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The extreme points of the set of belief measures

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Abstract

It is known that the σ-additive Möbius transform of a belief function (we prefer to call it belief measure) can be derived from Choquet's Theorem. One has to show that the extreme points of the compact convex set of belief measures are the $\{0,1\}$-valued belief measures, which are called filter games as well. A proof is implicit in the famous 1953/54 paper of Choquet but it is hard to read it. We present a direct proof and – for the sake of completeness – derive the Möbius transform.

1. Introduction

Within the theory of non-additive measures μ and their integrals there are two methods to reduce the problems to additive ones, which had both been introduced in cooperative game theory by Shapley. The first is to represent μ as inf or sup of additive measures in the core of μ. Here we are concerned with the other method to blow up the measurable space $(\Omega, \mathscr{A})$ to $(\mathrm{H}, 2^{\mathrm{H}})$ having a canonical injection $\mathscr{A} \to 2^{\mathrm{H}}, A \mapsto \widetilde{A}$, and a ($\sigma$-)additive measure μ^β on H with $\beta(A) = \mu^\beta(\widetilde{A})$ if β is a belief measure. A belief measure on $\mathscr{A}$ is a totally monotone set function with $\beta(\Omega) = 1$. For finite Ω, one can identify H with $2^\Omega \backslash \{\emptyset\}$ and the application $\beta \mapsto \mu^\beta$ is described through the combinatorial Möbius function of the poset 2^Ω, partially ordered by set inclusion.

The literature on the finitely additive and σ-additive Möbius transform μ^β is manyfold, see the introduction of [5] and the literature cited there. In [8] the σ-additive Möbius transform is derived from Choquet's Integral Representation Theorem. For this purpose it is necessary to determine the extreme points of the set TM_1 of belief measures. It has often been used that these are the $\{0,1\}$-valued belief measures (the elements of H), but all authors refer to [3,pp. 260–261]. But this proof is formulated in a very general setting, dual to our situation. Brüning [1] adopted Choquet's proof to the present problem and simplified it considerably,[1] the crucial step being Lemma 2.1.

The paper is organized as follows. The next section contains the notations and preliminary results. Section 3 provides the known [7] topological properties of the set of belief measures. The final section contains the main

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result (Proposition 4.1) that H is the set of extreme points of TM_1. Then existence and uniqueness of the σ-additive Möbius transform μ^β is derived from Choquet's Theorem along the lines in the literature [8,7].

2. Totally monotone measures

Here we give the basic definitions and notations and prove a lemma on decompositions of totally monotone measures.

Throughout the paper, Ω denotes a non-empty set and $\mathscr{S} \subset 2^\Omega$ a family of subsets containing Ω and the empty set, $\emptyset, \Omega \in \mathscr{S}$. An application $\mu : \mathscr{S} \to [0,\infty)$ is called a set function if $\mu(\emptyset) = 0$. μ is called normalized if $\mu(\Omega) = 1$. μ is called monotone, if $A, B \in \mathscr{S}$, $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.

A set function $\beta : \mathscr{A} \to [0,\infty)$ on an algebra $\mathscr{A}$ is called k-**monotone**, $k \in \mathbb{N}$, $k \geqslant 2$, if it is monotone and for any system $A_1, A_2, \ldots, A_k \in \mathscr{A}$ of k sets

$$\beta\left(\bigcup_{i=1}^k A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \beta\left(\bigcap_{i \in I} A_i\right) \geqslant 0.$$

2-monotonicity is also called **supermodularity**. **Submodularity** is the same condition with the inequality reversed. β is called **totally monotone**, if β is monotone and k-monotone for all $k \geqslant 2$. A **belief measure** is a normalised totally monotone set function.

For a set function $\beta : \mathscr{A} \to [0,\infty)$ the new set function β_A with $A \in \mathscr{A}$, $A \neq \emptyset$ is defined as $\beta_A(B) := \beta(A \cap B)$, $B \in \mathscr{A}$. The following lemma will play a key role in the proof of Proposition 4.1.

Lemma 2.1 [1, 4.4.2]. *Let $\beta : \mathscr{A} \to [0,1]$ be totally monotone. Then for $A \in \mathscr{A}$, $A \neq \emptyset$, the set functions β_A and $\beta' := \beta - \beta_A$ are totally monotone, too.*

Proof. Fix sets $A_1, A_2, \ldots A_k \in \mathscr{A}$. We will denote the algebra generated by these sets and the set A as $\mathscr{F}$. Now it will be shown that the discrete Möbius transforms of $\beta_A|\mathscr{F}$ and $\beta'|\mathscr{F}$ are non-negative which will be sufficient for the statement of the lemma.

For $B \in \mathscr{F}$ we define

$$\mu(B) := \begin{cases} \mu^{\beta|\mathscr{F}}(B) & \text{for } B \subset A, \\ 0 & \text{for } B \not\subset A, \end{cases}$$

where $\mu^{\beta|\mathscr{F}}$ denotes the Möbius transform of $\beta|\mathscr{F}$. Then μ coincides with the Möbius transform $\mu^{\beta_A|\mathscr{F}}$ of $\beta_A|\mathscr{F}$ because

$$\beta_A(C) = \beta(A \cap C) = \sum_{\substack{B \subset A \cap C \\ B \in \mathscr{F}}} \mu^{\beta|\mathscr{F}}(B) = \sum_{\substack{B \subset A \cap C \\ B \in \mathscr{F}}} \mu(B) = \sum_{\substack{B \subset C \\ B \in \mathscr{F}}} \mu(B).$$

So β_A is totally monotone.

Since the Möbius transform is linear in the set functions we get

$$\mu^{\beta'|\mathscr{F}}(B) = \begin{cases} 0 & \text{for } B \subset A, \\ \mu^{\beta|\mathscr{F}}(B) & \text{for } B \not\subset A. \end{cases}$$

Then $\mu^{\beta'|\mathscr{F}} \geqslant 0$ because β was assumed to be totally monotone. $\square$

Finally in this section we recall the definition of the Choquet integral w.r.t. a set function $\mu : \mathscr{A} \to [0,\infty)$ on an algebra $\mathscr{A} \subset \Omega$, needed only in Proposition 3.1 and Theorem 4.3. A function $h : \Omega \to \mathbb{R}$ is $\mathscr{A}$-measurable if $h^{-1}(B) \in \mathscr{A}$ for any Borel subset B of $\mathbb{R}$. In this case the **Choquet integral** of h w.r.t. μ is defined as

$$\int h \, d\mu := \int_{-\infty}^0 \mu(\Omega) - \mu(\{h \geqslant x\}) \, dx + \int_0^\infty \mu(\{h \geqslant x\}) \, dx$$

the integrals on the right-hand side being Riemann integrals. Of course, if μ is σ-additive, the Choquet integral is the usual integral w.r.t. the finite measure μ. For further details refer to [4].

3. The set of belief measures is compact

The σ-additive Möbius transform of a totally monotone measure will be derived from Choquet's Integral Representation Theorem in the next section. The topological requirements, mainly compactness for the set of belief measures are provided here.

Let $\mathscr{A} \subset 2^{\Omega}$ be an algebra, which we leave fixed, and

$$X := \{\mu : \mathscr{A} \to \mathbb{R} \mid \mu(\emptyset) = 0\}.$$

We do not require that the set functions in X are normalized and monotone, so X is a vector space. We endow X with the topology $\mathscr{V}$ of pointwise convergence, i.e. the relative topology of the product topology of $\mathbb{R}^{\mathscr{A}}$. It is a locally convex Hausdorff topology. On the convex cone

$$M := \{\mu \in X \mid \mu \text{ monotone}\}$$

of monotone set functions the Choquet integral with fixed integrand is continuous (see [7, Theorem 3] for domain $TM_1 \subset M$ of f_h as defined below).

Proposition 3.1. *For any bounded $\mathscr{A}$-measurable function $h : \Omega \to \mathbb{R}$ the functional*

$$f_h : M \to \mathbb{R}, \quad \mu \mapsto \int h \, d\mu$$

is continuous in the topology $\mathscr{V}$.

Proof. Since h is bounded we suppose $0 \leqslant h \leqslant 1$. h can be approximated uniformly by a sequence h_n of bounded $\mathscr{A}$-measurable functions, where h_n assumes only values $\frac{k}{n}$, $k \in \{0, 1, \ldots, n\}$, and $A_{n,k} := \{\omega \mid h_n(\omega) \geqslant \frac{k}{n}\} = \{\omega \mid h(\omega) \geqslant \frac{k}{n}\}$ [4, Lemma 6.2]. Then

$$\int h_n \, d\mu = \frac{1}{n} \sum_{k=1}^{n} \mu(A_{n,k}).$$

Given now $\mu_0 \in M$ and $\epsilon > 0$ we have to find a neighborhood U of μ_0 such that

$$|f_h(\mu) - f_h(\mu_0)| < \epsilon \quad \text{for all } \mu \in U.$$

Selecting n so that $\frac{1}{n}\mu_0(\Omega) < \frac{\epsilon}{4}$ and setting $U := \{\mu \in M \mid |\mu(A_{n,k}) - \mu_0(A_{n,k})| < \frac{\epsilon}{4} \text{ for } k \in \{0, 1, \ldots, n\}\}$ we get for all $\mu \in U$ first $\frac{1}{n}\mu(\Omega) = \frac{1}{n}(\mu(\Omega) - \mu_0(\Omega)) + \frac{1}{n}\mu_0(\Omega) < \frac{\epsilon}{4n} + \frac{\epsilon}{4}$ (recall $A_{n,0} = \Omega$) and then

$$\begin{aligned} |f_h(\mu) - f_h(\mu_0)| &\leqslant |f_h(\mu) - f_{h_n}(\mu)| + |f_{h_n}(\mu) - f_{h_n}(\mu_0)| + |f_{h_n}(\mu_0) - f_h(\mu_0)| \\ &\leqslant \frac{1}{n}\mu(\Omega) + \left| \frac{1}{n} \sum_{k=0}^{n-1} (\mu(A_{n,k}) - \mu_0(A_{n,k})) \right| + \frac{1}{n}\mu_0(\Omega) < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon \end{aligned}$$

what had to be shown. □

Next we set

$$\begin{aligned} M_1 &:= \{\mu \in M \mid \mu(\Omega) = 1\}, \\ TM_1 &:= \{\beta \in M_1 \mid \beta \text{ totally monotone}\}. \end{aligned}$$

Obviously, M_1 and TM_1 are convex.

Proposition 3.2. *The sets M_1 and TM_1 are compact in the topology $\mathscr{V}$.*

This result is implicitly contained in the proofs of [7]. To be complete we present the proof.

Proof. First it is shown that M_1 is compact with a similar idea used in the proof of the Banach–Alaoglu Theorem. This part of the proof is due to [7, Proposition 1]. Then it is sufficient to prove that TM_1 is a closed subset of M_1.

We define $P := \prod_{A \in \mathscr{A}} [0, 1]_A$ and the mapping

$$\tau := M_1 \to P, \quad \tau(\mu) = \prod_{A \in \mathscr{A}} \mu(A).$$

Now it is plain that τ is a homeomorphism onto $\tau(M_1)$ if P is endowed with the product topology. The set P is compact following Tychonoff's Theorem. We get that $\tau(M_1)$ is compact because it is a closed subset of P which is easily verified in constructing a neighborhood of any non-monotonic $\mu \in P$. Then M_1 is compact.

To see that TM_1 is closed in M_1 choose a net $(\beta_i)_{i \in I}$, $\beta_i \in TM_1$, which converges to an element $\beta' \in M_1$. We have to show

$$\beta'\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \beta'\left(\bigcap_{i \in I} A_i\right) \geqslant 0$$

for any system of sets $A_1, A_2, \ldots, A_k \in \mathscr{A}$ with $k \in \mathbb{N}$. Suppose the contrary, i.e. there exists a $k \in \mathbb{N},\ k \geqslant 1$, and sets $A_1, A_2, \ldots, A_k \in \mathscr{A}$ with

$$\beta'\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \beta'\left(\bigcap_{i \in I} A_i\right) =: c < 0.$$

Then the set

$$U := \left\{ \beta \in M_1 \,\middle|\, \left| \beta'\left(\bigcap_{i \in J} A_i\right) - \beta\left(\bigcap_{i \in J} A_i\right) \right| < \frac{-c}{2^k},\ J \subset \{1, \ldots, k\} \right\}$$

using the convention $\cap_{i \in \emptyset} A_i := \cup_{i \in \{1,\ldots,k\}} A_i$, is a neighborhood of β'. Since $\beta_i \to \beta'$ holds there exists an $i_0 \in I$ with $\beta_{i_0} \in U$. Then

$$\begin{aligned} \beta_{i_0}\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \beta_{i_0}\left(\bigcap_{i \in I} A_i\right) &< \beta'\left(\bigcup_{i=1}^{k} A_i\right) - \frac{c}{2^k} + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} \left[(-1)^{|I|} \beta'\left(\bigcap_{i \in I} A_i\right) - \frac{c}{2^k} \right] \\ &= \beta'\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \beta'\left(\bigcap_{i \in I} A_i\right) - c = 0. \end{aligned}$$

This is a contradiction to the assumption $\beta_{i_0} \in TM_1$. Therefore $\beta' \in TM_1$ and TM_1 is a closed subset of M_1. $\square$

4. The σ-additive Möbius transform of a totally monotone measure

The σ-additive Möbius transform of a totally monotone measure will now be derived from Choquet's Integral Representation Theorem. With the topological results from the last section the main task is to characterize the extremal points of the compact and convex set of belief functions. The concise proof of the latter result is the main achievement of the present paper.

Again let $\mathscr{A} \subset 2^{\Omega}$ be an algebra. We set

$$\mathrm{H} := \{\eta : \mathscr{A} \to \{0,1\} \mid \eta \text{ totally monotone}, \eta(\Omega) = 1\}.$$

An element η of H is called a **filter game** since the set $\{A \in \mathscr{A} \mid \eta(A) = 1\}$ is a filter [7,5]. The application

$$\mathscr{A} \to \widetilde{\mathscr{A}} := \{\widetilde{A} \mid A \in \mathscr{A}\} \subset 2^{\mathrm{H}}, \quad A \mapsto \widetilde{A} := \{\eta \in \mathrm{H} \mid \eta(A) = 1\}$$

is compatible with set inclusion and has the properties [5, Proposition 3.3]

$$\widetilde{A \cap B} = \widetilde{A} \cap \widetilde{B}, \quad \widetilde{A \cup B} \supset \widetilde{A} \cup \widetilde{B}, \quad A, B \in \mathscr{A}.$$

Proposition 4.1. H *is the set of extreme points of* TM_1.

We recall the definition of an extreme point. Let K be a convex subset of a linear space. Then a point $x \in K$ is called an **extreme point** of K if for $y, z \in K$ and $\alpha y + (1-\alpha)z = x$ with $0 \leqslant \alpha \leqslant 1$ we get $x = y = z$. The set of all extreme points of K is denoted by $\mathrm{ex}(K)$.

Proof. The filter games are extreme points of TM_1 because the values 0 and 1 are extreme points in $[0,1]$. Therefore it has to be shown that any extreme point $\gamma \in TM_1$ is a filter game, i.e. attains only the values 0 and 1.

Assume the contrary, i.e. there exists a set $A \in \mathscr{A}$ such that $0 < \gamma(A) < 1$. By Lemma 2.1 the set functions γ_A and $\gamma - \gamma_A$ are totally monotone. Normalizing them

$$\gamma_1 := \frac{\gamma_A}{\gamma(A)}, \quad \gamma_2 := \frac{\gamma - \gamma_A}{\gamma(\Omega) - \gamma(A)},$$

we get $\gamma_1, \gamma_2 \in TM_1$ and γ is a convex combination of γ_1, γ_2

$$\gamma = \alpha\gamma_1 + (1-\alpha)\gamma_2 \quad \text{with } 0 < \alpha := \gamma(A) < 1$$

a contradiction to γ being an extreme point. □

Now we are able to derive the existence of the Möbius transform from Choquet's Theorem which we cite in the version of [6].

Theorem 4.2. *Let X be a locally convex topological Hausdorff vector space and* $K \subset X$ *a compact and convex subset. Regard the set of functions*

$$A(K) := \{f : K \to \mathbb{R} \mid f \text{ affine and continuous}\}.$$

Let $\mathscr{B} \subset 2^{\mathrm{ex}(K)}$ *be the coarsest* σ*-algebra which makes all* $f|\mathrm{ex}(K)$ *with* $f \in A(K)$ *measurable. Then for each* $x \in K$ *there exists a probability measure* $\mu : \mathscr{B} \to [0,1]$ *such that*

$$f(x) = \int f|\mathrm{ex}(K)\,\mathrm{d}\mu \quad \text{for all } f \in A(K).$$

As usual we denote with $\sigma(\widetilde{\mathscr{A}})$ the σ-algebra generated by $\widetilde{\mathscr{A}}$.

Theorem 4.3. *Let* $\beta : \mathscr{A} \to [0,1]$ *be totally monotone on an algebra* $\mathscr{A} \subset 2^{\Omega}$ *and* $\beta(\Omega) = 1$. *Then there exists a probability measure* $\mu^\beta : \sigma(\widetilde{\mathscr{A}}) \to [0,1]$ *such that*

$$\int h\,\mathrm{d}\beta = \int \int h\,\mathrm{d}\eta\,\mathrm{d}\mu^\beta(\eta) \tag{1}$$

for all bounded and $\mathscr{A}$*-measurable functions* $h : \Omega \to \mathbb{R}$. *Furthermore*

$$\beta(A) = \mu^\beta(\widetilde{A}), \quad A \in \mathscr{A}$$

and μ^β *is the unique measure on* $\sigma(\widetilde{\mathscr{A}})$ *with this property.*

μ^β is called the **Möbius transform** of the belief measure β or, more precisely, the probability space $(\mathrm{H}, \sigma(\widetilde{\mathscr{A}}), \mu^\beta)$ is the Möbius transform of the 'belief space' $(\Omega, \mathscr{A}, \beta)$.

Proof. For a bounded and $\mathscr{A}$-measurable function $h : \Omega \to \mathbb{R}$ we define

$$f_h : TM_1 \to \mathbb{R}, \quad f_h(\beta) := \int h\,\mathrm{d}\beta.$$

This function is clearly affine and also continuous according to Proposition 3.1, hence $f_h \in A(TM_1)$. Let $\mathscr{B} \subset 2^{\mathrm{H}}$ be the coarsest σ-algebra which makes all $f|\mathrm{H}$ with $f \in A(TM_1)$ measurable. Since TM_1 is compact by Proposition 3.2 and $H = \mathrm{ex}(TM_1)$ by Proposition 4.1, we can apply Choquet's theorem for $\beta \in TM_1$. Hence there exists a measure $\mu : \mathscr{B} \to [0,1]$ such that

$$\int h\,\mathrm{d}\beta = \int \int h\,\mathrm{d}\eta\,\mathrm{d}\mu(\eta)$$

for all bounded and $\mathscr{A}$-measurable functions $h : \Omega \to \mathbb{R}$.

By choosing $h = 1_A$ with $A \in \mathscr{A}$ we get

$$f_{1_A}(\beta) = \int 1_A \, d\beta = \beta(A).$$

Since, for any $A \in \mathscr{A}$ the function $g := f_{1_A}|\mathrm{H}$ is $\mathscr{B}$-measurable it follows $\widetilde{A} = g^{-1}(\{1\}) \in \mathscr{B}$ and $\sigma(\widetilde{\mathscr{A}}) \subset \mathscr{B}$. In the next paragraph it will be shown that the functions $f_h|\mathrm{H}$ are $\sigma(\widetilde{\mathscr{A}})$-measurable. So, already the restricted set function $\mu^\beta := \mu|\sigma(\widetilde{\mathscr{A}})$ will do the job.

To prove $\sigma(\widetilde{\mathscr{A}})$-measurability of $g := f_h|\mathrm{H}$ it is sufficient to show that $g^{-1}((x,\infty)) \in \sigma(\widetilde{\mathscr{A}})$ for all $x \in \mathbb{R}$. But $g^{-1}((x,\infty)) = \{\eta \in \mathrm{H} \mid \int h \, d\eta > x\} = \{\omega \in \Omega \mid h(\omega) > x\} \in \widetilde{\mathscr{A}}$. For the last equation we used that η is $\{0,1\}$ valued, whence $\int h \, d\eta > x$ iff $\eta(h > x) = 1$.

For uniqueness of μ^β we apply (1) to indicator functions

$$\beta(A) = \int \int 1_A \, d\eta \, d\mu^\beta(\eta) = \int 1_{\widetilde{A}} \, d\mu^\beta = \mu^\beta(\widetilde{A}), \quad A \in \mathscr{A}.$$

By this equality μ^β is uniquely determined on $\widetilde{\mathscr{A}}$, hence on $\sigma(\widetilde{\mathscr{A}})$. □

Other proofs of the last theorem can be found in [7,5]. Finally some remarks on the measurability conditions. Proposition 3.1 and Theorem 4.3 remain valid if the measurability condition is weakened to Greco measurability (see [4,5]). Measurability conditions can be avoided at all in setting $\mathscr{A} = 2^\Omega$. If a totally monotone β is given only on a smaller algebra, then one may switch to the inner extension β_* on 2^Ω, which inherits total monotonicity [2, Proposition 1]. But one has to pay for this generalization, the space H on which the Möbius transform lives, becomes larger.

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