MATHEMATICS

Some multiple integral transformations involving the *H*-function of several variables*

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The object of the present paper is to give two general multiple integral transformations of the H-function of several complex variables, which was defined and studied elsewhere by the authors (*cf.*, *e.g.*, [10], [11] and [12]). It is shown how the main formulas (1.8) and (1.14) below, involving Fox's H-function ([4], p, 408) and the Laguerre polynomials, respectively, are related to each other and, of course, to a number of results given recently in the literature (see [1] and [5] through [9]). Several possible applications of the operational techniques provided by these results (and their various special cases) when viewed as multidimensional integral transformations are also indicated briefly.

1. INTRODUCTION AND THE MAIN RESULTS

Following the notations explained fairly fully in the earlier papers [11] and [12], let

$$(1.1)\qquad H\,{}^{0,\,\lambda:\ (\mu',\,\nu');\,\ldots;\,(\mu^{(n)},\,\nu^{(n)})}_{A,\,C:\,[B',\,D'];\,\ldots;\,[B^{(n)},\,D^{(n)}]}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

denote the H-function of n complex variables $z_1, \ldots, z_n$ (see also [10],

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p. 271 *et seq.*). Also, let the associated positive numbers

$$(1.2)\qquad \begin{cases} \theta_j^{(i)}, j=1, \dots, A;\ \phi_j^{(i)}, j=1, \dots, B^{(i)}; \\ \psi_j^{(i)}, j=1, \dots, C;\ \delta_j^{(i)}, j=1, \dots, D^{(i)};\ i=1, \dots, n, \end{cases}$$

be constrained by the inequalities

$$(1.3)\qquad \begin{cases} \Delta_i \equiv \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} \\ \quad - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \end{cases}$$

$$(1.4)\qquad \begin{cases} \Omega_i \equiv \sum_{j=1}^{A} \theta_j^{(i)} + \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} < 0, \\ \qquad \forall i \in \{1, \dots, n\}. \end{cases}$$

Then it is known that the multiple Mellin-Barnes contour integral defining the H-function (1.1) would converge absolutely when

$$(1.5)\qquad |\arg (z_i)| < \tfrac{1}{2}\Delta_i\, \pi, \quad i=1, \dots, n,$$

it being understood that the points $z_i=0$, $i=1, \dots, n$, and various exceptional parameter values, are excluded, and that (*cf.* [11], p. 131):

$$(1.6)\qquad H\begin{matrix} 0, \lambda:\ (\mu', \nu'); \dots; (\mu^{(n)}, \nu^{(n)}) \\ A, C: [B', D']; \dots; [B^{(n)}, D^{(n)}] \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= \begin{cases} O\left(|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n}\right), \max\left\{ |z_1|, \dots, |z_n| \right\} \to 0, \\ O\left(|z_1|^{\beta_1} \dots |z_n|^{\beta_n}\right), \lambda \equiv 0, \min\left\{ |z_1|, \dots, |z_n| \right\} \to \infty, \end{cases}$$

where, with $i=1, \dots, n$,

$$(1.7)\qquad \begin{cases} \alpha_i = \min\left\{ \operatorname{Re} (d_j^{(i)})/\delta_j^{(i)} \right\}, j=1, \dots, \mu^{(i)}, \\ \beta_i = \max\left\{ \operatorname{Re} (b_j^{(i)}-1)/\phi_j^{(i)} \right\}, j=1, \dots, \nu^{(i)}. \end{cases}$$

We now state our main results given by the following multiple integral transformations:

$$(1.8)\qquad \int_0^\infty \dots \int_0^\infty x_1^{\varrho_1-1} \dots x_n^{\varrho_n-1} \left(k_1 x_1^{\sigma_1} + \dots + k_n x_n^{\sigma_n}\right)^{\sigma}$$

$$\cdot H_{p,q}^{m,0}\left[\zeta\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)\middle|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\ (g_1,\gamma_1),\ldots,(g_q,\gamma_q)\end{matrix}\right]$$

$$\cdot H\begin{matrix}0,\lambda:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})\\ A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]\end{matrix}\begin{pmatrix}z_1X_1\\ \vdots\\ z_nX_n\end{pmatrix}dx_1\ldots dx_n$$

$$=\zeta^{-S}\Phi(k_1,\ldots,k_n)\,H\begin{matrix}0,n+m+\lambda:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})\\ n+q+A,1+p+C:[B',D'];\ldots;[B^{(n)},D^{(n)}]\end{matrix}$$

$$\begin{pmatrix}[1-\varrho_j/\sigma_j:s_j'/\sigma_j,\ldots,s_j^{(n)}/\sigma_j]_{1,n}, & [1-g_j-S\gamma_j:R_1\gamma_j,\ldots,R_n\gamma_j]_{1,q},\\ [1-S+\sigma:R_1-r_1,\ldots,R_n-r_n], & [1-e_j-S\varepsilon_j:R_1\varepsilon_j,\ldots,R_n\varepsilon_j]_{1,p},\end{matrix}$$

$$\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\ [(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ Z_1,\ldots,Z_n\Bigg),$$

where, for convenience,

$$\Phi(k_1,\ldots,k_n)=(\sigma_1\ldots\sigma_n)^{-1}\,k_1^{-\varrho_1/\sigma_1}\ldots k_n^{-\varrho_n/\sigma_n}, \tag{1.9}$$

$$\begin{cases}X_i=x_1^{s_1^{(i)}}\ldots x_n^{s_n^{(i)}}\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)^{r_i},\\ R_i=r_i+\dfrac{s_1^{(i)}}{\sigma_1}+\ldots+\dfrac{s_n^{(i)}}{\sigma_n},\\ Z_i=z_i\zeta^{-R_i}k_1^{-s_1^{(i)}/\sigma_1}\ldots k_n^{-s_n^{(i)}/\sigma_n},\ i=1,\ldots,n,\end{cases} \tag{1.10}$$

and

$$S=\sigma+\frac{\varrho_1}{\sigma_1}+\ldots+\frac{\varrho_n}{\sigma_n}, \tag{1.11}$$

it being understood, for example, that

$$\left[h_j:\mu_j',\ldots,\mu_j^{(n)}\right]_{1,m}$$

abbreviates the array of m groups of parameters:

$$\left[h_1:\mu_1',\ldots,\mu_1^{(n)}\right],\ldots,\left[h_m:\mu_m',\ldots,\mu_m^{(n)}\right],\quad m\geqq 1.$$

The integral formula (1.8) is valid under the following (sufficient) conditions:

(i) $k_i>0,\ \sigma_i>0,\ r_i\geqq 0,\ s_j^{(i)}>0,\ \forall i,j\in\{1,\ldots,n\}$;

(ii) Conditions corresponding appropriately to (1.3), (1.4) and (1.5) are satisfied by each of the multivariable H-functions occurring in (1.8);

(iii) $\operatorname{Re}(\varrho_i)>0$, $i=1, \ldots, n$, and

$$(1.12) \quad \operatorname{Re}(S) > -\sum_{i=1}^{n} R_i\alpha_i - \min_{1\leq j\leq m}\left\{\operatorname{Re}\left(\frac{g_j}{\gamma_j}\right)\right\},$$

where $\alpha_1, \ldots, \alpha_n$ are given by (1.7), and

(iv) m, p, q are integers such that $1\leqq m\leqq q$ and $p\geqq 0$, $\varepsilon_j>0$, $j=1, \ldots, p$, $\gamma_j>0$, $j=1, \ldots, q$,

$$(1.13) \quad \varkappa\equiv\sum_{j=1}^{p}\varepsilon_j-\sum_{j=1}^{q}\gamma_j<0,\ \eta\equiv\sum_{j=1}^{m}\gamma_j-\sum_{j=m+1}^{q}\gamma_j-\sum_{j=1}^{p}\varepsilon_j>0,$$

and $|\arg(\zeta)|<\frac{1}{2}\eta\pi$. {Here $H_{p,q}^{m,n}[z]$ denotes the familiar H-function of C. Fox ([4], p. 408; see also [9], p. 310).}

$$(1.14) \quad \int_0^\infty\cdots\int_0^\infty x_1^{\varrho_1-1}\cdots x_n^{\varrho_n-1}\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)^{\sigma}$$

$$\cdot\exp\left(-\gamma\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)\right)L_m^{(\alpha)}\left(\gamma\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)\right)$$

$$\cdot H\begin{matrix}0,\lambda:\ (\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})\\ A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]\end{matrix}\begin{pmatrix}z_1x_1^{s_1}\\ \vdots\\ z_nx_n^{s_n}\end{pmatrix}dx_1\ldots dx_n$$

$$=\frac{(-1)^m\gamma^{-S}}{m!}\Phi(k_1,\ldots,k_n)\,H\begin{matrix}0,\lambda+2:\ (\mu',\nu'+1);\ldots;(\mu^{(n)},\nu^{(n)}+1)\\ A+2,C+1:[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]\end{matrix}$$

$$\begin{pmatrix}[1-S:s_1/\sigma_1,\ldots,s_n/\sigma_n],[1-S+\alpha:s_1/\sigma_1,\ldots,s_n/\sigma_n],[(a):\theta',\ldots,\theta^{(n)}]:\\ [1-S+\alpha+m:s_1/\sigma_1,\ldots,s_n/\sigma_n],[(c):\psi',\ldots,\psi^{(n)}]:\end{pmatrix.$$

$$\left.\begin{matrix}[1-\varrho_1/\sigma_1:s_1/\sigma_1],[(b'):\phi'];\ldots;[1-\varrho_n/\sigma_n:s_n/\sigma_n],[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;\qquad[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \zeta_1,\ldots,\zeta_n\right),$$

where $L_m^{(\alpha)}(z)$ denotes the Laguerre polynomial of order α and degree m in z, $m\geqq 0$, $k_i>0$, $\sigma_i>0$, $s_i>0$, $\operatorname{Re}(\varrho_i)>0$, $\forall i\in\{1,\ldots,n\}$,

$$(1.15) \quad \operatorname{Re}(S)>-\sum_{i=1}^{n}(s_i\alpha_i/\sigma_i), \quad \operatorname{Re}(\gamma)>0,$$

$\Phi(k_1, \ldots, k_n)$, S and $\alpha_1, \ldots, \alpha_n$ being given, as before, by (1.9), (1.11) and (1.7), respectively,

$$(1.16) \quad \zeta_i=z_i(\gamma k_i)^{-s_i/\sigma_i}, \quad i=1,\ldots,n,$$

and conditions corresponding appropriately to (1.3), (1.4) and (1.5) are assumed to hold for the multivariable H-functions involved.

REMARK. Since $H^{m,0}_{p,q}[\zeta]$ vanishes *exponentially* when $|\zeta| \to \infty$ and the relevant parts of Condition (iv) of (1.8) hold true, and since $\mathrm{Re}(\gamma) > 0$ in (1.14), the convergence of our integral formulas (1.8) and (1.14) at their upper limit of integration can be guaranteed under the conditions stated already if we assume that, for *some* $\varkappa_1, \ldots, \varkappa_n$,

$$(1.17) \quad H\begin{matrix} 0, \lambda: & (\mu', \nu'); \ldots; (\mu^{(n)}, \nu^{(n)}) \\ A, C: & [B', D']; \ldots; [B^{(n)}, D^{(n)}] \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= O\left(|z_1|^{\varkappa_1} \ldots |z_n|^{\varkappa_n}\right), \lambda \neq 0, \min\{|z_1|, \ldots, |z_n|\} \to \infty,$$

which would evidently complement the asymptotic expansions given by (1.7).

2. PROOFS OF THE INTEGRAL FORMULAS (1.8) AND (1.14)

For convenience, let $\sum r_i \xi_i$ and $\sum s_j^{(i)} \xi_i$ denote the n-term sums

$$(2.1) \quad \sum_{i=1}^{n} r_i \xi_i \text{ and } \sum_{i=1}^{n} s_j^{(i)} \xi_i, \quad \forall j \in \{1, \ldots, n\},$$

respectively. Also let

$$(2.2) \quad \begin{cases} \Delta = \int_0^\infty \ldots \int_0^\infty x_1^{\varrho_1 - 1} \ldots x_n^{\varrho_n - 1} f\left(k_1 x_1^{\sigma_1} + \ldots + k_n x_n^{\sigma_n}\right) \\ \cdot H\begin{matrix} 0, \lambda: & (\mu', \nu'); \ldots; (\mu^{(n)}, \nu^{(n)}) \\ A, C: & [B', D']; \ldots; [B^{(n)}, D^{(n)}] \end{matrix}\begin{pmatrix} z_1 X_1 \\ \vdots \\ z_n X_n \end{pmatrix} dx_1 \ldots dx_n, \end{cases}$$

where the X_i are defined by (1.10), and the function f is so prescribed that the multiple integral converges.

On replacing the multivariable H-function occurring in (2.2) by its Mellin-Barnes contour integral given by [10, p. 271, Eq. (4.1)], if we interchange the order of the resulting $(x_1, \ldots, x_n)$- and $(\xi_1, \ldots, \xi_n)$-integrals, which is evidently justifiable under the various (sufficient) conditions stated with (1.8) in case f is specialized by (2.7) below, we find that

$$(2.3) \quad \begin{cases} \Delta = \dfrac{1}{(2\pi\omega)^n} \int_{\mathscr{L}_1} \ldots \int_{\mathscr{L}_n} z_1^{\xi_1} \ldots z_n^{\xi_n} \Phi_1(\xi_1) \ldots \Phi_n(\xi_n) \Psi(\xi_1, \ldots, \xi_n) \\ \cdot \left\{ \int_0^\infty \ldots \int_0^\infty x_1^{\varrho_1 + \Sigma s_1^{(i)} \xi_i - 1} \ldots x_n^{\varrho_n + \Sigma s_n^{(i)} \xi_i - 1} \left(k_1 x_1^{\sigma_1} + \ldots + k_n x_n^{\sigma_n}\right)^{\Sigma r_i \xi_i} \right. \\ \left. \cdot f\left(k_1 x_1^{\sigma_1} + \ldots + k_n x_n^{\sigma_n}\right) dx_1 \ldots dx_n \right\} d\xi_1 \ldots d\xi_n, \end{cases}$$

where $\omega=\sqrt{-1}$, the $\Phi_i(\xi_i)$ and $\Psi(\xi_1, \ldots, \xi_n)$ are defined by Equations (4.2) and (4.3) in [10, p. 272].

Now we interpret the innermost $(x_1, \ldots, x_n)$-integral by appealing to the following form of a known relationship [1, p. 173]:

$$
(2.4)\quad \begin{cases} \int_0^\infty \cdots \int_0^\infty x_1^{\varrho_1-1} \cdots x_n^{\varrho_n-1} f\left(k_1 x_1^{\sigma_1} + \ldots + k_n x_n^{\sigma_n}\right) dx_1 \ldots dx_n \\ = \Phi(k_1, \ldots, k_n) \dfrac{\Gamma\left(\frac{\varrho_1}{\sigma_1}\right) \cdots \Gamma\left(\frac{\varrho_n}{\sigma_n}\right)}{\Gamma\left(\frac{\varrho_1}{\sigma_1} + \ldots + \frac{\varrho_n}{\sigma_n}\right)} \int_0^\infty z^{\frac{\varrho_1}{\sigma_1} + \ldots + \frac{\varrho_n}{\sigma_n} - 1} f(z) dz, \end{cases}
$$

where, as before, $\Phi(k_1, \ldots, k_n)$ is given by (1.9), and

$$
\min_{1 \leq i \leq n} \left\{ k_i, \sigma_i, \operatorname{Re}(\varrho_i) \right\} > 0.
$$

Thus (2.3) assumes the form

$$
(2.5)\quad \begin{cases} \Delta = \dfrac{\Phi(k_1, \ldots, k_n)}{(2\pi\omega)^n} \int_{\mathscr{L}_1} \cdots \int_{\mathscr{L}_n} Y_1^{\xi_1} \cdots Y_n^{\xi_n} \Phi_1(\xi_1) \ldots \Phi_n(\xi_n) \Psi(\xi_1, \ldots, \xi_n) \\ \cdot \dfrac{\Gamma\left(\frac{\varrho_1^*}{\sigma_1}\right) \cdots \Gamma\left(\frac{\varrho_n^*}{\sigma_n}\right)}{\Gamma\left(\frac{\varrho_1^*}{\sigma_1} + \ldots + \frac{\varrho_n^*}{\sigma_n}\right)} \left\{ \int_0^\infty z^{S-\sigma+\Sigma R_i \xi_i - 1} f(z) dz \right\} d\xi_1 \ldots d\xi_n, \end{cases}
$$

where $\Phi(k_1, \ldots, k_n)$, R_i and S are given by (1.9), (1.10) and (1.11), respectively, and

$$
(2.6)\quad \begin{cases} Y_j = z_j k_j^{-\Sigma s_j^{(i)}/\sigma_j}, \\ \varrho_j^* = \varrho_j + \sum_{i=1}^n s_j^{(i)} \xi_i, \quad \forall j \in \{1, \ldots, n\}. \end{cases}
$$

If, in the integral relationship (2.6), we set

$$
(2.7)\qquad f(z) = z^\sigma H_{p,q}^{m,0}\left[\zeta z \,\middle|\, \begin{matrix} (e_1, \varepsilon_1), \ldots, (e_p, \varepsilon_p) \\ (g_1, \gamma_1), \ldots, (g_q, \gamma_q) \end{matrix} \right],
$$

evaluate the z-integral by means of the special case (when $n=0$) of a familiar formula expressing the Mellin transform of Fox's H-function (see, for example, [9], p. 311, Eq. (3.3) with z on its left-hand side replaced by zx), and then interpret the resulting $(\xi_1, \ldots, \xi_n)$-integral as an H-function of n variables, we shall be led fairly easily to our integral formula (1.8) under the various (sufficient) conditions stated already.

In order to derive the integral formula (1.14), we first replace the multivariable H-function occurring on its left-hand side by the Mellin-Barnes contour integral [10, p. 271, Eq. (4.1)], invert the order of the resulting multiple integrals, and then apply the known relationship (2.4). On evaluating the innermost z-integral by appealing to a slightly modified version of the well-known integral [3, p. 292, Eq. (1)], involving Laguerre polynomials, if we interpret the resulting multiple (Mellin-Barnes) contour integral as an H-function of n variables, we shall arrive at the desired integral formula (1.14) under the conditions stated already. The various steps just indicated are essentially similar to those in our derivation of (1.8), and we omit the details.

3. APPLICATIONS

It is fairly easy to observe that, when $\varepsilon_j = 1$, $j = 1, \ldots, p$, and $\gamma_j = 1$, $j = 1, \ldots, q$, the (single) H-function occurring in our integral formula (1.8) would reduce at once to the relatively more familiar G-function of Meijer (see, for example, [3], p. 434). {As a matter of fact, the H-function is known to reduce to the G-function in the not-too-obvious cases when $\varepsilon_j = \gamma_k = \delta$, $\delta > 0$, $j = 1, \ldots, p$ and $k = 1, \ldots, q$, or when the ε's and γ's are (positive) *rational* numbers.} And for special choices of the various parameters and the variable involved, the function $G_{p,q}^{m,0}(\zeta)$ can be further reduced not only to the Bessel functions $J_\nu(z)$, $Y_\nu(z)$ and $K_\nu(z)$, the Whittaker functions $M_{\varkappa,\mu}(z)$ and $W_{\varkappa,\mu}(z)$, and so on, but also to such products as

$$I_\nu(z)J_\nu(z),\ J_\nu(z)Y_\nu(z),\ J_\nu(z)K_\nu(z),\ K_\mu(z)K_\nu(z),\ W_{\varkappa,\mu}(z)W_{-\varkappa,\mu}(z),$$

et cetera. Thus, by appealing to the known relationships [3, pp. 440 and 442]

$$(3.1)\qquad e^{-z}K_\nu(z) = \sqrt{\pi}\, G_{1,2}^{2,0}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu,\ -\nu \end{matrix}\right) = \sqrt{\pi}\, H_{1,2}^{2,0}\left[2z \,\middle|\, \begin{matrix} (\frac{1}{2}, 1) \\ (\nu, 1), (-\nu, 1) \end{matrix}\right],$$

$$(3.2)\qquad \begin{cases} z^\sigma K_\mu(z)K_\nu(z) = \dfrac{\sqrt{\pi}}{2}\, G_{2,4}^{4,0}\left(z^2 \,\middle|\, \begin{matrix} \frac{1}{2}\sigma,\ \frac{1}{2}(\sigma+1) \\ \frac{1}{2}(\sigma \pm \mu \pm \nu) \end{matrix}\right), \\[2ex] \qquad = \dfrac{\sqrt{\pi}}{4}\, H_{2,4}^{4,0}\left[z \,\middle|\, \begin{matrix} (\frac{1}{2}\sigma, \frac{1}{2}), (\frac{1}{2}\sigma + \frac{1}{2}, \frac{1}{2}) \\ (\frac{1}{2}\sigma \pm \frac{1}{2}\mu \pm \frac{1}{2}\nu, \frac{1}{2}) \end{matrix}\right], \end{cases}$$

and

$$(3.3)\qquad \begin{cases} z^\sigma e^{-\frac{1}{2}z}\, W_{\varkappa,\mu}(z) = G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \sigma - \varkappa + 1 \\ \sigma \pm \mu + \frac{1}{2} \end{matrix}\right) \\[2ex] \qquad = H_{1,2}^{2,0}\left[z \,\middle|\, \begin{matrix} (\sigma - \varkappa + 1, 1) \\ (\sigma \pm \mu + \frac{1}{2}, 1) \end{matrix}\right], \end{cases}$$

we can easily obtain, as special cases of our result (1.8), certain multidimensional extensions of the double and triple integral transformations discussed in our earlier paper [9], in the references cited there [*op. cit.*, pp. 309 and 313], and in the recent paper by C. M. Joshi and M. L. Prajapat [5, p. 163, Eq. (2.5)]. Indeed, Joshi and Prajapat [*op. cit.*, p. 166, Eq. (3.6)] have also given a three-dimensional extension of our double integral transformation [9, p. 312, Eq. (3.6)], involving Fox's H-functions, and their triple integral transformation just mentioned would evidently reduce, in certain very special cases, to the main results in a *subsequent* paper by L. K. Bhagchandani [1, p. 131, Eq. (2.3); p. 132, Eq. (2.5)]. {Incidentally, Bhagchandani's main results [*loc. cit.*] are rather straightforward extensions of those in a 1967 paper by Srivastava and Joshi [8, p. 18, Eq. (2.2); p. 19, Eq. (2.3)], a reference to which is conspicuously missing in Bhagchandani's 1977 paper [1] notwithstanding the fact that these two papers (published exactly a decade apart) follow each other noticeably closely.} We remark in passing that the special cases of our results (1.8) and (1.14) when $n=2$ correspond essentially to the main integral formulas derived in a recent paper by S. L. Kalla, K. C. Gupta and S. P. Goyal [6].

We now show how our integral formula (1.14) can alternatively be deduced as a confluent case of (1.8). Indeed, if in (1.8) we let $r_i \to 0$ and $s_j^{(i)} \to 0$, $j \neq i$, where $i, j \in \{1, \ldots, n\}$, set $\zeta = \gamma$, $p=1$, $q=m=2$, and $s_i^{(i)} = s_i$, $\forall i \in \{1, \ldots, n\}$, and further specialize $H_{1,2}^{2,0}$ by appealing to (3.3) and the known relationship [3, p. 432]

$$(3.4)\qquad \begin{cases} W_{\mu+m+\frac{1}{2}, \pm\mu}(z) = (-1)^m m!\, z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} L_m^{(2\mu)}(z), \\ \qquad\qquad m = 0, 1, 2, \ldots, \end{cases}$$

we shall arrive at our integral formula (1.14) expressing the multidimensional Laguerre transformation of the H-function of several complex variables. As a matter of fact, this method of derivation of (1.14) from (1.8) can be applied *mutatis mutandis* to obtain similar multidimensional integral transforms with a fairly wide variety of special kernels.

For $\lambda = A = C = 0$, the multivariable H-function occurring on the left-hand sides of (1.8) and (1.14) would obviously reduce to the product of n H-functions of Fox, and if in (1.8) we further set $r_i = 0$, $\mu^{(i)} = D^{(i)} = 1$ and $\nu^{(i)} = B^{(i)} = 0$, $\forall i \in \{2, \ldots, n\}$, and proceed *appropriately* to the limits when $z_i, s_1^{(i)}, \ldots, s_n^{(i)} \to 0$, $i = 2, \ldots, n$, using the well-known fact that

$$(3.5)\qquad H_{0,1}^{1,0}\left[z \,\middle|\, \begin{matrix} - \\ (0, 1) \end{matrix}\right] = G_{0,1}^{1,0}\left(z \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) = e^{-z},$$

we shall obtain a multidimensional integral transformation which is easily written in the following (slightly modified) form:

$$(3.6)\qquad \int_0^\infty \cdots \int_0^\infty x_1^{\varrho_1 - 1} \cdots x_n^{\varrho_n - 1} \left(k_1 x_1^{\sigma_1} + \ldots + k_n x_n^{\sigma_n}\right)^\sigma$$

$$\cdot H_{p,q}^{l,m}\left[u\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)\middle|\begin{matrix}(a_j,\theta_j)_{1,p}\\(b_j,\phi_j)_{1,q}\end{matrix}\right]$$

$$\cdot H_{P,Q}^{L,M}\left[vx_1^{s_1}\ldots x_n^{s_n}\left(k_1x_1^{\sigma_1}+\ldots+k_nx_n^{\sigma_n}\right)^r\middle|\begin{matrix}(c_j,\psi_j)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]dx_1\ldots dx_n$$

$$=u^{-S}\Phi(k_1,\ldots,k_n)\,H_{p+q+n,Q+p+1}^{L+m,M+l+n}\left[w\middle|\begin{matrix}(1-\varrho_j/\sigma_j,\ s_j/\sigma_j)_{1,n},\\ (d_j,\delta_j)_{1,L},\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j,\psi_j)_{1,M},\ (1-b_j-S\phi_j,\ R\phi_j)_{1,q},\ (c_j,\psi_j)_{M+1,P}\\ (1-a_j-S\theta_j,\ R\theta_j)_{1,p},\ (d_j,\delta_j)_{L+1,Q},\ (1-S+\sigma,\ R-r)\end{matrix}\right],$$

where l, m, p, q and L, M, P, Q are integers constrained by $0\leqq l\leqq q$, $0\leqq m\leqq p$, $0\leqq L\leqq Q$, and $0\leqq M\leqq P$, $\theta_j>0$, $j=1,\ldots,p$, $\phi_j>0$, $j=1,\ldots,q$, $\psi_j>0$, $j=1,\ldots,P$, $\delta_j>0$, $j=1,\ldots,Q$, $k_i>0$, $\sigma_i>0$, $s_i>0$, $\mathrm{Re}(\varrho_i)>0$, $\forall i\in\{1,\ldots,n\}$, $r\geqq 0$, $|\arg(u)|<\frac{1}{2}U\pi$, $|\arg(v)|<\frac{1}{2}V\pi$, $u,v\neq 0$,

$$(3.7)\qquad\begin{cases}U\equiv\sum_{j=1}^{m}\theta_j-\sum_{j=m+1}^{p}\theta_j+\sum_{j=1}^{l}\phi_j-\sum_{j=l+1}^{q}\phi_j>0,\\ V\equiv\sum_{j=1}^{M}\psi_j-\sum_{j=M+1}^{P}\psi_j+\sum_{j=1}^{L}\delta_j-\sum_{j=L+1}^{Q}\delta_j>0,\\ \sum_{j=1}^{p}\theta_j-\sum_{j=1}^{q}\phi_j<0,\ \sum_{j=1}^{P}\psi_j-\sum_{j=1}^{Q}\delta_j<0,\end{cases}$$

and

$$(3.8)\qquad -\min_{1\leqq j\leqq l}\left\{\mathrm{Re}\left(\frac{b_j}{\phi_j}\right)\right\}<\mathrm{Re}(S)<-\max_{1\leqq j\leqq m}\left\{\mathrm{Re}\left(\frac{a_j-1}{\theta_j}\right)\right\},$$

$\Phi(k_1,\ldots,k_n)$ and S being defined, as before, by (1.9) and (1.11), respectively,

$$(3.9)\qquad R=r+\frac{s_1}{\sigma_1}+\ldots+\frac{s_n}{\sigma_n},\quad w=k_1^{-s_1/\sigma_1}\ldots k_n^{-s_n/\sigma_n}u^{-R}v,$$

and, for convenience, $(a_j,\theta_j)_{m+1,p}$ abbreviates the array of $p-m$ parameter pairs

$$(a_{m+1},\theta_{m+1}),\ldots,(a_p,\theta_p),$$

for integers m and p such that $0\leqq m\leqq p$, so that the abbreviation $(b_j,\phi_j)_{1,q}$ would represent the array of q parameter pairs

$$(b_1,\phi_1),\ldots,(b_q,\phi_q),\quad q\geqq 1,$$

and so on.

Obviously, this last result (3.6) can also be derived *directly* by applying the relationship (2.4) in conjunction with the Mellin transform of Fox's H-function.

Two further special cases of (3.6) are worthy of mention here. For $n=2$, it would evidently correspond to the main double integral transformation in an earlier paper by Srivastava, Gupta and Handa [7, p. 403, Eq. (6)]. On the other hand, if in (3.6) we set $k_i = \sigma_i = 1$, $\forall i \in \{1, \ldots, n\}$, it will yield yet another result due to Joshi and Prajapat [5, p. 167, Eq. (3.8)].

Finally, we should remark that, by appealing to the known relationship [10, p. 272, Eq. (4.7)], the multivariable H-functions occurring in our main results (1.8) and (1.14) can be reduced, under various special cases, to the generalized Lauricella functions of several complex variables, which include a great many of the useful functions of hypergeometric type (in one and more variables) as their particular cases. These and the other possible specializations indicated in this section would easily lead to several interesting applications (of the types discussed, for example, in [1], [5], [8] and [9]) of the operational techniques provided by (1.8) and (1.14) when viewed as multidimensional integral transformations, and we leave the details involved as conceivably fruitful exercises for the interested reader.

REFERENCES

1. Bhagchandani, L. K. – Some triple Whittaker transformations of certain hypergeometric functions, Rev. Mat. Hisp.-Amer. (4) 37, 129–146 (1977).
2. Edwards, J. – A Treatise on the Integral Calculus, Vol. II, Chelsea Publishing Co., New York, 1954.
3. Erdélyi, A., W. Magnus, F. Oberhettinger and F. G. Tricomi – Tables of Integral Transforms, Vol. II, McGraw-Hill Book Co., New York, Toronto and London, 1954.
4. Fox, C. – The G and H functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc. 98, 395–429 (1961).
5. Joshi, C. M. and M. L. Prajapat – Triple integral transformations of certain generalized hypergeometric functions, Jñānabha Sect. A 5, 161–176 (1975).
6. Kalla, S. L., K. C. Gupta and S. P. Goyal – On double integral transformations involving the H-function of two variables, Univ. Nac. Tucumán Rev. Ser. A 27 (1977), to appear.
7. Srivastava, H. M., K. C. Gupta and S. Handa – A certain double integral transformation, Nederl. Akad. Wetensch. Proc. Ser. A 78 = Indag. Math. 37, 402–406 (1975).
8. Srivastava, H. M. and C. M. Joshi – Certain double Whittaker transforms of generalized hypergeometric functions, Yokohama Math. J. 15, 17–32 (1967).
9. Srivastava, H. M. and R. Panda – Some operational techniques in the theory of special functions, Nederl. Akad. Wetensch. Proc. Ser. A 76 = Indag. Math. 35, 308–319 (1973).
10. Srivastava, H. M. and R. Panda – Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. 283/284, 265–274 (1976).
11. Srivastava, H. M. and R. Panda – Expansion theorems for the H function of several complex variables, J. Reine Angew. Math. 288, 129–145 (1976).
12. Srivastava, H. M. and R. Panda – Some expansion theorems and generating relations for the H function of several complex variables. I and II, Comment. Math. Univ. St. Paul. 24, fasc. 2, 119–137 (1975); *ibid.* 25, fasc. 2, 167–197 (1976).