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# Tighter bounds of the First Fit algorithm for the bin-packing problem

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ABSTRACT

In this paper, we present improved bounds for the First Fit algorithm for the bin-packing problem. We prove $C^{FF}(L) \leq \frac{17}{10}C^*(L) + \frac{7}{10}$ for all lists $L$, and the absolute performance ratio of *FF* is at most $\frac{12}{7}$.



## 1. Introduction

In the classical one-dimensional bin-packing problem, we are given a sequence $L = (a_1, a_2, \ldots, a_n)$ of items, each with a size in $(0, 1]$. We are required to pack them into a minimum number of unit-capacity bins. An excellent survey of the research on this problem is available in [2].

The bin-packing problem was one of the earliest to use an approximation algorithm and worst case analysis. For a given list $L$ and algorithm $A$, let $C^A(L)$ denote the number of bins used when $A$ is applied to list $L$, and $C^*(L)$ denote the optimum number of bins for a packing of $L$. We will omit the mention of $L$ if there is no ambiguity. The *asymptotic performance ratio* for $A$ is defined as

$$\inf\left\{r \geq 1 \middle| \text{ for some } N > 0,\ \frac{C^A(L)}{C^*(L)} \leq r \text{ for all } L \text{ with } C^*(L) \geq N\right\}.$$

The *absolute performance ratio* for $A$ is defined as

$$\inf\left\{r \geq 1 \,\middle|\, \frac{C^A(L)}{C^*(L)} \leq r \text{ for all list } L\right\}.$$

The bin-packing problem is also one of the few combinational optimization problems for which the asymptotic performance ratio and the absolute performance ratio of a given algorithm may not be the same.

For simplicity, we use $a_i$ to denote the size of item $a_i$. The *content* of a bin $B$, which is the total size of items packed in it, is also denoted as $B$, when this causes no confusion. *First Fit* (*FF* for short) and *First Fit Decreasing* (*FFD* for short) are two fundamental algorithms for addressing bin-packing problems [6]. The *FF* algorithm can be described as follows: When we are packing $a_i$, we place it in the lowest indexed bin whose current content does not exceed $1 - a_i$. Otherwise, we start a

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new bin with $a_i$ as its first item. Algorithm *FFD* first sorts the items in non-increasing order of their sizes and then performs *FF*.

In Johnson's pioneer work, he proved that $C^{FFD}(L) \le \frac{11}{9}C^*(L) + 4$ for all lists $L$ [6]. Note that the asymptotic performance ratio cannot be smaller than $\frac{11}{9}$ [7]. Later, the additive term was reduced to 3 by Baker [1], and 1 by Yue [11]. Recently, Dósa further reduced it to a tight value $\frac{6}{9}$ [3]. The absolute performance ratio $\frac{3}{2}$ of *FFD* was obtained by Simchi-Levi [9], and it is also tight since no polynomial time algorithm with absolute performance ratio less than $\frac{3}{2}$ exists unless $P = NP$ [5].

For the *FF* algorithm, Ullman proved $C^{FF}(L) \le \frac{17}{10}C^*(L) + 3$ for all lists $L$ [10]. Here the asymptotic performance ratio is asymptotically tight, and there also exists such a list $L$ that $C^*(L) = 10$ and $C^{FF}(L) = 17$ [7]. The additive term was reduced to 2 in [7], and $\frac{9}{10}$ in [4]. To the author's knowledge, no further improvement has been made since then. Simchi-Levi proved that the absolute ratio of *FF* is at most $\frac{7}{4}$ [9], but the bound is not tight. Though *FF* has a larger worst case ratio than *FFD*, it can be used for the online version, where the items arrive in some order and must be packed into a bin as soon as they arrive, without knowledge of the remaining items.

In this paper, we will give both a smaller additive term in the asymptotic performance ratio and a tighter absolute performance ratio of *FF*. Section 2 gives some definitions and useful lemmas. In Section 3, we prove $C^{FF}(L) \le \frac{17}{10}C^*(L) + \frac{7}{10}$ for all lists $L$. In Section 4, we prove that the absolute performance ratio of *FF* is at most $\frac{12}{7}$. Thus the gap between upper and lower bounds of the absolute performance ratio decreases by more than 70%.

## 2. Preliminaries

We define some terminology for convenience. An item greater than $\frac{1}{2}$ is called *large* while an item greater than $\frac{1}{4}$ is called *semilarge*. The number of large items is denoted as $l$. Note that a semilarge item also can be bigger than $\frac{1}{2}$.

Let $\mathcal{B}^*$ be the set of bins used in the optimal packing, and $\mathcal{B}^{FF}$ be the set of bins used by *FF*. If a bin $B_1$ is opened before another bin $B_2$ in the procedure of *FF*, then we say that $B_1$ is *before* $B_2$ and $B_2$ is *after* $B_1$. When algorithm *FF* terminates, a bin containing exactly one item (two items) is called an i-*bin* (ii-*bin*). A bin containing no less than two (three, four) items is called a II-*bin* (III-*bin*, IV-*bin*). An item in an i-bin (ii-bin, II-bin, III-bin, IV-bin) is called an i-*item* (ii-*item*, II-*item*, III-*item*, IV-*item*). Let $\mathcal{B}_{\rm i}$ ($\mathcal{B}_{\rm ii}$, $\mathcal{B}_{\rm II}$, $\mathcal{B}_{\rm III}$, $\mathcal{B}_{\rm IV}$) be the set of i-bins (ii-bins, II-bins, III-bins, IV-bins), and $N_{\rm i}$ ($N_{\rm ii}$, $N_{\rm II}$, $N_{\rm III}$, $N_{\rm IV}$) be the number of i-bins (ii-bins, II-bins, III-bins, IV-bins). Clearly,

$$\mathcal{B}^{FF} = \mathcal{B}_{\rm i} \cup \mathcal{B}_{\rm II} = \mathcal{B}_{\rm i} \cup \mathcal{B}_{\rm ii} \cup \mathcal{B}_{\rm III} \tag{1}$$

and

$$C^{FF} = N_{\rm i} + N_{\rm II} = N_{\rm i} + N_{\rm ii} + N_{\rm III}.$$

**Lemma 2.1.** $C^* \ge N_{\rm i}$.

**Proof.** Obviously, the total size of any two i-items exceeds 1. Otherwise, *FF* will not open a new bin for the item that arrived later. Accordingly, any two of them cannot be packed in one bin in the optimal packing. Hence $C^* \ge N_{\rm i}$. □

**Lemma 2.2.** *Given an integer $k \ge 1$, for any $M \ge k + 1$, if there are $M$ bins $B_1, B_2, \ldots, B_M$ in $\mathcal{B}^{FF}$ such that each of them contains at least $k$ items, then $\sum_{i=1}^{M} B_i > \frac{kM}{k+1}$.*

**Proof.** Without loss of generality, assume $B_s$ is before $B_t$ for any $1 \le s < t \le M$. For fixed $B_s$ and $B_t$, $s < t$, consider $k$ arbitrary items $a_{t_1}, a_{t_2}, \ldots, a_{t_k}$ in $B_t$. By *FF* we have $B_s + a_{t_j} > 1, j = 1, \ldots, k$. Summing the $k$ inequalities, we get

$$kB_s + B_t \ge kB_s + \sum_{j=1}^{k} a_{t_j} > k. \tag{2}$$

We will prove the lemma by induction on $M$. By (2), we have $kB_i + B_{k+1} > k$, $i = 1, \ldots, k$. Summing the $k$ inequalities, we get $k\sum_{i=1}^{k} B_i + kB_{k+1} > k^2$, i.e., $\sum_{i=1}^{k+1} B_i > k$. The result is true for $M = k + 1$. Suppose the result is true for $M = j \ge k + 1$, i.e., $\sum_{i=1}^{j} B_i > \frac{kj}{k+1}$. By (2), we have $kB_i + B_{j+1} > k$, $i = 1, \ldots, j$. Summing the $j$ inequalities, we get $k\sum_{i=1}^{j} B_i + jB_{j+1} > jk$. Combining this with the induction hypothesis, we have

$$\sum_{i=1}^{j+1} B_i = \frac{j\sum_{i=1}^{j} B_i + jB_{j+1}}{j} = \frac{k\sum_{i=1}^{j} B_i + jB_{j+1} + (j-k)\sum_{i=1}^{j} B_i}{j}$$
$$> \frac{jk + (j-k)\frac{kj}{k+1}}{j} = \frac{k(j+1)}{k+1}.$$

The result is also true for $M = j + 1$. The lemma is thus proved. □

**Corollary 2.1.** (i) *If* $\mathcal{B} \subseteq \mathcal{B}^{FF}$ *and* $|\mathcal{B}| \geq 2$, *then* $\sum_{B\in\mathcal{B}} B > \frac{1}{2}|\mathcal{B}|$.
(ii) *If* $\mathcal{B} \subseteq \mathcal{B}_{\text{II}}$ *and* $|\mathcal{B}| \geq 3$, *then* $\sum_{B\in\mathcal{B}} B > \frac{2}{3}|\mathcal{B}|$.
(iii) *If* $\mathcal{B} \subseteq \mathcal{B}_{\text{III}}$ *and* $|\mathcal{B}| \geq 4$, *then* $\sum_{B\in\mathcal{B}} B > \frac{3}{4}|\mathcal{B}|$.
(iv) *If* $\mathcal{B} \subseteq \mathcal{B}_{\text{IV}}$ *and* $|\mathcal{B}| \geq 5$, *then* $\sum_{B\in\mathcal{B}} B > \frac{4}{5}|\mathcal{B}|$.

## 3. The asymptotic performance ratio

We use the weighting function defined in [4], that is

$$W(x) = \begin{cases} \dfrac{6}{5}x, & 0 \leq x \leq \dfrac{1}{6}, \\ \dfrac{9}{5}x - \dfrac{1}{10}, & \dfrac{1}{6} < x \leq \dfrac{1}{3}, \\ \dfrac{6}{5}x + \dfrac{1}{10}, & \dfrac{1}{3} < x \leq \dfrac{1}{2}, \\ \dfrac{6}{5}x + \dfrac{4}{10}, & \dfrac{1}{2} < x \leq 1. \end{cases} \tag{3}$$

Clearly, $W(x)$ is an increasing function and

$$W(x) \geq \frac{6}{5}x. \tag{4}$$

Moreover, $W(a) > 1$ if $a$ is a large item. The weight of bin $B$, $W(B)$, is defined as the total weight of the items packed in it.

**Lemma 3.1** ([4]). *For every bin* $B$, $W(B) \leq \frac{17}{10}$. *Moreover, if* $B$ *does not contain large items, then* $W(B) \leq \frac{3}{2}$.

Let $\overline{W} = \sum_{a\in L} W(a)$ be the total weight of all items. By Lemma 3.1, we have

$$\overline{W} = \sum_{a\in L} W(a) = \sum_{B\in\mathcal{B}^*} W(B) \leq \sum_{B\in\mathcal{B}^*} \frac{17}{10} = \frac{17}{10}C^*. \tag{5}$$

**Lemma 3.2** ([4]). $\overline{W} > C^{FF} - 1 + \sum_{B\in\mathcal{B}^{FF}} \max\{0, W(B) - 1\}$.

Let $\mathcal{C} = \{B | B \in \mathcal{B}^{FF} \text{ and } W(B) < 1\}$ and $m = |\mathcal{C}|$. Label the bins in $\mathcal{C}$ as $C_1, C_2, \ldots, C_m$ such that $C_i$ is before $C_j$ for any $1 \leq i < j \leq m$. For each $C_i \in \mathcal{C}$, define $\alpha_i = \max\{\alpha | \text{ for some } j,\ 1 \leq j < i,\ C_j = 1 - \alpha\}$ with $\alpha_1$ taken to be 0.

**Lemma 3.3** ([4]). *If* $m \geq 2$, *then* $\sum_{i=1}^{m-1}(1 - W(C_i)) \leq \frac{6}{5}\alpha_m$.

The main result of this section is as follows.

**Theorem 3.1.** *For every list* $L$, $C^{FF}(L) \leq \frac{17}{10}C^*(L) + \frac{7}{10}$.

**Proof.** Suppose $C^{FF} > \frac{17}{10}C^* + \frac{7}{10}$, i.e. $C^{FF} \geq \frac{17}{10}C^* + \frac{4}{5}$. We have the following claims.

**Claim 3.1.** *If* $C^{FF} \geq \frac{17}{10}C^* + \frac{4}{5}$, *then there does not exist any large item in* II-*bins.*

**Proof.** Suppose there exists a large item $b_1$ in a II-bin $B'$. Since $B'$ is a II-bin, it contains another item $b_2$. If $B' \geq \frac{2}{3}$, by $b_1 > \frac{1}{2}$, (3) and (4), we have

$$W(B') \geq \left(\frac{6}{5}b_1 + \frac{4}{10}\right) + \frac{6}{5}(B' - b_1) = \frac{6}{5}B' + \frac{4}{10} \geq \frac{6}{5}.$$

Thus $\overline{W} > C^{FF} - 1 + \left(\frac{6}{5} - 1\right) = C^{FF} - \frac{4}{5} \geq \frac{17}{10}C^*$ by Lemma 3.2, which contradicts (5). Therefore, $B' < \frac{2}{3}$ and $b_2 \leq B' - b_1 < \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

For any bin $B$ before $B'$, $B > \frac{5}{6}$ since $b_2 < \frac{1}{6}$, so $W(B) \geq \frac{6}{5}B > 1$ by (4). For any II-bin $B$ after $B'$, any item in $B$ is greater than $\frac{1}{3}$ as $B' < \frac{2}{3}$. Hence $W(B) \geq 2W\left(\frac{1}{3}\right) = 1$. For any i-bin $B$ after $B'$, at most one bin does not containing large items. Recall that $B' < \frac{2}{3}$, the weight of $B$ is at least 1 if it contains a large item, and $W\left(\frac{1}{3}\right)$ otherwise. In other words, all bins except one in $\mathcal{B}^{FF}$ have weight at least 1, and the remaining one is at least $W\left(\frac{1}{3}\right) < 1$. Therefore, by (5),

$$\frac{17}{10}C^* \geq \overline{W} = \sum_{B\in\mathcal{B}^{FF}} W(B) \geq \sum_{i=1}^{C^{FF}-1} 1 + W\left(\frac{1}{3}\right) = C^{FF} - 1 + \frac{1}{2} = C^{FF} - \frac{1}{2} \geq \frac{17}{10}C^* + \frac{3}{10},$$

which is impossible. The claim is thus proved. □

**Claim 3.2.** *If* $C^{FF} \geq \frac{17}{10}C^* + \frac{4}{5}$, *then* $l \leq N_{\text{i}} - 1$, *where we recall that* $l$ *is the number of large items.*

**Proof.** By Claim 3.1, all large items are packed in i-bins by *FF*. Since no two large items can be packed into the same bin, we have $l \leq N_{\mathrm{i}}$. Suppose $l = N_{\mathrm{i}}$; then any i-bin contains exactly one large item, and its weight is at least 1. That is to say, each bin in $\mathcal{C}$ is a II-bin. We distinguish two cases according to the value of $m$.

*Case* 1. $m \geq 2$.

Note that $\alpha_m > \frac{1}{6}$. Otherwise, by the definition of $\alpha_m$ and (4), there exists $C_j \in \mathcal{C}$, $1 \leq j < m$, such that $C_j = 1 - \alpha_m$ and $W(C_j) \geq \frac{6}{5} C_j = \frac{6}{5}(1 - \alpha_m) \geq 1$, which is a contradiction.

Since $C_m$ is a II-bin, let $b_1, b_2$ be two items in $C_m$. Clearly, $b_1, b_2 > \alpha_m$. By Lemma 3.3, (4), (5) and $\alpha_m > \frac{1}{6}$, we have

$$\begin{aligned}
\frac{17}{10} C^* \geq \overline{W} &= \sum_{B \notin \mathcal{C}} W(B) + \sum_{i=1}^{m-1} W(C_i) + W(C_m) \\
&= \sum_{B \notin \mathcal{C}} W(B) + \sum_{i=1}^{m-1} (1 - (1 - W(C_i))) + W(C_m) \\
&\geq \sum_{B \neq C_m} 1 - \sum_{i=1}^{m-1} (1 - W(C_i)) + W(C_m) \\
&\geq \left(C^{FF} - 1\right) - \sum_{i=1}^{m-1} (1 - W(C_i)) + (W(b_1) + W(b_2)) \\
&> C^{FF} - 1 - \frac{6}{5}\alpha_m + \frac{6}{5}\alpha_m + \frac{6}{5}\alpha_m \\
&= C^{FF} - 1 + \frac{6}{5}\alpha_m > C^{FF} - \frac{4}{5},
\end{aligned}$$

which is impossible.

*Case* 2. $m = 1$.

If $W(C_1) > \frac{1}{5}$, then by (5),

$$\frac{17}{10} C^* \geq \overline{W} = \sum_{B \notin \mathcal{C}} W(B) + W(C_1) > C^{FF} - 1 + \frac{1}{5} = C^{FF} - \frac{4}{5},$$

which is a contradiction. Hence $W(C_1) \leq \frac{1}{5}$ and thus $C_1 \leq \frac{1}{6}$. If $l \geq 1$, then there exists an i-item $a \geq \frac{5}{6}$ since $N_{\mathrm{i}} = l \geq 1$ and $C_1 \leq \frac{1}{6}$. Note that all bins except $C_1$ have weight at least 1. By (5),

$$\frac{17}{10} C^* \geq \overline{W} \geq C^{FF} - 2 + W(a) + W(C_1) > C^{FF} - 2 + W\left(\frac{5}{6}\right) = C^{FF} - \frac{3}{5},$$

which is a contradiction. Then we know that $l = 0$, but this implies that $\overline{W} = \sum_{B \in \mathcal{B}^*} W(B) \leq \frac{3}{2} C^*$ by Lemma 3.1, and finally yields $C^{FF} - 1 < \overline{W} \leq \frac{3}{2} C^* \leq \frac{17}{10} C^* - \frac{1}{5}$ by Lemma 3.2. That is impossible as well. Then Claim 3.2 follows. □

Applying Lemma 2.1 and Claim 3.2, we obtain $l \leq C^* - 1$. In other words, there is at least one bin $B' \in \mathcal{B}^*$ which does not contain any large items. By Lemmas 3.1 and 3.2,

$$C^{FF} - 1 < \overline{W} = \sum_{B \in \mathcal{B}^* \setminus \{B'\}} W(B) + W(B') \leq \frac{17}{10}(C^* - 1) + \frac{3}{2} = \frac{17}{10} C^* - \frac{1}{5}.$$

It follows that $C^{FF} < \frac{17}{10} C^* + \frac{4}{5}$. The proof of Theorem 3.1 is thus completed. □

## 4. The absolute performance ratio

In this section, we prove that the absolute performance ratio of *FF* is no more than $\frac{12}{7}$.

**Lemma 4.1.** *If $N_{\mathrm{i}} \leq 1$ or $N_{\mathrm{i}} \geq C^{FF} - 2$, then $C^{FF} \leq \frac{5}{3} C^*$.*

**Proof.** As $C^{FF} = 1$ when $C^* = 1$ and $\frac{5}{3} C^* \geq 3$ when $C^* \geq 2$, our assertion is straightforward if $C^* = 1$ or $C^{FF} \leq 3$. Now assume $C^* \geq 2$ and $C^{FF} \geq 4$.

If $N_{\mathrm{i}} \leq 1$, then $N_{\mathrm{II}} = C^{FF} - N_{\mathrm{i}} \geq 3$. By Corollary 2.1, we have

$$C^* \geq \sum_{a \in L} a = \sum_{B \in \mathcal{B}_{\mathrm{i}}} B + \sum_{B \in \mathcal{B}_{\mathrm{II}}} B \geq \sum_{B \in \mathcal{B}_{\mathrm{II}}} B > \frac{2}{3} N_{\mathrm{II}} = \frac{2}{3}(C^{FF} - N_{\mathrm{i}}) \geq \frac{2}{3}(C^{FF} - 1),$$

i.e., $3C^* \geq 2C^{FF} - 1$. Recalling that $C^{FF} \geq 4$, we have $C^* \geq 3$ and thus $C^{FF} \leq \frac{3}{2} C^* + \frac{1}{2} \leq \frac{5}{3} C^*$.

If $N_{\rm i} \geq C^{FF} - 2$, then by Lemma 2.1, $C^{FF} \leq N_{\rm i} + 2 \leq C^* + 2$. If $C^* = 2$, then $C^{FF} = 4$ and $N_{\rm i} = 2$ since we assume $C^{FF} \geq 4$. Consider the two i-items $b_1, b_2$ which clearly follow $b_1 + b_2 > 1$. Then

$$\sum_{B \in \mathcal{B}_{\rm II}} B = \sum_{a \in L} a - (b_1 + b_2) \leq C^* - (b_1 + b_2) < 1,$$

which contradicts $N_{\rm II} = C^{FF} - N_{\rm i} = 2$. Hence $C^* \geq 3$ and we obtain $C^{FF} \leq C^* + 2 \leq \frac{5}{3}C^*$. □

**Lemma 4.2.** *If $C^{FF} \geq \frac{17}{10}C^*$, then $4N_{\rm i} \geq 18C^{FF} - 27C^* - 1$.*

**Proof.** Since $C^{FF} \geq \frac{17}{10}C^* > \frac{5}{3}C^*$, we obtain $N_{\rm i} \geq 2$ and $N_{\rm II} = C^{FF} - N_{\rm i} \geq 3$ by Lemma 4.1. In view of Corollary 2.1, we have

$$\begin{aligned} C^* &\geq \sum_{a \in L} a = \sum_{B \in \mathcal{B}_{\rm i}} B + \sum_{B \in \mathcal{B}_{\rm II}} B > \frac{1}{2}N_{\rm i} + \frac{2}{3}N_{\rm II} \\ &= \frac{1}{2}N_{\rm i} + \frac{2}{3}(C^{FF} - N_{\rm i}) = \frac{2}{3}C^{FF} - \frac{1}{6}N_{\rm i}. \end{aligned} \tag{6}$$

By Lemma 2.1, we further have $C^* > \frac{2}{3}C^{FF} - \frac{1}{6}C^*$, i.e., $C^{FF} < \frac{7}{4}C^*$, which is equivalent to

$$C^{FF} \leq \left\lceil \frac{7}{4}C^* \right\rceil - 1. \tag{7}$$

Direct calculation shows that $\lceil \frac{7}{4}C^* \rceil - 1 < \frac{17}{10}C^*$ when $C^* \leq 6$. Hence we assume $C^* \geq 7$ in the following.

Note that (6) is equivalent to

$$N_{\rm II} = C^{FF} - N_{\rm i} > 9C^{FF} - 12C^* - 3N_{\rm i}. \tag{8}$$

If $9C^{FF} - 12C^* - 3N_{\rm i} \leq 2$, then $C^{FF} \leq \frac{4}{3}C^* + \frac{1}{3}N_{\rm i} + \frac{2}{9} \leq \frac{5}{3}C^* + \frac{2}{63}C^* < \frac{17}{10}C^*$ by Lemma 2.1 and $C^* \geq 7$, which contradicts $C^{FF} \geq \frac{17}{10}C^*$. Therefore,

$$N_{\rm II} > 9C^{FF} - 12C^* - 3N_{\rm i} \geq 3. \tag{9}$$

**Claim 4.1.** *If $C^{FF} \geq \frac{17}{10}C^*$, then the last $9C^{FF} - 12C^* - 3N_{\rm i}$ II-bins contain only semilarge items.*

**Proof.** Suppose there exists an item which is not semilarge in one of the last $9C^{FF} - 12C^* - 3N_{\rm i}$ II-bins. Then the content of each of the first $N_{\rm II} - (9C^{FF} - 12C^* - 3N_{\rm i}) = 12C^* - 8C^{FF} + 2N_{\rm i}$ II-bins is at least $\frac{3}{4}$. Combining this with $N_{\rm i} \geq 2$, (9) and Corollary 2.1, we have

$$C^* > \frac{1}{2}N_{\rm i} + \frac{3}{4}(12C^* - 8C^{FF} + 2N_{\rm i}) + \frac{2}{3}(9C^{FF} - 12C^* - 3N_{\rm i}) = C^*,$$

which is a contradiction. □

**Claim 4.2.** *If $C^{FF} \geq \frac{17}{10}C^*$, then all* i-*items are semilarge items.*

**Proof.** Note that all the i-items are large except at most one. If the remaining one is not semilarge, then each of the remaining $N_{\rm i} - 1$ i-items should be greater than $\frac{3}{4}$. Then by Corollary 2.1 and $C^* \geq 7$, we have

$$C^* > \frac{3}{4}(N_{\rm i} - 1) + \frac{2}{3}N_{\rm II} = \frac{3}{4}(N_{\rm i} - 1) + \frac{2}{3}(C^{FF} - N_{\rm i}) > \frac{2}{3}C^{FF} - \frac{3}{4} \geq \frac{2}{3}C^{FF} - \frac{3}{28}C^*,$$

which leads to $C^{FF} < \frac{93}{56}C^* < \frac{17}{10}C^*$. □

**Claim 4.3.** *If $C^{FF} \geq \frac{17}{10}C^*$, then there are at most $3C^* - 2N_{\rm i} + 1$ semilarge* II-*items.*

**Proof.** Consider the packing of semilarge II-items in the optimal packing. Each of the $N_{\rm i} - 1$ bins containing a large i-item can accommodate at most one semilarge II-item. The bin containing the remaining i-item, which is semilarge by Claim 4.2, can accommodate at most two more semilarge II-items. Each of the remaining $C^* - N_{\rm i}$ bins can accommodate at most three semilarge II-items. Consequently, there are at most $(N_{\rm i} - 1) + 2 + 3(C^* - N_{\rm i}) = 3C^* - 2N_{\rm i} + 1$ semilarge II-items. □

By Claims 4.1 and 4.3, we have $2(9C^{FF} - 12C^* - 3N_{\rm i}) \leq 3C^* - 2N_{\rm i} + 1$, i.e. $4N_{\rm i} \geq 18C^{FF} - 27C^* - 1$. The lemma is thus proved. □

**Lemma 4.3.** *If $N_{\rm i} = C^*$, then $C^* \geq 2N_{\rm ii}$.*

**Proof.** Suppose $C^* < 2N_{\text{ii}}$. According to the pigeonhole principle, there exists a bin $B^* \in \mathcal{B}^*$ in which two ii-items $b_1$ and $b_2$ are packed. Since any two i-items cannot be packed in one bin in the optimal packing and $N_{\text{i}} = C^*$, there exists an i-item in $B^*$, say $a$. Since

$$b_1 + b_2 + a \leq 1, \tag{10}$$

$b_1$ and $b_2$ are packed in different bins by *FF*. Otherwise, *FF* will pack $a$, $b_1$ and $b_2$ together, contradicting the definition of an i-item. Let the two bins containing $b_1, b_2$ in $\mathcal{B}^{FF}$ be $B_1, B_2$ respectively, and $B_2$ be after $B_1$ without loss of generality. Since $b_1$ is a ii-item, there exists another item in $B_1$, say $b_1'$. Since $b_2$ is packed in a bin after $B_1$, we have

$$b_1 + b_1' + b_2 > 1. \tag{11}$$

It follows that $b_1'$ is not in $B^*$. Let $a'$ be the i-item which is packed in the same bin with $b_1'$ in $\mathcal{B}^*$; then

$$b_1' + a' \leq 1. \tag{12}$$

Since $a$ and $a'$ are both i-items,

$$a + a' > 1. \tag{13}$$

Therefore, by (12), (13) and (10),

$$b_1 + b_2 + b_1' \leq b_1 + b_2 + (1 - a') < b_1 + b_2 + a \leq 1,$$

which contradicts (11). □

**Lemma 4.4.** *If $N_{\text{i}} = C^*$ and $C^{FF} > \frac{12}{7}C^*$, then*

$$C^{FF} \leq \max\left\{\frac{1}{9}\left\lfloor\frac{C^*}{2}\right\rfloor + \frac{5}{3}C^* - \frac{1}{9}, \left\lfloor\frac{C^*}{2}\right\rfloor + C^* + 3\right\}.$$

**Proof.** Given $C^* \leq 10$, we get $C^{FF} \leq \lceil\frac{7}{4}C^*\rceil - 1 \leq \frac{12}{7}C^*$ by (7) and direct calculation. Hence we can assume $C^* \geq 11$. Moreover, given $C^{FF} \leq C^* + 7$, we get $C^{FF} \leq C^* + 7 \leq \frac{12}{7}C^*$ by $C^* \geq 11$. Then we can assume $C^{FF} \geq C^* + 8$ as well.

If $N_{\text{ii}} \leq 2$, there are at least $C^{FF} - N_{\text{i}} - N_{\text{ii}} \geq C^{FF} - C^* - 2 \geq 4$ III-bins. By Corollary 2.1, we have

$$C^* > \frac{3}{4}(C^{FF} - C^* - 2) + \frac{1}{2}(C^* + 2) = \frac{3}{4}C^{FF} - \frac{1}{4}C^* - \frac{1}{2},$$

i.e. $3C^{FF} < 5C^* + 2$. Hence $3C^{FF} \leq 5C^* + 1$ and thus $C^{FF} \leq \frac{5}{3}C^* + \frac{1}{3} \leq \frac{12}{7}C^*$. Therefore, we only need to consider the case when $N_{\text{ii}} \geq 3$.

If $N_{\text{ii}} \leq C^{FF} - C^* - 4$, then by Corollary 2.1 and $N_{\text{i}} = C^*$, we have

$$C^* > \frac{1}{2}N_{\text{i}} + \frac{2}{3}N_{\text{ii}} + \frac{3}{4}(C^{FF} - N_{\text{i}} - N_{\text{ii}}) = \frac{1}{2}C^* + \frac{2}{3}N_{\text{ii}} + \frac{3}{4}(C^{FF} - C^* - N_{\text{ii}}) = \frac{3}{4}C^{FF} - \frac{1}{4}C^* - \frac{1}{12}N_{\text{ii}},$$

i.e. $9C^{FF} < N_{\text{ii}} + 15C^*$. Hence $9C^{FF} \leq N_{\text{ii}} + 15C^* - 1$. Applying Lemma 4.3, we obtain $C^{FF} \leq \frac{1}{9}\lfloor\frac{C^*}{2}\rfloor + \frac{5}{3}C^* - \frac{1}{9}$. If $N_{\text{ii}} \geq C^{FF} - C^* - 3$, then $C^{FF} \leq \lfloor\frac{C^*}{2}\rfloor + C^* + 3$ by Lemma 4.3. Combining this with the two inequalities, the lemma is thus proved. □

**Lemma 4.5.** *There is not such a list that*

(i) $C^* = 11$ *and* $C^{FF} = 19$,
(ii) $C^* = 32$ *and* $C^{FF} = 55$,
(iii) $C^* = 39$ *and* $C^{FF} = 67$.

**Proof.** (i) Suppose there exists such a list. By Lemma 4.2, we have $4N_{\text{i}} \geq 18C^{FF} - 27C^* - 1 = 44 = 4C^*$. Therefore, $N_{\text{i}} = C^*$ by Lemma 2.1 and thus $N_{\text{ii}} \leq 5$ by Lemma 4.3. According to (8), $N_{\text{II}} > 9C^{FF} - 12C^* - 3N_{\text{i}} = 6 \geq N_{\text{ii}} + 1$. It follows that there is at least one III-bin in the last six II-bins. Hence, there are at least $2 \times (6-1) + 3 \times 1 = 13$ II-items in the last six II-bins, and these items are all semilarge by Claim 4.1. On the other hand, Claim 4.3 implies that there are at most $3C^* - 2N_{\text{i}} + 1 = 12$ semilarge II-items, which is a contradiction.

(ii) Suppose there exists such a list. By Lemma 4.2, we have $4N_{\text{i}} \geq 18C^{FF} - 27C^* - 1 = 125$. Therefore, by Lemma 2.1, $N_{\text{i}} = C^* = 32$ and thus $N_{\text{II}} = C^{FF} - N_{\text{i}} = 23$. Label all the II-bins as $B_1, B_2, \ldots, B_{23}$, so that $B_i$ is before $B_j$ for any $1 \leq i < j \leq 23$. The last $9C^{FF} - 12C^* - 3N_{\text{i}} = 15$ II-bins contain only semilarge II-items by Claim 4.1, but the number of semilarge II-items is no greater than $3C^* - 2N_{\text{i}} + 1 = 33$ by Claim 4.3. Then there are at most $33 - 15 \times 2 = 3$ semilarge items packed in the first $23 - 15 = 8$ II-bins. It follows that at least $8 - \lfloor\frac{3}{2}\rfloor = 7$ of the first eight II-bins contain not only semilarge items, and at least $8 - 3 = 5$ bins do not contain semilarge items. Let $B_{i_1}, B_{i_2}, \ldots, B_{i_s}$ be all the bins containing not only semilarge items, where $i_1 < i_2 < \cdots < i_t$ with $t = 7$ or 8. Choose five bins each of which does not contain any semilarge item, say $B_{j_1}, B_{j_2}, \ldots, B_{j_5}$, and $j_1 < j_2 < \cdots < j_5$. It is obvious that $J = \{j_1, j_2, \ldots, j_5\} \subseteq \{i_1, i_2, \ldots, i_s\} = I$.

If $j_5 < i_t$, then $B_{j_k} > \frac{3}{4}$ for $1 \le k \le 5$ as $B_{i_t}$ has items not greater than $\frac{1}{4}$ in it. For $1 \le k \le 5$, since $B_{j_k}$ does not contain any semilarge item, it must be a IV-bin. Accordingly, by Corollary 2.1,

$$32 = C^* > \sum_{k=1}^{5} B_{j_k} + \frac{2}{3}(N_{\mathrm{II}} - 5) + \frac{1}{2}N_{\mathrm{i}} > \frac{4}{5} \times 5 + \frac{2}{3} \times 18 + \frac{1}{2} \times 32 = 32,$$

which is a contradiction. Hence $j_5 = i_t$. Similarly to before, the $B_{j_k}$, $1 \le k \le 4$, which do not contain any semilarge item, must be IV-bins since $B_{j_5}$ has items not greater than $\frac{1}{4}$ in it. In the light of $t \ge 7$, there exist $s_1, s_2 \in I \setminus J$. $B_{s_1}, B_{s_2}$ are before $B_{j_5}$ since $j_5 = i_t$. If $B_{j_5}$ is a III-bin, then by (2)

$$\begin{aligned} B_{s_1} + B_{s_2} + \sum_{k=1}^{5} B_{j_k} &= \frac{1}{3}(3B_{s_1} + B_{j_5}) + \frac{1}{3}(3B_{s_2} + B_{j_5}) + \frac{1}{4}(4B_{j_1} + B_{j_2}) + \frac{3}{16}(4B_{j_2} + B_{j_3}) \\ &\quad + \frac{13}{80}(4B_{j_3} + B_{j_4}) + \frac{13}{240}(3B_{j_3} + B_{j_5}) + \frac{67}{240}(3B_{j_4} + B_{j_5}) \\ &> \frac{1}{3} \times 3 + \frac{1}{3} \times 3 + \frac{1}{4} \times 4 + \frac{3}{16} \times 4 + \frac{13}{80} \times 4 + \frac{13}{240} \times 3 + \frac{67}{240} \times 3 = \frac{27}{5}, \end{aligned}$$

but that will lead us to

$$32 = C^* > B_{s_1} + B_{s_2} + \sum_{k=1}^{5} B_{j_k} + \frac{2}{3}(N_{\mathrm{II}} - 7) + \frac{1}{2}N_{\mathrm{i}} > \frac{27}{5} + \frac{2}{3} \times 16 + \frac{1}{2} \times 32 > \frac{481}{15},$$

which is a contradiction. Therefore $B_{j_5}$ must be a ii-bin, so its content is less than $\frac{1}{2}$ since it does not contain any semilarge items. It follows that items in a II-bin after $B_{j_5}$ are all large, which causes its content to be greater than 1.

(iii) Suppose there exists such a list. By Lemma 4.2, we have $4N_{\mathrm{i}} \ge 18C^{FF} - 27C^* - 1 = 152 = 4C^* - 4$. We distinguish two cases according to the value of $N_{\mathrm{i}}$.

*Case* 1. $N_{\mathrm{i}} = C^* = 39$.

By Claims 4.1 and 4.3, the last $9C^{FF} - 12C^* - 3N_{\mathrm{i}} = 18$ II-bins contain only semilarge items, while the number of semilarge II-items cannot exceed $3C^* - 2N_{\mathrm{i}} + 1 = 40$. Moreover, $N_{\mathrm{II}} = C^{FF} - N_{\mathrm{i}} = 28$. Then there are at most $40 - 18 \times 2 = 4$ semilarge items in the first $28 - 18 = 10$ II-bins. Therefore, at least $10 - 4 = 6$ of the first ten II-bins do not contain any semilarge items. Among these bins, the content of each of the first five bins is at least $\frac{3}{4}$, and they must be IV-bins as a consequence. On the other hand, by Lemma 4.3, $N_{\mathrm{ii}} \le 19$ and thus $N_{\mathrm{III}} \ge 9$. Therefore, by Corollary 2.1,

$$39 = C^* > \frac{4}{5} \times 5 + \frac{3}{4}(N_{\mathrm{III}} - 5) + \frac{2}{3}N_{\mathrm{ii}} + \frac{1}{2}N_{\mathrm{i}} \ge \frac{4}{5} \times 5 + \frac{3}{4} \times 4 + \frac{2}{3} \times 19 + \frac{1}{2} \times 39 = \frac{235}{6},$$

which is a contradiction.

*Case* 2. $N_{\mathrm{i}} = C^* - 1 = 38$.

By Claim 4.1, there are at least $9C^{FF} - 12C^* - 3N_{\mathrm{i}} = 21$ II-bins, and thus at least 42 semilarge II-items. If all the 38 i-items are large, then each bin in $\mathcal{B}^*$ containing an i-item can contain at most one semilarge II-item in the optimal packing. However, the remaining $42 - N_{\mathrm{i}} = 4$ semilarge II-items cannot be packed in the remaining $C^* - N_{\mathrm{i}} = 1$ bin. Hence there exists an i-item which is not large, but it is still semilarge by Claim 4.2. Moreover, there aren't any large II-items. Otherwise, there are at least 37 large i-items and one large II-item. Each of them can be packed with at most one semilarge II-item in the optimal packing. However, the remaining $42 - (37 + 1) - 1 = 3$ semilarge II-items and one semilarge i-item cannot be packed in the remaining $C^* - (37 + 1) = 1$ bin, which is a contradiction. Therefore, we have $l \le N_{\mathrm{i}} - 1 = C^* - 2$. Thus by Lemmas 3.1 and 3.2, we have

$$66 = C^{FF} - 1 < \overline{W} \le \frac{17}{10}(C^* - 2) + \frac{3}{2} \times 2 = \frac{659}{10},$$

which is a contradiction. □

In order to get Theorem 4.1, we also need the following lemma concerning the diophantine equation.

**Lemma 4.6** ([8] *Diophantine Equation*)**.** *If a and b are coprime, u is an integer. The linear diophantine equation $ax + by = u$ has infinitely many solutions. If the pair $(x_0, y_0)$ is one integral solution, then all others are of the form*

$$x = x_0 + bv, \qquad y = y_0 - av,$$

*where $v$ is an integer.*

**Theorem 4.1.** *For every list $L$, $C^{FF}(L) \le \frac{12}{7}C^*(L)$.*

**Proof.** If $C^{FF} \le \frac{17}{10}C^*$ or $C^* \le 10$, the result clearly follows by the previous discussion. We assume $C^{FF} > \frac{17}{10}C^*$ and $C^* \ge 11$ in the following. Let

$$31C^* - 18C^{FF} = u \tag{14}$$

be a diophantine equation relating to $C^*$ and $C^{FF}$, where $u$ is an integer and

$$u = 31C^* - 18C^{FF} \ge 27C^* + 4N_{\mathrm{i}} - 18C^{FF} \ge -1 \tag{15}$$

by Lemmas 2.1 and 4.2. Since $(7u, 12u)$ is a solution of (14), any integral solution of (14) can be written as

$$\begin{cases} C^* = 7u + 18v, \\ C^{FF} = 12u + 31v, \end{cases} \tag{16}$$

by Lemma 4.6, where $v$ is an integer. Taking the expressions for $C^*$ and $C^{FF}$ in Theorem 3.1, this requires

$$u + 4v \le 7. \tag{17}$$

When $u \ge 4$ we get $v \le 0$ from (17), so by (16)

$$\frac{C^{FF}}{C^*} = \frac{12u + 31v}{7u + 18v} = \frac{31}{18} - \frac{1}{18(7 + \frac{18v}{u})} \le \frac{31}{18} - \frac{1}{18 \times 7} = \frac{12}{7}.$$

When $u \le 3$, due to (15) and (17), the possible pairs $(u, v)$ are

$$(-1, 1), (-1, 2), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1),$$

and the corresponding pairs $(C^*, C^{FF})$ are

$$(11, 19), (29, 50), (18, 31), (7, 12), (25, 43), (14, 24), (32, 55), (21, 36), (39, 67).$$

Lemma 4.5 excludes the possibility of (11, 19), (32, 55), (39, 67). For the pairs of (29, 50), (18, 31), (25, 43), we have

$$4N_{\mathrm{i}} \ge 18C^{FF} - 27C^* - 1 \ge 4C^* - 3,$$

by Lemma 4.2 and direct calculation. So $N_{\mathrm{i}} = C^*$, and thus Lemma 4.4 implies that such a list will not exist. The remaining pairs all fulfill $C^{FF} = \frac{12}{7}C^*$. Then we complete the proof of Theorem 4.1. □

Since there exists such a list that $C^* = 10$ and $C^{FF} = 17$ [7], Theorem 4.1 shows that the gap between the lower and upper bounds of the absolute performance ratio of *FF* is less than 0.0143. We conjecture that the absolute performance ratio of *FF* is exactly $\frac{17}{10}$, which implies that the absolute performance ratio and asymptotic performance ratio of *FF* are identical. This is not common among bin-packing algorithms. To settle the conjecture, the first step is to determine whether there exists such a list that $C^* = 7$ and $C^{FF} = 12$, or not.

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