrix

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}$$

belongs to $\mathscr{D}_k$. Let c' be any integer $\geqslant 1$ and let M' be the matrix defined by $M = \left(\begin{smallmatrix} c' & 1 \\ 1 & 0 \end{smallmatrix}\right) M'$. It is obvious that the first line of M' is strictly lower than the first one of M. Let

$$c = \min_{\substack{1 \leqslant i \leqslant k \\ \beta_i \neq 0}} \left(\left[\frac{\alpha_i}{\beta_i} \right] \right).$$

Clearly,

(a) if $c' < c$, then $M' \in \mathscr{D}'_k$;

(b) if $c' = c$, then $M' \in \mathscr{D}_k \cup \mathscr{E}_k$;

(c) if $c' > c$, then $M' \notin \mathscr{M}_{k, \mathbf{N}}$.

Moreover, if $\beta_j = 0$ but $\alpha_j \neq 0$ for one index j then, in the above case (b), M' belongs to $\mathscr{E}_k$. Therefore, there exists a finite sequence of integers $c_0, c_1, ..., c_n$ with $c_0 \geqslant 0$ and $c_j \geqslant 1$ for $1 \leqslant j \leqslant n$, such that $(\Pi_{c_0 c_1 \cdots c_i})^{-1} A \in \mathscr{D}_k$ for $i = 0, ..., n-1$ and $(\Pi_{c_0 c_1 \cdots c_n})^{-1} A = A' \in \mathscr{E}_k$. Formula (F) is established and the uniqueness readily follows from Lemmas 1 and 2.

Let i be an integer, $1 \leqslant i \leqslant k$, with $\beta_i \neq 0$ and let $\left(\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right)$ be the ith column of A'. Therefore,

$$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \Pi_{c_0 c_1 \cdots c_n} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

From Lemma 3, if $u' \geqslant v'$, the depth of α_i/β_i is $\geqslant n$, its n first partial quotients are $c_0, c_1, ..., c_{n-1}$ and the $(n+1)$th partial quotient is $\geqslant c_n$ $(= c_n$ if $u' > v')$. Now, if $u' < v'$ the Euclidean division $v' = qu' + r$ $(q \geqslant 1,$ $0 \leqslant r < u')$ gives

$$\begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} c_n + q & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u' \\ r \end{pmatrix}.$$

Hence the n first partial quotients of α_i/β_i are $c_0, c_1, ..., c_{n-1}$ and the next one is $> c_n$. Finally, if $\beta_i = 0$ (and $\alpha_i \neq 0$) for at least one index i, the factorization (F) is reduced to $A = \Pi_{c_0} A'$ with $c_0 \neq 0$. ∎

3. FINITE TRANSDUCERS AND MÖBIUS MAPS

3.1. To each matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbf{Z}$, and $ab - bc = \pm D$, $D > 1$, we associate the Möbius map

$$[A]: x \mapsto \frac{ax+b}{cx+d}$$

and the norm $\rho(A) = \max\{|a| + |b|, |c| + |d|\}$. Equalities $ad - bc = d(a-c) + c(d-b) = a(d-b) + b(a-c)$ show that $\rho(A) \leqslant |\det(A)|$ for any A in $\mathscr{E}_2$. In particular

Lemma 5. *The subset of all matrices A of $\mathscr{E}_2$ such that* $\det(A) = \pm D$ *is finite.*

Lemma 6. *Let $x = [c_0; c_1, c_2, ...]$ be an irrational real number and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a row-balanced matrix in $\mathscr{M}_{2,\mathbf{N}}$. Then there exists an integer $n \leqslant 2\log_2 |\det(A)|$ such that $A\Pi_{c_0 c_1 \cdots c_n} \notin \mathscr{E}_2$. Moreover, if (by (F)), $A\Pi_{c_0 c_1 \cdots c_n} = \Pi_{c'_0 c'_1 \cdots c'_r} A'$, then $(ax+b)/(cx+d) = [c'_0; c'_1, ..., c'_{r-1}, ...]$ and the rth partial quotient of this continued fraction expansion is $\geqslant c'_r$.*

Proof. The result is clear for $\det(A) = \pm 1$. Henceforth, we assume $|\det(A)| \geqslant 2$ and let $(p_k/q_k)_{k \in \mathbf{N}}$ be the sequence of the convergents of x. Define A_n by

$$A_n = A\pi_{c_0 c_1 \cdots c_n} = \begin{pmatrix} ap_n + bq_n & ap_{n-1} + bq_{n-1} \\ cp_n + dq_n & cp_{n-1} + dq_{n-1} \end{pmatrix}$$

and notice that

$$\lim_{n\to\infty} \frac{ap_n + bq_n}{cp_n + dq_n} = \lim_{n\to\infty} \frac{ap_{n-1} + bq_{n-1}}{cp_{n-1} + dq_{n-1}} = \frac{ax+b}{cx+d} = [c'_0; c'_1, c'_2, ...].$$

Therefore, there exists n such that $A_n \notin \mathscr{E}_2$. But we have to say a little bit more. In fact, for any matrix B with nonnegative entries and any integers $x, y \geqslant 1$, a straightforward computation shows that $\rho(B\Pi_{xy}) \geqslant 2\rho(B)$. Consequently $\rho(A\Pi_{c_0 \cdots c_n}) \geqslant 2^{[n/2]+1}$ and $A_n \notin \mathscr{E}_2$ as soon as $n \geqslant 2\log_2 |\det(A)|$. By Theorem 1, $A_n = \Pi_{c''_0 c''_1 \cdots c''_r} A'$ and the r first partial quotients of $(ap_n + bq_n)/(cp_n + dq_n)$ and $(ap_{n-1} + bq_{n-1})/(cp_{n-1} + dq_{n-1})$ are $c''_0, c''_1, ..., c''_{r-1}$.

Since $(ax+b)/(cx+d)$ is between $(ap_n+bq_n)/(cp_n+dq_n)$ and $(ap_{n-1}+bq_{n-1})/(cp_{n-1}+dq_{n-1})$, one derives $c'_k=c''_k$ for all indices $k\leqslant r-1$ and $c'_r\geqslant c''_r$. ∎

Remark 3. In order to compute the continued fraction expansion of $(ax+b)/(cx+d)$ in the general case ($a, b, c, d\in\mathbf{Z}$), we mainly have to consider matrices $A=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $\mathscr{E}_2$ and $x>1$. Indeed, the other cases can be reduced to this one. In fact, suppose that $x=[c_0; c_1, c_2, ...]<1$ ($c_0\in\mathbf{Z}$); then $x'=[c_1; c_2, c_3, ...]>1$ and $(ax+b)/(cx+d)=(a'x'+b')/(c'x'+d')$ with

$$\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}c_0 & 1\\ 1 & 0\end{pmatrix}.$$

Now assume that $[(a'x'+b')/(c'x'+d')]=c'_0<0$ with $x'>1$. Then

$$\begin{pmatrix}0 & 1\\ 1 & -c'_0\end{pmatrix}\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix}a'' & b''\\ c'' & d''\end{pmatrix}$$

with $(a''x'+b'')/(c''x'+d'')>1$; the first partial quotient of $(a'x'+b')/(c'x'+d')$ is c'_0 and the next ones are those of $(a''x'+b'')/(c''x'+d'')$. Moreover, continued fraction expansions of t and $-t$ are closely related. In fact if $t=[t_0; t_1, t_2, ...]$ with $t_1>1$ then $-t=[-(t_0+1); 1, (t_1-1), t_2, t_3, ...]$ and for $t_1=1$, $-t=[-(t_0+1); t_2+1, t_3, ...]$. Therefore we may assume $a''>0$ and $c''>0$. Finally, suppose that $\left(\begin{smallmatrix}a'' & b''\\ c'' & d''\end{smallmatrix}\right)\notin\mathscr{M}_{2,\mathbf{N}}$ but $x'=[c_1; c_2, c_3, ...]>1$ and $(a''x'+b'')/(c''x'+d'')>0$. It easily follows that there exists integer r such that either $\left(\begin{smallmatrix}a'' & b''\\ c'' & d''\end{smallmatrix}\right)\Pi_{c_1c_2\cdots c_r}$ or its opposite is in $\mathscr{D}_2\cup\mathscr{D}'_2$. By Theorem 1,

$$\begin{pmatrix}a'' & b''\\ c'' & d''\end{pmatrix}\Pi_{c_1c_2\cdots c_r}=\Pi_{c'_1c'_2\cdots c'_s}A'$$

and $A'=\left(\begin{smallmatrix}a' & b'\\ c' & d'\end{smallmatrix}\right)\in\mathscr{E}_2$. After $c'_1, c'_2, ..., c'_s$, the next partial quotients of $(a''x'+b'')/(c''x'+d'')$ are those of $(a'x_r+b')/(c'x_r+d')$ with $x_r=[c_{r+1}; c_{r+2}, c_{r+3}, ...]>1$.

3.2. *Transducers Associated to Möbius Maps*

Let $\mathscr{M}_D$ be the finite set of all row-balanced matrices in $\mathscr{M}_{2,\mathbf{N}}$ whose determinants are $\pm D$. We consider a transducer $\mathscr{T}_D=(\mathscr{C}, \mathscr{B}, \mathscr{A}, \Phi, \Psi)$ with $\mathscr{C}=\mathbf{N}\backslash\{0\}$, $\mathscr{B}=\mathscr{M}_D$, $\mathscr{A}=(\mathbf{N}\backslash\{0\})\cup\{0a\}_{a\in\mathbf{N}\backslash\{0\}}$. For all $c\in\mathscr{C}$ and $B\in\mathscr{B}$, the instructions ϕ_c and ψ_c of $\mathscr{T}_D$ are defined as follows:

— if $B\Pi_c=B'\in\mathscr{M}_D$, then $\phi_c(B)=B'$ and $\psi_c(B)=\wedge$;

— if $B\Pi_c\notin\mathscr{M}_D$, according to Theorem 1, there is a unique factorization $B\Pi_c=\Pi_{c'_1c'_2\cdots c'_r}B'$ with $B'\in\mathscr{E}_k$, $c'_1\in\mathbf{N}$ and $c'_j\in\mathbf{N}\backslash\{0\}$, $j=2, 3, ..., r$. Moreover, if $c'_1=0$, then $r\geqslant 2$.

Consequently we set by definition $\phi_c(B) = B'$ and

$$\psi_c(B) = \begin{cases} p_1 p_2 \cdots p_r & \text{with } p_k = c'_k \text{ for } 1 \leqslant k \leqslant r \text{ if } c'_1 \neq 0 \\ p_1 \cdots p_{r-1} & \text{with } p_1 = 0c'_2 \text{ and } p_k = c'_{k+1} \\ & \text{for } 2 \leqslant k \leqslant r-1 \text{ if } c'_1 = 0. \end{cases}$$

Now choose any matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathscr{M}_D$ and any irrational real number $x = [c_0; c_1, c_2, ...]$ greater than 1. Taking A as the initial state of $\mathscr{T}_D$ we get $[\Psi, \Phi]_{c_0 c_1 \cdots c_n}(A) = P_1 P_2 \cdots P_s$ with $P_i \in \mathscr{A}$. Words in $\mathscr{A}^*$ will be reduced according to the relation

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Two cases arise; $P_1 P_2 \cdots P_s = (0)\, P'_1 P_2 \cdots P_s$ or $P_1 P_2 \cdots P_s = (\wedge) P'_1 P_2 \cdots P_s$ with $P'_1 \in \mathbf{N} \backslash \{0\}$. Notice that $P'_1 P_2 \cdots P_s$ does not end with the letter 0 so that, due to relation 3, $\alpha(P'_1 P_2 \cdots P_s) \in \mathscr{N}_2$. Finally, there exists a unique word $w \in (\mathbf{N} \backslash \{0\})^*$ such that $\alpha(P_1 P_2 \cdots P_s) = \alpha(w)$ if $P_1 \in \mathbf{N} \backslash \{0\}$ and $\alpha(P_1 P_2 \cdots P_s) = \alpha(0)\, \alpha(w)$ otherwise. The word w is obtained by reading $P'_1 P_2 \cdots P_s$ from left to right, replacing subwords of the form $a(0b)$ ($a, b \in \mathbf{N} \backslash \{0\}$), by the letter $(a+b)$. This defines a map $\mu: \mathscr{A}^* \mapsto (\mathbf{N} \backslash \{0\})^* \cup (0(\mathbf{N} \backslash \{0\})^*)$ by $\mu(P_1 P_2 \cdots P_s) = w$ if $P_1 = P'_1$ and $\mu(P_1 P_2 \cdots P_s) = 0w$ if $P_1 = 0P'_1$. Notice that $|w| = |P'_1 P_2 \cdots P_s| - 2\,|P'_1\, P_2 \cdots P_s|_0$.

Let τ be the map defined on $(\mathbf{N} \backslash \{0\})^* \cup (0(\mathbf{N} \backslash \{0\})^*)$ by $\tau(c'_0 c'_1 \cdots c'_k) = c'_0 c'_1 \cdots c'_{k-1}$. Now we define the so-called reducing map

$$\beta = \tau \circ \mu. \tag{4}$$

THEOREM 2. *For any Möbius map $[A]$ associated to the matrix $A \in \mathscr{M}_D$ and for any irrational real number $x = [c_0; c_1, c_2, ...] > 1$, the continued fraction expansion $[A](x) = [c'_0; c'_1, c'_2, ...]$ is given from the transducer $\mathscr{T}_D$ (defined above) and the reducing map β (see (4)) by the formula*

$$\beta([\Psi, \Phi]_{c_0 c_1 \cdots c_n}) = c'_0 \cdots c'_{k_n}$$

with $k_n + 2 \geqslant n/\mathrm{card}(\mathscr{M}_D)$.

Proof. By the construction of $\mathscr{T}_D$ and Lemma 6, it remains to prove that $k_n + 2 \geqslant [n/\mathrm{card}(\mathscr{M}_D)]$. Let $B = B_0$ be any state of $\mathscr{T}_D$ and let $a_0 \cdots a_{m-1}$ be a word on $\mathbf{N} \backslash \{0\}$ of length $m \geqslant \mathrm{card}(\mathscr{M}_D)$. Put $B_{j+1} = \phi_{a_j}(B_j)$ and $\psi(B_j) = a'_j$ for $j = 0, ..., m-1$. We claim that there is an index $j \in \{0, ..., m-1\}$ such that $a'_j \notin \{\wedge\} \cup \{0b;\, b \in \mathbf{N} \backslash \{0\}\}$. In fact, there exist k and ℓ, $0 \leqslant k < \ell \leqslant \mathrm{card}(\mathscr{M}_D)$, with $B_k = B_l$. Therefore, if we assume that all outputs a_j are in $\{\wedge\} \cup \{0b; b \in \mathbf{N} \setminus \{0\}\}$, we can exhibit a state B and a word $a_0 \cdots a_{\ell-1}$ of length $\ell \leqslant \mathrm{card}(\mathcal{M}_D)$, with $a_i \in \mathbf{N} \setminus \{0\}$ and $B = B_0 = B_\ell$. This equality implies

$$B\Pi_{a_0 \cdots a_{\ell-1}} = \Pi_{a'_0 \cdots a'_{\ell-1}} B, \tag{5}$$

where each a'_i is the empty word or a word of the form $0b_i$, $b_i \in \mathbf{N} \setminus \{0\}$. But $a'_0 \cdots a'_{\ell-1}$ cannot be the empty word, hence $\Pi_{a'_0 \cdots a'_{\ell-1}} = \Pi_{0N}$ where N is the sum of integers b_i. Equality (5) gives

$$B(\Pi_{a_0 \cdots a_{\ell-1}})^n = \begin{pmatrix} 1 & 0 \\ nN & 1 \end{pmatrix} B \qquad (n \in \mathbf{N}), \tag{6}$$

but the entries of the matrix given by the left side of (6) increase exponentially with respect to n while the right side says that these entries are linear in n; a contradiction which proves our claim.

Going back to the hypothesis and notations of Theorem 2, set $[\Psi, \Phi]_{c_0 \cdots c_n}(A) = a'_0 \cdots a'_n$ and consider indices $i_1 < \cdots < i_{r_n}$, where the relation $a'_i \in \mathbf{N} \setminus \{0\}$ holds. One has $\mu(a'_0 \cdots a'_n) = c'_0 \cdots c'_{r_n - 1} c''_n$ with $c'_0 = \mu(a'_0 \cdots a'_{i_1})$, $c'_k = \mu(a'_{i_k+1} \cdots a'_{i_{k+1}})$ for $k = 1, ..., r_n - 1$ and $c''_n = \mu(a'_{i_{r_n}+1} \cdots a'_n)$. From above, integers $i_1 + 1$, $i_{k+1} - i_k$, and $n - i_{r_n}$ are less than $\mathrm{card}(\mathcal{M}_D)$ so that $n \leqslant (r_n + 1)\mathrm{card}(\mathcal{M}_D)$. Now, the equality $k_n = r_n - 1$ concludes the proof. ∎

EXAMPLE 1. Multiplication by 2 is associated to the matrix $H_2 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We distinguish four classes of inputs c, namely $c = 1$, $c = 2$, c odd $\geqslant 3$, and c even $\geqslant 4$.

There are eight row-balanced matrices whose determinants are ± 2, but starting with the state H_2, only five states can be reached by reading words in $(\mathbf{N} \setminus \{0\})^*$, namely

$$H_2, \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

The transducer $\mathcal{T}_2$ restricted to these states will be denoted by $\mathcal{T}'_2$. In fact, for $X \in \mathcal{M}_2$ and $c \in \mathbf{N} \setminus \{0\}$, $\phi_c(X)$ is one of the above states so that, after the first instruction, we are only concerned with the transducer $\mathcal{T}'_2$. Figure 1 gives a symbolic representation of $\mathcal{T}'_2$.

Let us compare this with the Raney construction. Continued fractions $x = [c_0; c_1, c_2, ...]$ are replaced by infinite strings

$$\xi = R^{c_0} L^{c_1} R^{c_2} L^{c_3} \cdots$$

with $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. The transducer corresponding to the multiplication by 2 is then a two-tape automaton $\Theta_2 = (Q, E)$ over the alphabet

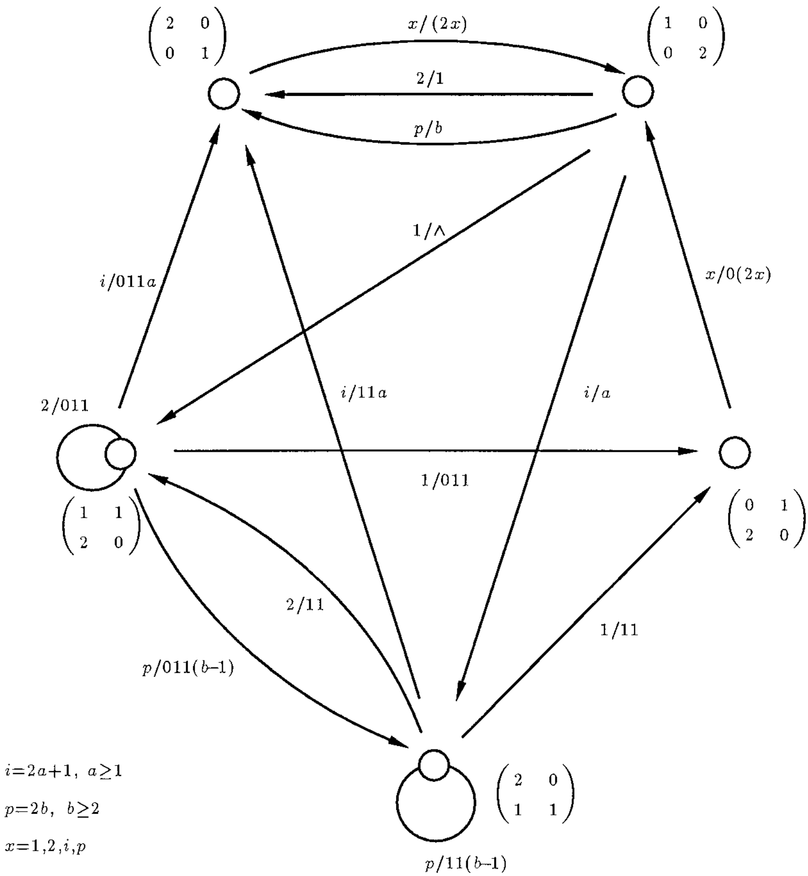

FIG. 1. Transducer $\mathscr{T}'_2$ associated to the multiplication by 2.

$\{R, L\}$, where the space of states Q has two elements, namely $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and the set $E \subset Q \times \{R, L\}^* \times \{R, L\}^* \times Q$ of input–output instructions has elements $(A, L^2/L, A)$, $(A, LR/RL, B)$, $(A, R/R^2, A)$, and those obtained by interchanging A with B and R with L (see Fig. 2). Notice that there are input and output codes for each state. Starting from A, we first read the prefix P of ξ which is a word of the input code $\{L^2, LR, R\}$ associated to A, then go to the next state X and output W if $(A, P/W, X) \in E$. The next step runs in the same way replacing A by X. One sees that the string ξ is factorized by the automaton, step by step, according to the input code corresponding to the current state.

Using transducer $\mathscr{T}'_2$, a straightforward computation leads to the following by-product:

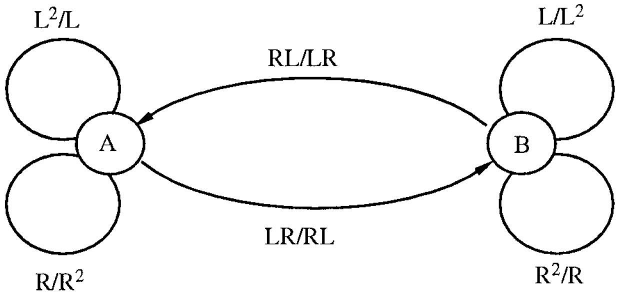

FIG. 2. The Raney transducer Θ_2 associated to the multiplication by 2.

(i) *Let x be the purely quadratic number $[\bar{n}]$. Then the greatest possible length for the period of $2x$ is 5 and this bound is reached in the particular case where n is odd and $\geqslant 3$. In fact, for $n = 2i + 1$,*

$$2x = [\overline{2n, i, 1, 1, i}].$$

(ii) *For any input partial quotient $c \geqslant 3$ and any state X of $\mathcal{T}'_2$, the output obtained after reduction contains a partial quotient which belongs to* $\{2c, 2c+1, 2c+2, [c/2], [c/2]-1\}$.

Part (ii) explains some results of Lang and Trotter [La–Tr] who computed the continued fraction expansion for several algebraic numbers, in particular $2^{1/3}$ and $4^{1/3}$. Since $4^{1/3}$ is the image of $2^{1/3}$ by the Möbius map associated with the matrix $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$, then large partial quotients are related: *e.g.*, the 36th partial quotient of $2^{1/3}$ is 534 and leads to the 41st one of $4^{1/3}$ which is 266. The 572th partial quotient of $2^{1/3}$ is 7451 and leads to the 579th one of $4^{1/3}$ which is 14902, etc.

3.3. *A Formal Two-Tape Automaton*

The transducer $\mathcal{T}'_D$ can be reorganized into a finite transducer $\mathcal{F}_D$ with the same space of states $\mathcal{M}_D$, but the input alphabet is

$$\mathcal{S}_D = \{\bar{1}, ..., \bar{D}\} \cup \{\ell_1, ..., \ell_D\},$$

where $\bar{s}$ denotes the constant map with constant value s and ℓ_s, for $1 \leqslant s \leqslant D$, is the linear polynomial defined by

$$\ell_s(x) = Dx + s \qquad (x \in \mathbf{N} \backslash \{0\}).$$

Finally, the output alphabet $\mathcal{W}_D$ is a finite set of words on an alphabet which contains a finite number of nonnegative integers and a finite number of linear polynomials p which depend on the input letter, the current state and such that

$$p(\mathbf{N}\backslash\{0\}) \subset \mathbf{N}\backslash\{0\}.$$

In fact this transducer can be viewed as a two-tape automaton $(\mathcal{M}_D, E_D)$, where E_D is a subset of $\mathcal{M}_D \times \mathcal{S}_D \times \mathcal{W}_D \times \mathcal{M}_D$ such that any element (B, u, w, B') (also denoted by $(B, u/w, B')$) in E_D has the following form:

(j) If $u = \bar{s}$ with $1 \leqslant s \leqslant D$, then $w = \psi_s(B)$ and $B' = \phi_s(B)$.

In the case where $u = \ell_s$, we need a result proved in [St2, Lemma 2], namely:

Proposition 1. *Let $B \in \mathcal{M}_D$ and $s \in \{1, ..., D\}$. Then there exists $B' \in \mathcal{M}_D$ such that $\phi_{\ell_s(x)}(B) = B'$ for all $x \in \mathbf{N}\backslash\{0\}$. Moreover there exists $\alpha = \alpha(s, B) \in 0(\mathbf{N}\backslash\{0\})^* \cup (\mathbf{N}\backslash\{0\})^*$, $\alpha \neq 0$ and there exists a linear polynomial $p = p_{sB}$, both α and p depending only on s, B (and D), with $p(\mathbf{N}\backslash\{0\}) \subset \mathbf{N}\backslash\{0\}$ such that the matrix equality*

$$B \begin{pmatrix} \ell_s(x) & 1 \\ 1 & 0 \end{pmatrix} = \Pi_\alpha \begin{pmatrix} p(x) & 1 \\ 1 & 0 \end{pmatrix} B'$$

holds for all $x \in \mathbf{N}\backslash\{0\}$.

We are ready to complete the definition of E_D, using notations of Proposition 1:

(jj) If $u = \ell_s$, then $B' = \phi_{s+D}(B)$ and $w = \alpha(s, B)\, p_{sB}$.

The computation of $[A](x)$ for $A \in \mathcal{M}_D$ and $x = [c_0; c_1, c_2, ...] > 1$ with the two-tape automaton $(\mathcal{M}_D, E_D)$ runs as follows: We first introduce the substitution θ: $\mathbf{N}\backslash\{0\} \to \mathcal{S}_D$ defined by

$$\theta(c) = \begin{cases} \bar{c}, & \text{if } \ 1 \leqslant c \leqslant D \\ \ell_c, & \text{otherwise.} \end{cases}$$

Let $(d_n)_n$ be the output sequence given by $\mathcal{T}_D$ from the input sequence $(c_n)_n$, but without applying the reducing map. Running the two-tape automaton $(\mathcal{M}_D, E_D)$ with the input sequence $(\theta(c_n))_n$ gives the output sequence $(w_n)_n$ with $w_n = v_n p_n$, where p_n is the empty word (if $c_n \leqslant D$) or a suitable linear polynomial given by Proposition 1. Then

$$d_n = \begin{cases} v_n p_n(x_n), & \text{with } \ x_n = [c_n/D] \text{ if } c_n > D, \\ v_n, & \text{otherwise.} \end{cases}$$

4. CONTINUED FRACTION EXPANSIONS OF BIQUADRATIC FRACTIONAL TRANSFORMATIONS

4.1. Let $x = [c'_0; c'_1, c'_2, ...]$ and $y = [c_0; c_1, c_2, ...]$ be two irrational real numbers and let a, b, c, d, e, f, g, h be integers such that the biquadratic fractional quantity

$$z := \frac{axy + bx + cy + d}{exy + fx + gy + h}$$

is irrational. In this section, an algorithm for computing the continued fraction expansion of z is exhibited and described by means of a transducer. Previous studies in this direction has been carried out by Gosper [Go]. The algorithm depends on two group homomorphisms from $\mathbf{GL_2(R)}$ to $\mathbf{GL_4(R)}$, which commute.

LEMMA 7. *Let Γ and Ξ be maps from $\mathbf{GL_2(R)}$ to $\mathbf{GL_4(R)}$ defined by*

$$\Gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}, \qquad \Xi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Then Γ and Ξ are homomorphisms such that

$$\Gamma(A)\, \Xi(B) = \Xi(B)\, \Gamma(A).$$

Moreover, $\det(\Gamma(A)\, \Xi(B)) = (\det A \det B)^2$ *and* $\operatorname{tr}(\Gamma(A)\, \Xi(B)) = \operatorname{tr} A \operatorname{tr} B$.

Proof. Clearly, Γ and Ξ are group homomorphisms and equalities $\Gamma(A)\,\Gamma(B) = \Gamma(AB)$, $\Xi(A)\,\Xi(B) = \Xi(AB)$ and $\Gamma(A)\,\Xi(B) = \Xi(B)\,\Gamma(A)$ are obtained by easy computations. This is also the case for the formulae on determinants and traces. ∎

LEMMA 8. *Let $(p_k/q_k)_{k \in \mathbf{N}}$ and $(p'_k/q'_k)_{k \in \mathbf{N}}$ be the sequences of convergents of y and x, respectively. Define the matrices*

$$M_k = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}, \qquad M'_k = \begin{pmatrix} p'_k & p'_{k-1} \\ q'_k & q'_{k-1} \end{pmatrix}, \qquad A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

and

$$A_k = \begin{pmatrix} a_k & b_k & c_k & d_k \\ e_k & f_k & g_k & h_k \end{pmatrix} = A\Gamma(M_k)\,\Xi(M'_k).$$

Assume that $exy + fx + gy + h \neq 0$, *then the four sequences* $(a_k/e_k)_k$, $(b_k/f_k)_k$, $(c_k/g_k)_k$, *and* $(d_k/h_k)_k$ *tend to* $(axy + bx + cy + d)/(exy + fx + gy + h)$.

Proof. A straightforward computation gives

$$\frac{a_k}{e_k} = \frac{ap_k p'_k + bq_k p'_k + cp_k q'_k + dq_k q'_k}{ep_k p'_k + fq_k p'_k + gp_k q'_k + hq_k q'_k},$$

and consequently $\lim_k a_k/e_k = (axy + bx + cy + d)/(exy + fx + gy + h)$. The other cases are similar. ∎

4.2. *Transducer* $\mathcal{T}_{\mathcal{R}}$ *Associated to a Biquadratic Fractional Transformation* $\mathcal{R}$: $(x, y) \mapsto (axy + bx + cy + d)/(exy + fx + gy + h)$

We associate to $\mathcal{R}$ a transducer $\mathcal{T}_{\mathcal{R}} = (\mathcal{C}, \mathcal{B}, \mathcal{A}, \Phi, \Psi)$ with infinite space of states $\mathcal{B} = \mathcal{E}_4$. As above (Remark 3), we may study essentially the case with $x > 1$, $y > 1$ and

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \in \mathcal{E}_4.$$

The initial state for $\mathcal{T}_{\mathcal{R}}$ will be $b_0 = A$ and alphabets are respectively $\mathcal{C} = (\mathbf{N}\backslash\{0\}) \times (\mathbf{N}\backslash\{0\})$ and $\mathcal{A} = (\mathbf{N}\backslash\{0\}) \cup \{0a\}_{a \in \mathbf{N}\backslash\{0\}}$. The matrix

$$N_{(c, c')} := \Gamma\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \Xi\begin{pmatrix} c' & 1 \\ 1 & 0 \end{pmatrix}$$

is associated to any input letter $(c, c') \in \mathcal{C}$. We are ready to complete the definition of $\mathcal{T}_{\mathcal{R}}$. For all $B \in \mathcal{B}$, if $BN_{(c, c')} = B' \in \mathcal{E}_4$, we set $\phi_{(c, c')}(B) = B'$ and $\psi_{(c, c')}(B) = \wedge$. If $BN_{(c, c')} \notin \mathcal{E}_4$, by applying (F) and Theorem 1, we get $BN_{(c, c')} = \Pi_{a'_1 a'_2 \cdots a'_r} B'$ and we put $\phi_{(c, c')}(B) = B'$, $\psi_{(c, c')}(B) = a'_1 a'_2 \cdots a'_r$. Recall that $a'_1 a'_2 \cdots a'_r$. belongs to $0(\mathbf{N}\backslash\{0\})^* \cup (\mathbf{N}\backslash\{0\})^*$.

THEOREM 3. *Let* $\mathcal{R}$: $(x, y) \mapsto (axy + bx + cy + d)/(exy + fx + gy + h)$ *be a biquadratic fractional transformation with positive integer coefficients and let* $x > 1$ *and* $y > 1$ *be any irrational numbers such that* $z = \mathcal{R}(x, y)$ *is irrational. Then the continued fraction expansion of z is given by the transducer* $\mathcal{T}_{\mathcal{R}}$ *and the reducing map* β *(see (4)). More precisely, using notations of Lemma 8 with* $x = [c'_0; c'_1, c'_2, ...]$ *and* $y = [c_0; c_1, c_2, ...]$, *there exists an index* $m_0 \geqslant 1$

such that for index $m \geqslant m_0$, *the matrix* A_m *belongs to* $\mathscr{D} \cup \mathscr{D}'$, $\beta \circ [\Psi, \Phi]_{(c_0, c'_0)(c_1, c'_1) \cdots (c_m, c'_m)} = a_0 a_1 \cdots a_{n_m}$, *and* $z = [a_0; a_1, ..., a_{n_m}, ...]$ *with* $\lim_{m \to \infty} n_m = \infty$.

Proof. From Lemma 8 there exists m_0 such that $A_m \in \mathscr{D}_4$ if $m \geqslant m_0$. For $m' \geqslant m \geqslant m_0$, Theorem 1 (part (i)) implies that $\beta \circ [\Psi, \Phi]_{(c_0, c'_0)(c_1, c'_1) \cdots (c_m, c'_m)}$ is a prefix of the word $\beta \circ [\Psi, \Phi]_{(c_0, c'_0)(c_1, c'_1) \cdots (c_{m'}, c'_{m'})}$. Now, both Lemma 8 and Theorem 1 (part (ii)) show that the sequence $(n_m)_m$ increases to infinity. ▮

Remark 4. If z is assumed to be rational, then Theorem 1 implies that there exists an index m_0 and a word $\alpha \in 0(\mathbf{N} \backslash \{0\})^* \cup (\mathbf{N} \backslash \{0\})^*$ such that for $m \geqslant m_0$ the equality $A_m = \Pi_\alpha B_m$ holds with $B_m \in \mathscr{E}_4$. This situation, and also some cases, where x, y, and z are irrational quadratic numbers, will be examined in a forthcoming paper [Li-St], where the sequences of states, obtained by applying the transducer $\mathscr{T}_{\mathscr{R}}$, satisfy linear recurrences.

EXAMPLE 2. The number $3\sqrt{2}$ $(= [4; \overline{4, 8}]^\infty)$ can be computed from $\sqrt{2}$ $(= [1, \overline{2}]^\infty)$ using Section 3. We can also however apply the above transducer associated to $(x, y) \mapsto x + y$ with $x = \sqrt{2}$ and $y = 2\sqrt{2}$ $(= [2; \overline{1, 4}]^\infty)$. In that case the initial state is $A = \left(\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. Let $(b_n)_n$ be the sequence of successive states of the transducer $\mathscr{T}_{\mathscr{R}}$ obtained from the input sequence $(1, 2)((2, 1)(2, 4))^\infty$ which corresponds to $x = \sqrt{2}$ and $y = 2\sqrt{2}$. A straightforward computation shows that $(b_n)_n$ verifies a linear recurrence of order 8, namely

$$b_{n+8} = 34 b_{n+4} - b_n, \qquad n \geqslant 1.$$

Moreover, $\psi_{(2,1)}(b_{4n+1}) = \wedge$, $\psi_{(2,4)}(b_{4n+2}) = 012$, $\psi_{(2,1)}(b_{4n+3}) = 021$, and $\psi_{(2,4)}(b_{4n+4}) = 06$ for $n \geqslant 0$.

5. CONTINUED FRACTION EXPANSION FOR VALUES OF RATIONAL MAPS

5.1. *A Homomorphism from* $\mathbf{GL}_2(\mathbf{R})$ *to* $\mathbf{GL}_{n+1}(\mathbf{R})$

In this section, irrational quantities of the form $P(x)/Q(x)$ are considered where x is an irrational real number and $P(\cdot)$, $Q(\cdot)$ are polynomials with rational coefficients. We will define an algorithm as a transducer for computing the continued fraction expansion of $R(x) = P(x)/Q(x)$. It is quite natural to replace $R(x)$ by the homogeneous form $R(u, v) = v^n P(u/v)/v^n Q(u/v)$, where n is the projective degree of $R(\cdot)$. If $P(X) = p_n X^n + p_{n-1} X^{n-1} + \cdots + p_0$ and $Q(X) = q_n X^n + q_{n-1} X^{n-1} + \cdots + q_0$ then

$$\begin{pmatrix} v^n P(u/v) \\ v^n Q(u/v) \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 \end{pmatrix} \begin{pmatrix} u^n \\ u^{n-1}v \\ \vdots \\ n \end{pmatrix}.$$

This suggests a linkage of $R(u, v)$ with $R(au+cv, bu+dv)$ for any matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We will get it without any reference to $R(\cdot)$. First, let $\mathscr{P}_n(x, y)$ (or for short $\mathscr{P}_n$) be the vector space of homogeneous real polynomials in variables x and y of degree n and let the map $\Pi_n: \mathscr{P}_1^2 \to \mathscr{P}_n^{n+1}$ be defined by

$$\Pi_n(u, v) = (u^n \;\; u^{n-1}v \;\; \cdots \;\; uv^{n-1} \;\; v^n).$$

Notice that Π_n is homogeneous of degree n.

Lemma 9. *The map Π_n sends a row of two linearly independent polynomials of $\mathscr{P}_1$ to a row of $n+1$ linearly independent polynomials of $\mathscr{P}_n$.*

The proof is left to the reader. Following our plan, we introduce a new definition:

Definition 2. Let u and v be two linearly independent polynomials of $\mathscr{P}_1$ and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in $\mathbf{GL_2}(\mathbf{R})$. From Lemma 9

$$\Pi_n((u \quad v)\, A)$$
$$= ((au+cv)^n \;\; (au+cv)^{n-1}(bu+dv) \;\; \cdots \;\; (au+cv)(bu+dv)^{n-1} \;\; (bu+dv)^n)$$

is a row of $n+1$ linearly independent polynomials. Therefore, there exists a unique matrix in $\mathbf{GL_{n+1}}(\mathbf{R})$ denoted by $f_n(A)$ and defined by the relation

$$\Pi_n[(u \quad v)\, A] = (\Pi_n[(u \quad v)])\, f_n(A).$$

Remark 5. (a) If I_k denotes the identity in $\mathbf{GL_k}(\mathbf{R})$, then $f_n(I_2) = I_{n+1}$.

(b) $f_1(A) = A$, $f_2\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

$$= \begin{pmatrix} a^2 & ab & c^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \text{ and more generally}$$

$$f_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \begin{pmatrix} a^n & a^{n-1}b & a^{n-2}b^2 & \cdots & ab^{n-1} & b^n \\ \binom{n}{1} a^{n-1}c & \cdot & \cdot & \cdots & \cdot & \binom{n}{1} b^{n-1}d \\ \binom{n}{2} a^{n-2}c^2 & \cdot & \cdot & \cdots & \cdot & \binom{n}{2} b^{n-2}d^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{n-1} ac^{n-1} & \cdot & \cdot & \cdots & \cdot & \binom{n}{n-1} bd^{n-1} \\ c^n & c^{n-1}d & c^{n-2}d^2 & \cdots & cd^{n-1} & d^n \end{pmatrix}.$$

THEOREM 4. *The map* f_n *is a* 1–1 *homomorphism from* $\mathbf{GL_2}(\mathbf{R})$ *to* $\mathbf{GL_{n+1}}(\mathbf{R})$ *and for* $A \in \mathbf{GL_2}(\mathbf{R})$, $\det(f_n(A)) = (\det(A))^{n(n+1)/2}$.

Proof. Let A and B be matrices in $\mathbf{GL_2}(\mathbf{R})$. From the definition,

$$\begin{aligned}(\Pi_n[(u \quad v)])\, f_n(AB) &= \Pi_n[(u \quad v)\, AB] = \Pi_n([(u \quad v)\, A]\, B) \\ &= \Pi_n[(u \quad v)\, A]\, f_n(B) = (\Pi_n[(u \quad v)]\, f_n(A))\, f_n(B) \\ &= \Pi_n[(u \quad v)]\, f_n(A)\, f_n(B).\end{aligned}$$

Thus $f_n(A)\, f_n(B) = f_n(AB)$. Now, to compute $\det(f_n(A))$ it is enough to assume that A is triangular. In that case, if the diagonal coefficients of A are a, b, then $f_n(A)$ is also a triangular matrix with diagonal coefficients a^n, $a^{n-1}b$, ..., ab^{n-1} and b^n. In particular $\det(f_n(A)) = (\det(A))^{n(n+1)/2}$. ∎

Remark 6. The proof shows that if λ and μ are eigenvalues of A, then λ^n, $\lambda^{n-1}\mu$, ..., $\lambda\mu^{n-1}$, μ^n are the eigenvalues of $f_n(A)$.

In the sequel it will be useful to introduce the Hilbert representation of polynomials [Ac-He].

DEFINITION 3. Let $\mathscr{R}_n$ be the linear space of all real polynomials in x of degree $\leqslant n$. To every element P of $\mathscr{R}_n$ we associate the row-matrix $M_{n,P} = (a_n \;\; a_{n-1} \;\; \cdots \;\; a_1 \;\; a_0)$ defined by $P(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i$.

Our next step is to give a convenient algebraic construction of a polynomial whose roots are the images by a Möbius map of those of a given polynomial.

DEFINITION 4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $\mathbf{GL_2}(\mathbf{R})$. We define the map $g_{n,A}: \mathcal{R}_n \mapsto \mathcal{R}_n$ by the relation $(M_{n,P})\, f_n(A) = M_{n,\, g_{n,A}(P)}$.

As we shall see, $g_{n,A}(P)$ has the required property and even more.

PROPOSITION 2. *The map $A \mapsto g_{n,A}$ is a 1–1 antimorphism from $GL_2(\mathbf{R})$ to the space* Aut$(\mathcal{R}_n)$ *of linear automorphisms of $\mathcal{R}_n$, i.e. $g_{n,A} \circ g_{n,B} = g_{n,BA}$ for all A and B in $\mathbf{GL_2}(\mathbf{R})$.*

Proof. For all $P \in \mathcal{R}_n$, we have successively $M_{n,\, g_{n,BA}(P)} = M_{n,P} f_n(BA) = [M_{n,P} f_n(B)]\, f_n(A) = M_{n,\, g_{n,B}(P)} f_n(A) = M_{n,\, g_{n,A} \circ g_{n,B}(P)}$. Hence $g_{n,A} \circ g_{n,B} = g_{n,BA}$. Finally, $g_{n,I_2} = Id_{\mathcal{R}_n}$ and $g_{n,A^{-1}} = (g_{n,A})^{-1}$ for all $A \in \mathbf{GL_2}(\mathbf{R})$. ∎

PROPOSITION 3. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL_2}(\mathbf{R})$ and assume that P is a real polynomial of degree n. Then $g_{n,A}(P)$ is of degree n if and only if, for any root α of P, $a - \alpha c \neq 0$.*

Proof. Let the polynomial P be defined by $P(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i$, $(a_n \neq 0)$ and let $g_{n,a}(P)(x) = \sum_{i=0}^{n} a'_i \binom{n}{i} x^i$. Hence $a'_n = \sum_{i=0}^{n} a_i \binom{n}{i} a^i c^{n-i}$. If $c = 0$, then $a \neq 0$ and $a'_n = a_n a^n \neq 0$. If $c \neq 0$, then $a'_n = (1/c^n)\, P(a/c)$ and $a'_n \neq 0$ if and only if a/c is not a root of P. ∎

THEOREM 5. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL_2}(\mathbf{R})$ and $P(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i$. Consider the Möbius map $[A]: x \mapsto (ax+b)/(cx+d)$. Assume that P and $g_{n,A}(P)$ have the same degree n and let α be a root of P. Then*

$$g_{n,A}(P)(x) = \sum_{i=0}^{n} a_i \binom{n}{i} (ax+b)^i\, (cx+d)^{n-i}$$

and $[A]^{-1}(\alpha)$ is a root of $g_{n,A}(P)$ with the same order of multiplicity as α.

Proof. Assume that $g_{n,A}(P)(x) = \sum_{i=0}^{n} a'_i \binom{n}{i} x^i$ and set

$$C_n(x) = \begin{pmatrix} x^n \\ \binom{n}{1} x^{n-1} \\ \vdots \\ \binom{n}{n-1} x \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} g_{n,A}(P)(x) &= (a'_n \quad a'_{n-1} \quad \cdots \quad a'_1 \quad a'_0)\, C_n(x) \\ &= (a_n \quad a_{n-1} \quad \cdots \quad a_1 \quad a_0)\, f_n(A)\, C_n(x). \end{aligned}$$

Now, $f_n(A)\, C_n(x)$ is the first column of the square matrix $f_n(A)\, f_n((\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}))$ which is also equal to $f_n(A(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}))$ or $f_n((\begin{smallmatrix} ax+b & 0 \\ cx+d & 0 \end{smallmatrix}))$. Therefore, $f_n(A)\, C_n(x) = (cx+d)^n\, C_n([A](x))$ and $g_{n,A}(P)(x) = \sum_{i=0}^{n} a_i \binom{n}{i} (ax+b)^i (cx+d)^{n-i}$. If $ct+d \neq 0$ for a given value t, then $g_{n,A}(P)(t) = (ct+d)^n\, P((at+b)/(ct+d))$. Consequently, if α is a root of P and if P and $g_{n,A}(P)$ have the same degree n, then $[A]^{-1}(\alpha)$ is a root of $g_{n,A}(P)$ with the same order of multiplicity. ▌

The above constructions lead to the following fundamental theorem.

THEOREM 6. *Let* $M = (\begin{smallmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ a'_n & a'_{n-1} & \cdots & a'_1 & a'_0 \end{smallmatrix})$ *be a matrix such that* a_i *and* a'_i *are integers for* $0 \leqslant i \leqslant n$. *Let* P, Q, *and* R *be defined by*

$$P(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i, \qquad Q(x) = \sum_{i=0}^{n} a'_i \binom{n}{i} x^i, \qquad R(x) = \frac{P(x)}{Q(x)}.$$

Let $\xi = [c_0; c_1, c_2, ...]$ *be a real number such that* $P(\xi) \neq 0$, $Q(\xi) \neq 0$ *and define*

$$M_k = M f_n(\Pi_{c_0 c_1 \cdots c_k}) = \begin{pmatrix} a_{k,n} & a_{k,n-1} & \cdots & a_{k,1} & a_{k,0} \\ a'_{k,n} & a'_{k,n-1} & \cdots & a'_{k,1} & a'_{k,0} \end{pmatrix}$$

for all integers $k \geqslant 0$. *Then*, $\lim_{k \to \infty} (a_{k,i}/a'_{k,i}) = R(\xi)$ $(0 \leqslant i \leqslant n)$.

Proof. Let $(p_k/q_k)_{k \in \mathbf{N}}$ be the sequence of convergents of ξ. Then $M_k = M f_n((\begin{smallmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{smallmatrix}))$. Let P_k and Q_k be defined by

$$P_k(x) = \sum_{i=0}^{n} a_{k,i} \binom{n}{i} x^i, \qquad Q_k(x) = \sum_{i=0}^{n} a'_{k,i} \binom{n}{i} x^i.$$

Theorem 5 implies that

$$P_k(x) = \sum_{i=0}^{n} a_i \binom{n}{i} (p_k x + p_{k-1})^i (q_k x + q_{k-1})^{n-i}$$

and

$$Q_k(x) = \sum_{i=0}^{n} a'_i \binom{n}{i} (p_k x + p_{k-1})^i (q_k x + q_{k-1})^{n-i}.$$

Comparing terms of degree n and degree 0, one derives

$$a_{k,n} = q_k^n P\left(\frac{p_k}{q_k}\right), \qquad a'_{k,n} = q_k^n Q\left(\frac{p_k}{q_k}\right),$$

$$a_{k,0} = q_{k-1}^n P\left(\frac{p_{k-1}}{q_{k-1}}\right), \qquad a'_{k,0} = q_{k-1}^n Q\left(\frac{p_{k-1}}{q_{k-1}}\right).$$

Therefore $\lim_{k\to\infty}(a_{k,n}/a'_{k,n}) = \lim_{k\to\infty}(a_{k,0}/a'_{k,0}) = R(\xi)$. From Theorem 5, any root α of P with multiplicity v corresponds to a root α_k of P_k with multiplicity v, which is given by $\alpha_k = (q_{k-1}\alpha - p_{k-1})/(-q_k\alpha + p_k) = -q_{k-1}/q_k + (-1)^{k+1}/q_k(p_k - q_k\alpha)$. By assumption $|p_k/q_k - \alpha| \geqslant |\xi - \alpha|/2 > 0$ for k large enough. In other words, $\alpha_k = -q_{k-1}/q_k + \mathcal{O}(q_k^{-2})$. In the same way, any root β of Q corresponds to a root β_k of Q_k (with the same multiplicity) which is given by $\beta_k = -q_{k-1}/q_k + (-1)^{k+1}/q_k(p_k - q_k\beta)$ so that $\beta_k = -q_{k-1}/q_k + \mathcal{O}(q_k^{-2})$. Let $\alpha^{(1)}, ..., \alpha^{(n)}$ be the roots of P. Coefficients $a_{k,i}$ of P_k verify (replacing i by $n-\ell$):

$$\binom{n}{n-\ell}\frac{a_{k,n-\ell}}{a_{k,n}} = (-1)^\ell \sum_{1\leqslant i_1 < \cdots < i_\ell \leqslant n} \alpha_k^{(i_1)} \cdots \alpha_k^{(i_\ell)}.$$

But

$$\begin{aligned}
\alpha_k^{(i_1)} \cdots \alpha_k^{(i_\ell)} &= \left(-\frac{q_{k-1}}{q_k}\right)^\ell + \sum_{s=0}^{\ell-1}\left(\frac{q_{k-1}}{q_k}\right)^s \mathcal{O}\left(\frac{1}{q_k^{2(\ell-s)}}\right) \\
&= \left(-\frac{q_{k-1}}{q_k}\right)^\ell \left(1 + \mathcal{O}\left(\sum_{s=0}^{\ell-1}\left(\frac{1}{q_k q_{k-1}}\right)^{\ell-s}\right)\right) \\
&= \left(-\frac{q_{k-1}}{q_k}\right)^\ell \left(1 + \mathcal{O}\left(\frac{1}{q_k q_{k-1}}\right)\right).
\end{aligned}$$

Using this latter equality, we get

$$\frac{a_{k,n-\ell}}{a_{k,n}} = \left(\frac{q_{k-1}}{q_k}\right)^\ell \left(1 + \mathcal{O}\left(\frac{1}{q_k q_{k-1}}\right)\right).$$

The same formula holds for the ratio $a'_{k,n-\ell}/a'_{k,n}$. Therefore,

$$\lim_{k\to\infty} \frac{a_{k,n-\ell}}{a_{k,n}}\left(\frac{a'_{k,n-\ell}}{a'_{k,n}}\right)^{-1} = 1$$

and consequently, for all i, $0 \leqslant i < n$,

$$\lim_{k\to\infty}\frac{a_{k,i}}{a'_{k,i}} = \lim_{k\to\infty}\frac{a_{k,n}}{a'_{k,n}} = R(\xi). \quad \blacksquare$$

5.2. *Transducer $\mathcal{T}_R$ Associated to the Map $x \mapsto R(x) = P(x)/Q(x)$*

Let P, Q, and R be given as in Theorem 6. Let $\mathcal{T}_R = (\mathcal{C}, \mathcal{B}, \mathcal{A}, \Phi, \Psi)$ be the transducer with input alphabet $\mathcal{C} = \mathbf{N} \setminus \{0\}$, output alphabet $\mathcal{A} = (\mathbf{N} \setminus \{0\}) \cup \{0a;\, a \in \mathbf{N} \setminus \{0\}\}$ and space of states $\mathcal{B} = \mathcal{E}_{n+1}$. As in Remark 3, we may assume that $\xi = [c_0; c_1, c_2, ...] > 1$ and $M = \left(\begin{smallmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ a'_n & a'_{n-1} & \cdots & a'_1 & a'_0 \end{smallmatrix}\right) \in \mathcal{E}_{n+1}$. The initial state will be $b_0 = M$. For all $c \in \mathcal{C}$ and $B \in \mathcal{B}$, instructions ϕ_c and ψ_c of $\mathcal{T}_R$ are defined as follows: if $Bf_n((\begin{smallmatrix} c & 1 \\ 1 & 0 \end{smallmatrix})) = B'$ belongs to $\mathcal{B}$, then $\phi_c(B) = B'$ and $\psi_c(B) = \wedge$. Otherwise, by applying (F) and Theorem 1, we get $Bf_n((\begin{smallmatrix} c & 1 \\ 1 & 0 \end{smallmatrix})) = \Pi_{c'_1 c'_2 \cdots c'_r} B'$ and set $\phi_c(B) = B'$, $\psi_c(B) = c'_1 c'_2 \cdots c'_r$. We first use this transducer in the case $R(\xi)$ is not a rational number.

THEOREM 7. *For any rational map R with integer coefficients and for any real number $\xi = [c_0; c_1, c_2, ...]$ such that $R(\xi) \notin \mathbf{Q} \cup \{\infty\}$, the continued fraction expansion of $R(\xi)$ is given by the transducer $\mathcal{T}_R$ and the reducing map β (see (4)). If $\beta \circ [\Psi, \Phi]_{c_0 c_1 \cdots c_m} = c'_0 c'_1 \cdots c'_{p_m}$, then*

$$R(\xi) = [c'_0; c'_1, ..., c'_{p_m}, ...]$$

with $\lim_{m \to \infty} p_m = \infty$.

The proof is analogous to that of Theorem 3, using Theorem 4 and Theorem 6. Since $R(\xi) \notin \mathbf{Q} \cup \{\infty\}$, the length of $c'_0 c'_1 \cdots c'_{p_m}$ tends to infinity with m. ∎

5.3. *An Algorithm Which Computes the Continued Fraction Expansion of $2^{1/3}$*

Let $g: \mathbf{R} \mapsto \mathbf{R}$ be the rational map $g(x) = (x^4 + 4x)/(2x^3 + 2)$ and let $(u_k)_{k \in \mathbf{N}}$ be the sequence defined by $u_0 = 1$ and $u_{k+1} = g(u_k)$, $k \geqslant 1$. Classically, $\lim_{k \to \infty} u_k = 2^{1/3}$. Set $2^{1/3} = [c_0; c_1, c_2, ...]$ and let $(p_k/q_k)_{k \in \mathbf{N}}$ be the corresponding sequence of the convergents. The matrix $M = \left(\begin{smallmatrix} 2 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{smallmatrix}\right)$ is associated to g. Starting with the two first partial quotients of $2^{1/3}$ (namely $c_0 = 1$ and $c_1 = 3$), we get successively from the transducer $\mathcal{T}_g$

$$\phi_3 \circ \phi_1(b_0) = \begin{pmatrix} 44 & 12 & 4 & 1 & 0 \\ 20 & 10 & 0 & 1 & 2 \end{pmatrix} = b_1, \qquad [\Psi, \Phi]_{13}(b_0) = 1315,$$

$$\phi_5 \circ \phi_1(b_1) = \begin{pmatrix} 704 & 121 & 22 & 3 & 0 \\ 654 & 121 & 12 & 9 & 6 \end{pmatrix} = b_2, \qquad [\Psi, \Phi]_{15}(b_1) = 15114118,$$

$$\phi_8 \circ \phi_1 \circ \phi_1 \circ \phi_4 \circ \phi_1 \circ \phi_1(b_2) = \begin{pmatrix} 966948 & 112743 & 13426 & 1465 & 124 \\ 299826 & 35580 & 3818 & 598 & 138 \end{pmatrix}$$

and now $[\Psi, \Phi]_{114118}(b_2) = 1141181(14)\,1(10)\,214(12)\,2321$; hence,

$$2^{1/3} = [1; 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots].$$

In three steps, 21 partial quotients have been obtained! This fact leads to the natural question: "is it possible to compute all partial quotients of $2^{1/3}$ using as above the transducer $\mathscr{T}_g$ and only the two first partial quotients?" The next theorem gives a positive answer.

THEOREM 8. *Let $2^{1/3} = [c_0; c_1, c_2, \ldots]$ be the continued fraction expansion of $2^{1/3}$ and for any integer $k > 1$ let c be an integer such that $1 \leqslant c \leqslant c_k$. Then the transducer $\mathscr{T}_g$ gives $[\Psi, \Phi]_{c_0 c_1 \cdots c_{k-1} c}(b_0) = c_0 c_1 \cdots c_{k-1} c_k \cdots c_{m-1} c'$ with $m \geqslant k$ and $1 \leqslant c' \leqslant c_m$. Moreover, if $c = c_k$ then $m > k$ and if $c < c_k$ with $m = k$, then $c < c' \leqslant c_k$.*

In other words, Theorem 7 shows that the output of the transducer is always "better" than the input. The proof uses a measure of irrationality of $2^{1/3}$. In fact, we just need the following.

LEMMA 10. *With the notations in Theorem 7, let q_k be the denominator of the kth convergent of $2^{1/3}$. Then $q_{k+1} < q_k^2$ for $k \geqslant 4$. In particular $c_{k+1} < q_k$ and* $\max\{c_{k+2}, c_{k+3}\} < q_k^4$.

Proof. According to a result of Baker [Ba], the inequality

$$\left|2^{1/3} - \frac{p}{q}\right| > \frac{10^{-6}}{q^{2,955}} \tag{7}$$

holds for all rational numbers p/q (p and q co-primes). In particular, for all convergents p_n/q_n of $2^{1/3}$ satisfying $q_n > 10^{134}$, one has $|2^{1/3} - p_n/q_n| > q_n^{-3}$. From the rough estimate $((1+\sqrt{5})/2)^n < q_n$, (7), and the classical inequality $|2^{1/3} - p_n/q_n| < 1/(q_n q_{n+1})$, it follows that $q_{n+1} < q_n^2$ for all $n > 642$. The remaining cases are checked directly using the thousand first partial quotients of $2^{1/3}$ given in [La-Tr]. ∎

Proof of Theorem 8. For any integer $k \geqslant 1$, we define $x_k = [c_{k+1}; c_{k+2}, c_{k+3}, \ldots]$ $(= (q_{k-1} 2^{1/3} - p_{k-1})/(p_k - q_k 2^{1/3}))$. For $k \geqslant 5$, we have $x_k > 1 + 1/(c_{k+2} + 1)$ and by Lemma 10, $x_k > 1 + 1/q_{k+1}$ but we shall use the weaker inequality $x_k > 1 + 1/q_{k-1}^4$. Now we set

$$Mf_4\left(\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}\right) = \begin{pmatrix} a'_4 & a'_3 & a'_2 & a'_1 & a'_0 \\ b'_4 & b'_3 & b'_2 & b'_1 & b'_0 \end{pmatrix}$$

and

$$\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}^{-1} Mf_4\left(\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}\right) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{pmatrix} = M_k.$$

A straightforward computation gives

$$\left|\frac{a_j}{b_j} - x_k\right| = \frac{|a'_j - b'_j 2^{1/3}|}{|p_k b'_j - q_k a'_j|\ |p_k - q_k 2^{1/2}|},$$

$$\left|\frac{b_j}{a_j} - \frac{1}{x_k}\right| = \frac{|a'_j - b'_j 2^{1/3}|}{|p_{k-1} b'_j - q_{k-1} a'_j|\ |p_{k-1} - q_{k-1} 2^{1/3}|}.$$

We claim that $M_k \in \mathscr{D}_5$. After easy but cumbersome calculations we obtain

(a) $a'_4 - b'_4 2^{1/3} = 2(p_k - q_k 2^{1/3})^3\,(p_k + q_k 2^{1/3})$,

(b) $a'_3 - b'_3 2^{1/3} = (p_k - q_k 2^{1/3})^2 \left[\frac{1}{2}(p_k - q_k 2^{1/3})(p_{k-1} + q_{k-1} 2^{1/3}) + \frac{3}{2}(p_k + q_k 2^{1/3})(p_{k-1} - q_{k-1} 2^{1/3})\right]$,

(c) $a'_2 - b'_2 2^{1/3} = (p_k - q_k 2^{1/3})^2\,(p_{k-1} - q_{k-1} 2^{1/3})(p_{k-1} + q_{k-1} 2^{1/3}) + (p_{k-1} - q_{k-1} 2^{1/3})^2\,(p_k - q_k 2^{1/3})(p_k + q_k 2^{1/3})$,

(d) $a'_1 - b'_1 2^{1/3} = (p_{k-1} - q_{k-1} 2^{1/3})^2 \left[\frac{1}{2}(p_{k-1} - q_{k-1} 2^{1/3})(p_k + q_k 2^{1/3}) + \frac{3}{2}(p_{k-1} + q_{k-1} 2^{1/3})(p_k - q_k 2^{1/3})\right]$,

(e) $a'_0 - b'_0 2^{1/3} = 2(p_{k-1} - q_{k-1} 2^{1/3})^3\,(p_{k-1} + q_{k-1} 2^{1/3})$.

and

(a′) $p_k b'_4 - q_k a'_4 = 2 p_k q_k (p_k - q_k 2^{1/3})(p_k^2 + p_k q_k 2^{1/3} + q_k^2 4^{1/3})$

(b′) $p_k b'_3 - q_k a'_3 = (p_k q_{k-1} + p_{k-1} q_k)(p_k - q_k 2^{1/3}) \times (p_k^2 + p_k q_k 2^{1/3} + q_k^2 4^{1/3})$

(c′) $p_k b'_2 - q_k a'_2 = 2 p_{k-1} q_{k-1} (p_k - q_k 2^{1/3})(p_k^2 + p_k q_k 2^{1/3} + q_k^2 4^{1/3})$

$$\text{(d}'\text{)}\quad p_{k-1}b'_1-q_{k-1}a'_1=(p_{k-1}q_k+p_kq_{k-1})(p_{k-1}-q_{k-1}2^{1/3})\times(p_{k-1}^2+p_{k-1}q_{k-1}2^{1/3}+q_{k-1}^2 4^{1/3})$$

$$\text{(e}'\text{)}\quad p_{k-1}b'_0-q_{k-1}a'_0=2p_{k-1}q_{k-1}(p_{k-1}-q_{k-1}2^{1/3})\times(p_{k-1}^2+p_{k-1}q_{k-1}2^{1/3}+q_{k-1}^2 4^{1/3}).$$

From these formulae we derive, after simplifications,

$$\left|\frac{a_4}{b_4}-x_k\right|=\frac{|p_k-q_k2^{1/3}|\,|p_k+q_k2^{1/3}|}{p_kq_k(p_k^2+p_kq_k2^{1/3}+q_k^2 4^{1/3})},$$

$$\left|\frac{a_3}{b_3}-x_k\right|=\frac{\left[\begin{array}{l}|[1/2(p_k-q_k2^{1/3})(p_{k-1}+q_{k-1}2^{1/3})\\ \qquad +3/2(p_k+q_k2^{1/3})(p_{k-1}-q_{k-1}2^{1/3})|]\end{array}\right]}{(p_kq_{k-1}+p_{k-1}q_k)(p_k^2+p_kq_k2^{1/3}+q_k^2 4^{1/3})},$$

$$\left|\frac{a_2}{b_2}-x_k\right|=\frac{\left[\begin{array}{l}|(p_k-q_k2^{1/3})(p_{k-1}-q_{k-1}2^{1/3})(p_{k-1}+q_{k-1}2^{1/3})\\ \qquad +(p_{k-1}-q_{k-1}2^{1/3})^2\,(p_k+q_k2^{1/3})|\end{array}\right]}{2p_{k-1}q_{k-1}\,|p_k-q_k2^{1/3}|\,(p_k^2+p_kq_k2^{1/3}+q_k^2 4^{1/3})},$$

$$\left|\frac{b_1}{a_1}-\frac{1}{x_k}\right|=\frac{\left[\begin{array}{l}|[1/2(p_k+q_k2^{1/3})(p_{k-1}-q_{k-1}2^{1/3})\\ \qquad +3/2(p_k-q_k2^{1/3})(p_{k-1}+q_{k_1}2^{1/3})|]\end{array}\right]}{(p_kq_{k_1}+p_{k-1}q_k)(p_{k-1}^2+p_{k-1}q_{k-1}2^{1/3}+q_{k-1}^2 4^{1/3})},$$

$$\left|\frac{b_0}{a_0}-\frac{1}{x_k}\right|=\frac{|p_{k-1}-q_{k-1}2^{1/3}|\,|p_{k-1}+q_{k-1}2^{1/3}|}{p_{k-1}q_{k-1}(p_{k-1}^2+p_{k-1}q_{k-1}2^{1/3}+q_{k-1}^2 4^{1/3})}.$$

Using inequalities $|p_n-q_n2^{1/3}|<1/q_{n+1}<1/q_n$, $|p_n-q_n2^{1/3}|<|p_{n-1}-q_{n-1}2^{1/3}|$, and taking into account the above lower bound for x_k, we obtain $|a_j/b_j-x_k|<1/q_{k-1}^4$ for $j=2$, 3, 4 and $|b_j/b_j-1/x_k|<1/q_{k-1}^4$ for $j=0$, 1. Therefore $a_j/b_j>1$ for $j=0$, 1, 2, 4 and our claim is proved.

From Theorems 1 and 7, $[\Psi,\Phi]_{c_0c_1\cdots c_k}=c_0c_1\cdots c_k\cdots c_{m-1}c'$ and $c'\leqslant c_m$. To complete the proof, assume that $c_k\geqslant 2$, let c be an integer such that $0<c<c_k$ and let p and q be defined by $p=cp_{k-1}+p_{k-2}$ and $q=cq_{k-1}+q_{k-2}$. For all integers $k\geqslant 1$, put $x'_k=(q_{k-1}2^{1/3}-p_{k-1})/(p-q2^{1/3})=[0;c_k-c,c_{k+1},c_{k+2},...]$ and

$$N_k:=\begin{pmatrix}p & p_{k-1}\\ q & q_{k-1}\end{pmatrix}^{-1}Mf_4\left(\begin{pmatrix}p & p_{k-1}\\ q & q_{k-1}\end{pmatrix}\right)$$

$$=\begin{pmatrix}1 & 0\\ c_k-c & 1\end{pmatrix}M_kf_4\left(\begin{pmatrix}1 & 0\\ -(c_k-c) & 1\end{pmatrix}\right)$$

$$=\begin{pmatrix}\alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0\\ \alpha'_4 & \alpha'_3 & \alpha'_2 & \alpha'_1 & \alpha'_0\end{pmatrix}.$$

Using $c_{k+1} < q_{k+1}^4$, we get $x'_k < 1 - 1/(q_{k-1}^4 + 1)$ with $|\alpha_j/\alpha'_i - x'_k| = \rho_i$ and analogous computations as above lead to $0 < \alpha_i/\alpha'_i < 1$ $(i = 0, 1, 2, 4)$. Hence $N_k \in \mathscr{D}'_5$ and $[\Psi, \Phi]_{c_0 c_1 \cdots c_{k-1} c} = c_0 c_1 \cdots c_{k-1} \cdots c_{m-1} c'$ with either $m > k$ and $c' \leqslant c_m$ or $m = k$ and $c < c' \leqslant c_k$. ∎

5.4. *The Rational Case*

The two next theorems study the case where $R(\xi) \in \mathbf{Q}$. This corresponds to the fact that performing the transducer $\mathscr{T}_R$, there exists an integer n_0 such that after the input word $c_0 c_1 \cdots c_{n_0}$ (recall that $\xi = [c_0; c_1, c_2, ...]$), all the next inputs c_n $(n \geqslant n_0)$ give rise to the same outputs, namely the empty word $\wedge$.

THEOREM 9. *Let* $B = \begin{pmatrix} b_n & b_{n-1} & \cdots & b_1 & b_0 \\ b'_n & b'_{n-1} & \cdots & b'_1 & b'_0 \end{pmatrix}$ *be a matrix in* $\mathscr{M}_{n+1, \mathbf{N}}$, *let* P, Q, R *be defined as in Theorem* 6 *and let* $\xi = [c_0; c_1, c_2, ...]$. *Consider the sequence* $(B_k)_{k \in \mathbf{N}}$, *where* $B_0 = B$ *and* $B_{k+1} = B_k f_n(\begin{pmatrix} c_k & 1 \\ 1 & 0 \end{pmatrix})$ *for* $k \geqslant 0$ *and let* Σ *be the polynomial* $\Sigma(x) = \sum_{i=0}^{n} (b_i - b'_i)\binom{n}{i} x^i$. *Assume that* $B_k \in \mathscr{E}_{n+1}$ *for all* $k \geqslant 0$ (*i.e., the input sequence* $c_0 c_1 c_2 c_3 \cdots$ *for the transducer* $\mathscr{T}_R$ *gives the output* $\wedge \ \wedge \ \wedge \ \cdots$). *Then* ξ *is a root of* Σ.

Proof. Set $B_k = \begin{pmatrix} b_{k,n} & b_{k,n-1} & \cdots & b_{k,1} & b_{k,0} \\ b'_{k,n} & b'_{k,n-1} & \cdots & b'_{k,1} & b'_{k,0} \end{pmatrix}$ and $\Sigma_k(x) = \sum_{i=0}^{n} (b_{k,i} - b'_{k,i}) \binom{n}{i} x^i$, $k \geqslant 0$. Let $(p_k/q_k)_k$ be the sequence of convergents of ξ. As in the proof of Theorem 6, if β is a root of Σ with multiplicity v, then $\beta_k = (q_{k-1}\beta - p_{k-1})/(-q_k \beta + p_k)$ is a root of Σ_k with the same multiplicity v and $\beta_k = -q_{k-1}/q_k + \mathscr{O}(1/q_k^2)$ if $\beta \neq \xi$. Thus, if ξ is not a root of Σ, then $\mathscr{Re}(\beta_k) < 0$ for any root β of Σ and k large enough. This implies that the integers $b_{k,i} - b'_{k,i}$ $(0 \leqslant i \leqslant n)$ have identical signs, but $B_k \in \mathscr{E}_{n+1}$, a contradiction. ∎

THEOREM 10. *Let* M, R, *and* ξ *be defined as in Theorem* 6. *Suppose that by applying* (F) *and Theorem* 1, *we obtain*

$$M f_n(\Pi_{c_0 c_1 \cdots c_p}) = \Pi_{c'_0 c'_1 \cdots c'_r} B.$$

Let $(B_k)_{k \in \mathbf{N}}$ *be the sequence defined by* $B_0 = B$ *and* $B_{k+1} = B_k f_n(\begin{pmatrix} c_{p+k+1} & 1 \\ 1 & 0 \end{pmatrix})$ *for* $k \geqslant 0$. *Moreover, assume that* $B_k \in \mathscr{E}_{n+1}$ *for all integers* $k \geqslant 0$. *Then* $R(\xi) = [c'_0; c'_1, ..., c'_r, 1]$.

Proof. For any integer p', $p' > p$, the output of $\mathscr{T}_R$ at step p' is $\wedge$; consequently, $R(\xi) \in \mathbf{Q}$ (Theorem 7). Now, let $(M_k)_k$ be the sequence of matrices defined as in Theorem 6. Then, $\lim_k (a_{k,i}/a'_{k,i}) = R(\xi)$. On the other hand, applying Theorem 9 to matrix A and $\xi_p = [c_{p+1}; c_{p+2}, c_{p+3}, ...]$ (in place of ξ) we get (with the corresponding notations) $\Sigma(\xi_p) = 0$ and from Theorem 6, $\lim_k (b_{k,i}/b'_{k,i}) = 1$ $(0 \leqslant i \leqslant n)$. Therefore, $R(\xi) = [\Pi_{c'_0 c'_1 \cdots c'_r}](1)$ $= [c'_0; c'_1, ..., c'_r, 1]$. ∎

6. FURTHER CONSEQUENCES OF THE FACTORIZATION THEOREM

6.1. *Continued Fraction Expansion of Some Algebraic Numbers*

Let M be a square matrix with nonnegative entries. Assume that M is primitive, i.e., there exists an integer k_0 such that $M^{k_0} > 0$. By the classical Perron–Frobenius theorem, there exists a unique maximal eigenvalue λ of multiplicity 1. Moreover $\lambda > 0$ and the eigenvector V associated to λ can be chosen with positive entries. Let P be the matrix defined by $M = \lambda[V]\, P[V]^{-1}$, where $[V]$ denotes the diagonal matrix with $[V]_{ii} = V_i$; P is stochastic and primitive. Therefore, classically (see, for example, [Ga]), $\lim_{m\to\infty} P^m = P^\infty$, where P^∞ is a matrix with identical lines $[P_0, ..., P_n]$, each P_i belonging to $\mathbf{Q}[\lambda]$ if the entries of M are integers, and $\lim_{m\to\infty}(\lambda^{-m}M^m)_{ij} = V_i V_j^{-1} P_i$. This result will be used in the following weaker form.

LEMMA 11. *Let M be a square matrix with nonnegative integer entries and assume that M is primitive. Then for integers i, i' $(0 \leqslant i, i' \leqslant n)$, the $n+1$ sequences of ratios $k \mapsto (M^k)_{i,l}/(M^k)_{i',l}$ $(0 \leqslant l \leqslant n)$ converge to the same limit $\xi_{i,i'}$ $(= V_i P_i / V'_i P'_i)$ which is an algebraic number.*

Let U be the matrix with two rows where all entries $u_{m,p}$ are null except $u_{0,i}$ and $u_{1,i'}$ which are equal to 1. We are interested in the continued fraction expansion of $\xi_{i,i'}$. To this aim we introduce a new transducer.

DEFINITION 5. Let $\mathscr{T}_M = (\mathscr{C}, \mathscr{B}, \mathscr{A}, \Phi, \Psi)$ be the transducer defined as follows:

— the input alphabet $\mathscr{C}$ is reduced to $\{M\}$;
— the output alphabet is $\mathscr{A} = (\mathbf{N}\backslash\{0\}) \cup \{0a;\, a \in \mathbf{N}\backslash\{0\}\}$;
— the space of states is $\mathscr{B} = \mathscr{E}_{n+1}$ with initial state $b_0 = U$.

The instructions ϕ_M and ψ_M are defined by $\phi_M(B) = B'$ and $\psi_M(B) = \wedge$ if $BM = B' \in \mathscr{E}_{n+1}$. Otherwise, applying (F) and Theorem 1 we get $BM = \Pi_{c_1 c_2 \cdots c_r} B'$ and set $\phi_M(B) = B'$, $\psi_M(B) = c_1 c_2 \cdots c_r$.

THEOREM 11. *With the above notations, assume that $\xi_{i,i'}$ is irrational. Then its continued fraction expansion is obtained by the transducer $\mathscr{T}_M$ and the reducing map β: if $\beta \circ [\Psi, \Phi]_{M^n} = c'_0 c'_1 \cdots c'_{p_n}$, then $\xi_{i,i'} = [c'_0; c'_1, ..., c'_{p_n}, ...]$ and* $\lim_{n\to\infty} p_n = \infty$.

The proof is an easy consequence of Theorem 1.

Notice that Adam and Rhin [Ad-Rh] study analogous questions, but involving the Jacobi–Perron algorithm.

EXAMPLE 3. We go back to the $2^{1/3}$ by considering the matrix

$$M=\begin{pmatrix}1&1&2\\2&1&2\\1&1&1\end{pmatrix}$$

whose dominant eigenvalue is $1+2^{1/3}+4^{1/3}$. The powers of M have the form

$$M^k=\begin{pmatrix}c_k&b_k&2a_k\\2a_k&c_k&2b_k\\b_k&a_k&c_k\end{pmatrix}.$$

It is easy to see that $\lim_{k\to\infty}(c_k/b_k)=\lim_{k\to\infty}(b_k/a_k)=\lim_{k\to\infty}(2a_k/c_k)=2^{1/3}$. An easy calculation by hand with the transducer $\mathcal{T}_M$ gives the following first outputs: 11, 02, 13, 02, 1, 13, ... Hence we get $2^{1/3}=[1;3,1,5,1,1,...]$, the next partial quotient being $\geqslant 3$. Transducer $\mathcal{T}_M$ is very simple but it realizes an algorithm which seems to compute the continued fraction expansion of $2^{1/3}$ more slowly than that given in Section 5.3.

6.2. The analytic theory of continued fractions furnishes expansions of real irrational numbers of the form

$$\xi=a_0+\cfrac{b_0}{a_1+\cfrac{b_1}{a_2+\cfrac{b_2}{a_3+\cfrac{b_3}{\ddots}}}}$$

where a_k and b_k are integers. Independently of its convergence, this expansion can be at least related by two products of matrices, namely

$$\begin{pmatrix}a_0&b_0\\1&0\end{pmatrix}\cdots\begin{pmatrix}a_n&b_n\\1&0\end{pmatrix}\qquad\begin{pmatrix}1&a_0\\0&1\end{pmatrix}\begin{pmatrix}0&b_0\\1&a_1\end{pmatrix}\cdots\begin{pmatrix}0&b_n\\1&a_{n+1}\end{pmatrix}.$$

Eventually these products are reorganized as the product of a 2×2-matrix A followed by a sequence of 2×2-matrices M_k with integer entries. The product $AM_1M_2\cdots M_k=\left(\begin{smallmatrix}p_k&p'_k\\q_k&q'_k\end{smallmatrix}\right)$ is related to ξ by

$$\lim_{k\to\infty}\frac{p_k}{q_k}=\lim_{k\to\infty}\frac{p'_k}{q'_k}=\xi. \tag{8}$$

For example (see Wall [Wa]),

(i) $\xi=\pi$: $A=\begin{pmatrix}0&4\\1&1\end{pmatrix}$, $M_k=\begin{pmatrix}0&k^2\\1&2k+1\end{pmatrix}$,

(ii) $\xi=\log 2$: $A=\begin{pmatrix}0&1\\1&1\end{pmatrix}$, $M_k=\begin{pmatrix}k&k(2k+1)\\2&5k+2\end{pmatrix}=\begin{pmatrix}0&k\\1&2\end{pmatrix}\begin{pmatrix}0&k\\1&2k+1\end{pmatrix}$, or

(iii) $\xi=2^{1/3}$: $A=\begin{pmatrix}3&4\\2&4\end{pmatrix}$, $M_k=\begin{pmatrix}15k+5&6k+8\\6k+3&3k+4\end{pmatrix}=\begin{pmatrix}2&3k-1\\1&0\end{pmatrix}\begin{pmatrix}6k+3&3k+4\\1&0\end{pmatrix}$.

Hurwitz numbers (see [Pe, St1, St2]) are also related to this form, but the entries of M_k are polynomials in k with $\det(M_k)=\pm 1$.

For any irrational number ξ given by (8), we associate the transducer $\mathscr{T}_\xi=(\mathscr{C}, \mathscr{B}, \mathscr{A}, \Phi, \Psi)$, where the space of states is $\mathscr{B}=\mathscr{E}_2$, the initial state is $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, the input alphabet is $\mathscr{C}=A\cup\{M_k;\, k\geqslant 1\}$ and the output alphabet is $\mathscr{A}=(\mathbf{N}\backslash\{0\})\cup\{0a;\, a\in\mathbf{N}\backslash\{0\}\}$. The instructions associated to $M\in\mathscr{C}$ are given for $B\in\mathscr{B}$ by $\phi_M(B)=B'$ and $\psi_M(B)=\wedge$ if $BM=B'\in\mathscr{E}_2$ or $\phi_M(B)=B'$ and $\psi_M(B)=c_1c_2\cdots c_r$ if $BM=\Pi_{c_1c_2\cdots c_r}B'$ by Theorem 1.

THEOREM 12. *The continued fraction expansion of the irrational number* ξ *given by* (8) *can be computed by the transducer* $\mathscr{T}_\xi$ *with the reducing map* β. *If* $\beta\circ[\Psi, \Phi]_{AM_1M_2\cdots M_k}=c'_0c'_1\cdots c'_{p_k}$, *then* $\xi=[c'_0; c'_1, ..., c'_{p_k}, ...]$ *and* $\lim_{k\to\infty} p_k=\infty$.

The proof is a straightforward application of Theorem 1.

EXAMPLE 4. Consider the case $\xi=\pi$. The transducer $\mathscr{T}_\pi$ starts with $\phi_A(I)=A=b_1$ and $\psi_A(I)=\wedge$ and then gives the following first outputs: 31, 05, $\wedge$, 013, 0(11), $\wedge$, 01, $\wedge$, $\wedge$, 1(205), ... Using β we get $\pi=[3; 7, 15, 1, ...]$ with the next partial quotient $\geqslant$205 (in fact, it is equal to 292). Notice that this algorithm is not efficient to compute the continued fraction of π. This is due to the fact that (i) does not furnish good rational approximations.

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