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NOTE

A Characterization of Graphs Having All (g, f)-Factors

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Let G be a graph with vertex set V and let $g, f\colon V \to \mathbb{Z}^+$. We say that G has all (g, f)-factors if G has an h-factor for every $h\colon V \to \mathbb{Z}^+$ such that $g(v) \leqslant h(v) \leqslant f(v)$ for every $v \in V$ and at least one such h exists. In this note, we derive from Tutte's f-factor theorem a similar characterization for the property of having all (g, f)-factors. An analogous result for parity-factors is presented also. © 1998 Academic Press

Let G be a finite graph with possible multiple edges and loops and let $g, f\colon V \to \mathbb{Z}^+$, where $V = V(G)$ is the vertex set of G and $\mathbb{Z}^+$ denotes the set of nonnegative integers. For a vertex $v \in V$ we let $d_G(v)$ denote the degree of v in G. A (g, f)-factor of G is a spanning subgraph F such that $g(v) \leqslant d_F(v) \leqslant f(v)$ for all $v \in V$. An (f, f)-factor is called an f-factor. Let $U, W \subseteq V$ be disjoint sets. We write $f(U)$ instead of $\sum_{v \in U} f(v)$ and $d_G(U)$ instead of $\sum_{v \in U} d_G(v)$. By $e_G(U, W)$ we denote the number of edges of G joining a vertex of U to a vertex of W.

Lovász [6] gave a characterization of graphs having a (g, f)-factor and thereby he generalized Tutte's f-factor theorem [8]. In [9] Tutte showed that the (g, f)-factor theorem can be derived from the f-factor theorem. Given positive integers a and b, the f-factor theorem has been applied in [4] and [5] to obtain conditions implying the existence of h-factors for every $h\colon V \to \{a, a+1, ..., b\}$ with $h(V) \equiv 0 \pmod 2$. More generally, one can ask for the existence of h-factors, where $h\colon V \to \mathbb{Z}^+$ is any function such that $g(v) \leqslant h(v) \leqslant f(v)$ for every $v \in V$ and $h(V) \equiv 0 \pmod 2$. The aim of this note is to present a characterization of graphs having these factors. The result will be proved using Tutte's theorem, and so the f-factor theorem is also self-refining in this direction.

In the following let $g, f\colon V \to \mathbb{Z}^+$ such that there exists a function $h\colon V \to \mathbb{Z}^+$ with $g(v) \leqslant h(v) \leqslant f(v)$ for every vertex $v \in V$ and $h(V) \equiv 0$ (mod 2). We will say that G has all (g, f)-factors if and only if G has an h-factor for every h described above.

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THEOREM 1. *G has all (g, f)-factors if and only if*

$$g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f) \geqslant \begin{cases} -1, & \text{if } f \neq g \\ 0, & \text{if } f = g \end{cases} \tag{1}$$

for all disjoint sets $D, S \subseteq V$, where $q^*_G(D, S, g, f)$ denotes the number of components C of $G-(D \cup S)$ such that there exists a vertex $v \in V(C)$ with $g(v) < f(v)$ or $e_G(V(C), S) + f(V(C)) \equiv 1 \pmod 2$.

We excluded above the particular situation in which $g = f$ and $f(V) \equiv 1 \pmod 2$, that is, a situation where no (g, f)-factor exists. By means of Lovász' result we can see the necessity for that.

THEOREM 2 ((g, f)-factor theorem [6]). *G has a (g, f)-factor if and only if*

$$f(D) - g(S) + d_{G-D}(S) - \hat{q}_G(D, S, g, f) \geqslant 0$$

for all disjoint sets $D, S \subseteq V$, where $\hat{q}_G(D, S, g, f)$ denotes the number of components C of $G-(D \cup S)$ with $g(v) = f(v)$ for all $v \in V(C)$ and $e_G(V(C), S) + f(V(C)) \equiv 1 \pmod 2$.

Note that every component counted by $\hat{q}_G(D, S, g, f)$ is also counted by $q^*_G(D, S, g, f)$, and hence it holds that $q^*_G(D, S, g, f) \geqslant \hat{q}_G(D, S, g, f)$ for all disjoint sets $D, S \subseteq V$. Moreover, it holds that $q^*_G(D, S, g, f) = \hat{q}_G(D, S, g, f)$ if and only if $g(v) = f(v)$ for all $v \in V-(D \cup S)$. So, we have

$$\begin{aligned} &g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f) \\ &\qquad \leqslant f(D) - g(S) + d_{G-D}(S) - \hat{q}_G(D, S, g, f), \end{aligned}$$

with equality if and only if $f = g$. Therefore, a graph satisfying (1) for all disjoint sets $D, S \subseteq V$ has a (g, f)-factor by Theorem 2, but if $g = f$ and $f(V) \equiv 1 \pmod 2$, no (g, f)-factor exists.

THEOREM 3. (f-Factor Theorem [8].) *G has an f-factor if and only if*

$$\Theta_G(D, S, f) := f(D) - f(S) + d_{G-D}(S) - q_G(D, S, f) \geqslant 0$$

for all disjoint sets $D, S \subseteq V$, where $q_G(D, S, f)$ denotes the number of components C of $G-(D \cup S)$ such that $e_G(V(C), S) + f(V(C)) \equiv 1 \pmod 2$. Moreover, $\Theta_G(D, S, f) \equiv f(V) \pmod 2$ for all disjoint sets $D, S \subseteq V$.

Proof of Theorem 1. We first verify that (1) guarantees that G has all (g, f)-factors. Therefore let $h: V \to \mathbb{Z}^+$ be such that $g(v) \leqslant h(v) \leqslant f(v)$ for every $v \in V$ and $h(V) \equiv 0 \pmod 2$. Then we have for disjoint sets $D, S \subseteq V$

$$\begin{aligned}\Theta_G(D, S, h) &= h(D) - h(S) + d_{G-D}(S) - q_G(D, S, h)\\ &\geqslant g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f)\\ &\geqslant -1.\end{aligned}$$

So, G has an h-factor by Theorem 3.

Next we show that (1) is satisfied if G has all (g, f)-factors. Let $D, S \subseteq V$ be disjoint sets. We define h: $V \to \mathbb{Z}^+$ as follows. For all $v \in D$, we let $h(v) = g(v)$, and for all $v \in S$, we let $h(v) = f(v)$. Now let C be a component of $G - (D \cup S)$. If $f(v) = g(v)$ for all $v \in V(C)$, then we let $h(v) = f(v)$ for all $v \in V(C)$. Otherwise there exists a vertex $u \in V(C)$ such that $g(u) < f(u)$. Then we let $h(v) = f(v)$ for all $v \in V(C) - \{u\}$ and choose $h(u)$ to be $f(u)$ or $f(u) - 1$ such that $e_G(V(C), S) + h(C)$ becomes odd. Thereby, we have $q^*_G(D, S, g, f) = q_G(D, S, h)$, and so

$$g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f) = \Theta_G(D, S, h).$$

Hence (1) is satisfied for D and S by Theorem 3, if $h(V)$ is even, since G has an h-factor. Finally, if $h(V)$ is odd, we have $f \neq g$, since G has a (g, f)-factor. So there exists a vertex $u \in V$ such that $g(u) < h(u)$ or $h(u) < f(u)$. Now we define h^*: $V \to \mathbb{Z}^+$ by $h^*(v) = h(v)$ for all $v \in V - \{u\}$ and

$$h^*(u) = \begin{cases} h(u) - 1, & \text{if} \quad h(u) = f(u)\\ h(u) + 1, & \text{otherwise.}\end{cases}$$

Therefore, $h^*(V)$ is even and $g(v) \leqslant h^*(v) \leqslant f(v)$ holds for all $v \in V$. So, G has an h^*-factor and thus $\Theta_G(D, S, h^*) \geqslant 0$. Moreover, it holds that

$$\Theta_G(D, S, h) - \Theta_G(D, S, h^*) = \begin{cases} h(u) - h^*(u), & \text{if} \quad u \in D\\ h^*(u) - h(u), & \text{if} \quad u \in S\\ q_G(D, S, h^*) - q_G(D, S, h), & \text{otherwise,}\end{cases}$$

and so we have $\Theta_G(D, S, h) - \Theta_G(D, S, h^*) \geqslant -1$. Thus

$$\begin{aligned}g(D) - f(S) + d_{G-D}(S) - q^*_G(D, S, g, f) &= \Theta_G(D, S, h)\\ &\geqslant \Theta_G(D, S, h^*) - 1 \geqslant -1.\end{aligned}$$

Since $f \neq g$, this shows that (1) is also satisfied, if $h(V)$ is odd.

This completes the proof of Theorem 1. ∎

It is quite natural to proceed similarily in related situations. As a simple example, we consider parity-factors. Let therefore g, f: $V \to \mathbb{Z}^+$ such that

$$g(v) \leqslant f(v) \qquad \text{and} \qquad g(v) \equiv f(v) \pmod 2 \tag{2}$$

for every $v \in V$. Then a spanning subgraph F of G is called a (g, f)-parity-factor, if $g(v) \leqslant d_F(v) \leqslant f(v)$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V$.

THEOREM 4 [2, 7]. *G has a* (g, f)*-parity-factor if and only if*

$$f(D) - g(S) + d_{G-D}(S) - q_G(D, S, f) \geqslant 0$$

for all disjoint sets $D, S \subseteq V$.

We say that G has all (g, f)-parity-factors, if G has an h-factor for every $h\colon V \to \mathbb{Z}^+$, such that $g(v) \leqslant h(v) \leqslant f(v)$ and $h(v) \equiv f(v) \pmod 2$ for all $v \in V$.

THEOREM 5. *G has all* (g, f)*-parity-factors if and only if*

$$g(D) - f(S) + d_{G-D}(S) - q_G(D, S, f) \geqslant 0 \tag{3}$$

for all disjoint sets $D, S \subseteq V$.

Proof. We first show that G has all (g, f)-parity-factors, if (3) is satisfied. Therefore let $h\colon V \to \mathbb{Z}^+$ be such that $g(v) \leqslant h(v) \leqslant f(v)$ and $h(v) \equiv f(v) \pmod 2$ for all $v \in V$. Then it holds that $q_G(D, S, h) = q_G(D, S, f)$ for all disjoint sets $D, S \subseteq V$, and thus

$$\begin{aligned} \Theta_G(D, S, h) &= h(D) - h(S) + d_{G-D}(S) - q_G(D, S, h) \\ &\geqslant g(D) - f(S) + d_{G-D}(S) - q_G(D, S, f) \geqslant 0. \end{aligned}$$

So, G has an h-factor by Theorem 3.

Next we verify (3), if G has all (g, f)-parity-factors. Let $D, S \subseteq V$ be disjoint sets. We define $h\colon V \to \mathbb{Z}^+$ as follows. For all $v \in D$, we let $h(v) = f(v)$, and for all $v \notin D$, we let $h(v) = g(v)$. Then we have $q_G(D, S, h) = q_G(D, S, g) = q_G(D, S, f)$ by (2), and since G has an h-factor, we get

$$g(D) - f(S) + d_{G-D}(S) - q_G(D, S, f) = \Theta_G(D, S, h) \geqslant 0,$$

as desired. ∎

Remarks. Both referees pointed out that our results are related to a more general factor problem. Suppose that for every vertex $v \in V$ of a graph G a set $B_v \subset \mathbb{Z}^+$ is given. The *general factor problem* asks whether there exists a factor F of G such that $d_F(v) \in B_v$ for every $v \in V$. A set $B \subset \mathbb{Z}$ is said to have a *gap of length* $p \geqslant 1$ if there exists an integer $k \in B$ such that $k+1, ..., k+p \notin B$ and $k+p+1 \in B$. The general factor problem is well understood, if every B_v has no gap of length greater than 1 (or, using the terminology of [1], every B_v is a one-dimensional jump system). Characterizations in the spirit of Tutte's f-factor theorem are presented in [7, 3]. Note that Theorems 2–4 are special cases thereof. It seems however that there is no obvious generalization of Theorems 1 and 5 for the property of having all $(B_v\colon v \in V)$-factors.

Another problem posed by one of the referees remains open also: Is there a polynomial algorithm for testing whether a graph G has all (g, f)-factors?

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REFERENCES

1. A. Bouchet and W. H. Cunningham, Delta-matroids, jump systems, and bisubmodular polyhedra, *SIAM J. Discrete Math.* **8** (1995), 17–32.
2. C. Chen and M. Kano, Odd and even factors with given properties, *Chinese Quart. J. Math.* **7** (1992), 65–71.
3. G. Cornuéjols, General factors of graphs, *J. Combin. Theory Ser. B* **45** (1988), 185–198.
4. P. Katerinis, Toughness of graphs and the existence of factors, *Discrete Math.* **80** (1990), 81–92.
5. M. Kano and N. Tokushige, Binding numbers and *f*-factors of graphs, *J. Combin. Theory Ser. B* **54** (1992), 213–221.
6. L. Lovász, Subgraphs with prescribed valencies, *J. Combin. Theory* **8** (1970), 391–416.
7. L. Lovász, The factorization of graphs, II, *Acta Math. Sci. Hungar.* **23** (1972), 223–246.
8. W. T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952), 314–328.
9. W. T. Tutte, Graph factors, *Combinatorica* **1** (1981), 79–97.