Note

A note on the gradient of heat semigroup✩

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Abstract

A new representation for the gradient of heat semigroup on Riemannian manifold is given by using the integration by parts formula on Wiener space, which reflects in some sense the asymptotic property of Brownian motion.

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1. Introduction

Let \((X, H, \mu)\) be the classical Wiener space, where the elements in \(X\) consist of the continuous functions from \([0, 1]\) to \(\mathbb{R}^d\) starting from 0, \(H \subset X\) is the Cameron–Martin space, the elements in \(H\) are absolutely continuous and whose derivatives are square integrable, and \(\mu\) is the Wiener measure. We use \(\{x_t\}_{t \geq 0}\) to denote the \(d\)-dimensional standard Brownian motion on \(X\).

Let us consider the following interesting problem: for fixed \(t > 0\) and \(x, a \in \mathbb{R}^d\)

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[ f(x_t + x) \langle x_\varepsilon, a \rangle \right] = \gamma
\]

where \(f\) is a smooth function with compact support on \(\mathbb{R}^d\), \(\langle \cdot, \cdot \rangle\) denotes the usual inner product in \(\mathbb{R}^d\).

We can give a simple answer for this problem. Define

\[
\eta_\varepsilon(s) := \frac{\varepsilon \wedge s}{\varepsilon} \cdot a \in H.
\]

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Then for $\varepsilon < t$,
\[ \eta_\varepsilon(t) = a \]
and
\[ \langle x_\varepsilon, a \rangle = \varepsilon \int_0^t \langle \dot{\eta}_\varepsilon(s), dx_s \rangle = \varepsilon \delta(\eta_\varepsilon). \]
(1)

Here $\delta(\eta_\varepsilon)$ denotes the Skorohod integral of $\eta_\varepsilon$ and $\dot{\eta}_\varepsilon(s)$ is the derivative of $\eta_\varepsilon(s)$ with respect to $s$. By the integration by parts formula on Wiener space (cf. [2,3]), we have
\[\frac{1}{\varepsilon} \mathbb{E}[f(x_t + x) \langle x_\varepsilon, a \rangle] = \mathbb{E}[f(x_t + x) \delta(\eta_\varepsilon)] = \mathbb{E}[D^X_{\eta_\varepsilon} f(x_t + x)]\]
\[= \frac{d}{d\tau} \bigg|_{\tau=0} \mathbb{E}[f(x_t + x + \tau \eta_\varepsilon(t))] = \frac{d}{d\tau} \bigg|_{\tau=0} \mathbb{E}[f(x_t + x + \tau a)] \]
\[= \langle \partial_x T_t f, a \rangle, \]
where $D^X$ denotes the Malliavin derivative on Wiener space $X$ and $T_t f(x) := \mathbb{E}[f(x_t + x)]$ is the Brownian semigroup.

We now turn to the diffusion process on a Riemannian manifold $M$, and can ask whether this type limit holds if we replace $x_t + x$ above by the diffusion process $m_t(m_0)$ on $M$. Now let us state our theorem in detail and give a positive answer. In particular, this limit will give a new representation for the gradient of heat semigroup.

Let $M$ be a compact Riemannian manifold with dimension $d$. We denote by $\Delta$ the Laplace–Beltrami operator on $M$, and by $\nabla$ the Levi-Civita connection. A smooth vector field $Z$ on $M$ is given. We consider the following elliptic operator (cf. [1]):
\[Lf := \frac{1}{2} \Delta f - \partial Z f.\]
The diffusion process $m_t(t, m_0)$ associated with $L$ is given by the projection of the solution to the following Stratonovich stochastic differential equation on orthogonal frame bundle $O(M)$:
\[
\begin{cases}
    dx(t) = \sum_{a=1}^{d} A_a(r_x(t)) [\circ dx^a_t - z^a(r_x(t)) \, dt], \\
    r_x(0) = 0 \in \pi^{-1}(m_0),
\end{cases}
\]
where $\{A_a\}_{a=1,\ldots,d}$ are the canonical horizontal vector fields on $O(M)$ corresponding to the Levi-Civita connection, $\pi: O(M) \mapsto M$ is the bundle projection, and $z^a(r) := [r^{-1} Z]^a$ is the scalarization of $Z$. In particular, the horizontal lift of $Z$ is just given by $z^a(r) A_a(r)$.

The heat semigroup associated to $L$ is defined by
\[ (T_t f)(m_0) := \mathbb{E}[f(m_x(t, m_0))] = \mathbb{E}[f(\pi(r_x(t, r_0)))] \]

We have the following result.

**Theorem 1.** Let $f \in C^\infty(M)$. For any $m_0 \in M$, $t > 0$ and $U \in T_{m_0}(M)$, we have
\[ \partial_U (T_t f)(m_0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[ f(m_x(t, m_0)) \langle x_\varepsilon, u \rangle \right], \]
where $u = r_0^{-1}(U) \in \mathbb{R}^d$ and $r_0 \in \pi^{-1}(m_0)$ is fixed.

**Remark 2.** Since the law of Brownian motion is invariant under the orthogonal transformation, the above limit is independent of the choice of initial frame $r_0$. 

2. Proof of theorem

We define
\[ \eta_\varepsilon(s) := \varepsilon \wedge s \cdot u \in H. \]

Then by the integration by parts formula on Wiener space, we have by (1)
\[ \mathbb{E}[D_X^{m_x} f(m_x(t, m_0))] = \mathbb{E}[f(m_x(t, m_0)) \sigma(\eta_\varepsilon)] = \frac{1}{\varepsilon} \mathbb{E}[f(m_x(t, m_0))(x_\varepsilon, u)]. \]  

(3)

Now we prove
\[ \lim_{\varepsilon \to 0} \mathbb{E}[D_X^{m_x} f(m_x(t, m_0))] = \partial U(T; f). \]  

(4)

One embeds \(O(M)\) in Euclidean space \(\mathbb{R}^{n_0}\) for some \(n_0 \in \mathbb{N}\). The usual norm in \(\mathbb{R}^{n_0}\) is denoted by \(\| \cdot \|\). Then we have the following assertion: for fixed \(t > 0\)
\[ \lim_{\varepsilon \to 0} \limsup_{\tau \to 0} \frac{1}{\tau^2} \mathbb{E}\|r_{x + \tau \eta_\varepsilon}(t, r_0) - r_x(t, r_0 + \tau U^*)\|^2 = 0, \]  

(5)

where \(U^*(r_0) = u^\alpha A_\alpha(r_0)\) is the horizontal lift of \(U\) on \(O(M)\), and \(r_0 + \tau U^*\) denotes the integral curve in \(O(M)\) tangent to \(U^*\) at starting point \(r_0\).

For the simplicity of notation, we write \(r_{x + \tau \eta_\varepsilon}(t, r_0)\) (respectively \(r_x(t, r_0 + \tau U^*)\)) as \(r_{1, \tau}(t)\) (respectively \(r_{2, \tau}^2(t)\)). From Eq. (2), we know
\[ r_{1, \tau}(t) - r_{2, \tau}^2(t) = \sum_{\alpha=1}^d \int_0^t \left[ A_\alpha(r_{1, \tau}^2(s)) - A_\alpha(r_{1, \tau}^2(s)) \right] \circ d\kappa^\alpha_{x}(s) \]
\[ + \sum_{\alpha=1}^d \int_0^t [A_\alpha(r_{1, \tau}^2(s)) - A_\alpha(r_{1, \tau}^2(s)) - A_\alpha(r_{1, \tau}^2(s))] ds \]
\[ + \sum_{\alpha=1}^d \left( \tau \int_0^t A_\alpha(r_{1, \tau}^2(s)) \kappa^\alpha_{x}(s) ds \right) + r_0 - (r_0 + \tau U^*) \]
\[ := I_1(\varepsilon, \tau) + I_2(\varepsilon, \tau) + I_3(\varepsilon, \tau). \]

By the relation of Stratonovich’s integral and Itô’s integral and Burkholder’s inequality, it is easy to get
\[ \mathbb{E}\|I_1(\varepsilon, \tau)\|^2 \leq C \int_0^t \mathbb{E}\|r_{1, \tau}^1(s) - r_{1, \tau}^2(s)\|^2 ds. \]

Hereafter \(C\) denotes a generic constant independent of \(\tau\) and \(\varepsilon\).

Moreover, by the compactness of \(O(M)\), we also have
\[ \mathbb{E}\|I_2(\varepsilon, \tau)\|^2 \leq C \int_0^t \mathbb{E}\|r_{1, \tau}^1(s) - r_{1, \tau}^2(s)\|^2 ds. \]

Now we deal with \(I_3(\varepsilon, \tau)\). For \(\varepsilon < t\), we have
\[ \| I_3(\varepsilon, \tau) \| \leq \tau \left\{ \sum_{\alpha=1}^{d} \left\| \int_0^t \left[ A_\alpha(r_{\varepsilon, \tau}(s)) - A_\alpha(r_0) \right] \dot{\eta}_\varepsilon^\alpha(s) \, ds \right\| + \sum_{\alpha=1}^{d} \left\| \int_0^1 \dot{\eta}_\varepsilon^\alpha(s) \, ds A_\alpha(r_0) \right\| + \frac{1}{\tau} \left[ r_0 - (r_0 + \tau U^*) \right] \right\} \]
\[ \leq \tau \left\{ \frac{C}{\varepsilon} \int_0^\varepsilon \| r_{\varepsilon, \tau}(s) - r_0 \| \, ds + \sum_{\alpha=1}^{d} u_\alpha A_\alpha(r_0) + \frac{1}{\tau} \left[ r_0 - (r_0 + \tau U^*) \right] \right\}. \]

In view of the fact \( \lim_{\tau \to 0} \frac{1}{\tau}[r_0 - (r_0 + \tau U^*)] = -\sum_{\alpha=1}^{d} u_\alpha A_\alpha(r_0) \), we therefore have
\[ \lim_{\tau \to 0} \frac{1}{\tau^2} \mathbb{E} \| I_3(\varepsilon, \tau) \|^2 \leq C \cdot \lim_{\tau \to 0} \sup_{0 \leq s \leq \varepsilon} \mathbb{E} \| r_{\varepsilon, \tau}(s) - r_0 \|^2 \leq C \cdot \varepsilon, \]
where the second step follows from Eq. (2) and is standard.

Finally, we set
\[ g_\varepsilon(s) := \lim_{\tau \to 0} \frac{1}{\tau^2} \mathbb{E} \| r_{\varepsilon, \tau}(s) - r_0 \|^2, \]
then Fatou's lemma gives
\[ g_\varepsilon(t) \leq C \int_0^t g_\varepsilon(s) \, ds + C \cdot \varepsilon. \]

Thus, Gronwall’s inequality yields the desired limit (5).

Hence, we have by (5)
\[ \lim_{\varepsilon \to 0} \mathbb{E} \left[ D_{\eta_\varepsilon}^X \tilde{f}(r_x(t, r_0)) \right] = \lim_{\varepsilon \to 0} \left( d \frac{d}{d\tau} \bigg|_{\tau=0} \mathbb{E} \left[ \tilde{f}(r_x + \tau \eta_\varepsilon(t, r_0)) \right] \right) \]
\[ = d \bigg|_{\tau=0} \mathbb{E} \left[ \tilde{f}(r_x(t, r_0 + \tau U^*)) \right] = \partial_U (T_t f)(m_0), \quad (6) \]
where \( \tilde{f}(r) := f(\pi(r)) \) is the lift of \( f \) in \( O(M) \).

Combining (3), (4) and (6), we complete the proof.

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**References**

