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Letter

Convergence of a preconditioned iterative method for H-matrices

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Abstract

In this paper, we consider a preconditioned iterative method for solving the linear system $Ax = b$, which is a generalization of a method proposed in Kotakemori et al. [3] and prove its convergence for the case when A is an H-matrix.

Keywords: Gauss–Seidel method; Preconditioning; H-matrix

1. Introduction

For solving the linear system $Ax = b$ or its preconditioned form

$$PAx = Pb, \tag{1}$$

we consider the iterative process

$$x_{k+1} = M_p^{-1}N_p x_k + M_p^{-1}Pb, \quad k = 0,1,\ldots, \tag{2}$$

which corresponds to a splitting $PA = M_p - N_p$, where A is an $n \times n$ matrix with unit diagonal elements, P is an $n \times n$ preconditioning matrix and x and b are n-dimensional vectors. Let $A = I - L - U$ and $B = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{n-1}, 0)$ with $\beta_i \geqslant 0, 1 \leqslant i \leqslant n-1$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular matrices of A, respectively. We propose here a preconditioned iterative method with $P = I + BU$, $M_p = I - B\bar{D} - L - B\bar{E}$ and $N_p = U - BU + BU^2 + B\bar{F}$, where $\bar{D}$, $\bar{E}$ and $\bar{F}$ are the diagonal, strictly lower and strictly

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upper triangular parts of $UL = \bar{D} + \bar{E} + \bar{F}$, respectively. If $\beta_1 = \cdots = \beta_{n-1} = \beta$, then the method reduces to the one discussed in a previous paper [3]. The purpose of this paper is to prove a convergence theorem for the method for the case where A is an H-matrix.

We first recall the following: A real vector $x = (x_1, \ldots, x_n)^T$ is called nonnegative (positive) and denoted by $x \geqslant 0$ ($x > 0$), if $x_i \geqslant 0$ ($x_i > 0$) for all i. Similarly, a matrix $A = (a_{ij})$ is called nonnegative and denoted by $A \geqslant O$, if $a_{ij} \geqslant 0$ for all i,j.

Definition 1.1. An $n \times n$ matrix A is an M-matrix, if $a_{ii} > 0$ for all i, $a_{ij} \leqslant 0$ for $i \neq j$ and $A^{-1} \geqslant O$.

Definition 1.2. An $n \times n$ matrix A is an H-matrix, if its comparison matrix $\langle A \rangle = (\alpha_{ij})$ is an M-matrix, where α_{ij} is

$$\alpha_{ii} = |a_{ii}|, \qquad \alpha_{ij} = -|a_{ij}|,\ i \neq j.$$

Definition 1.3 (Frommer [2]). The splitting $A = M - N$ is called H-splitting if $\langle M \rangle - |N|$ is an M-matrix.

Then the following results are known:

Theorem 1.4 (Fan [1]). *Let A have nonpositive off-diagonal entries. Then a real matrix A is M-matrix if and only if there exists some vector $u = (u_1, \ldots, u_n)^T > 0$ such that $Au > 0$.*

Theorem 1.5 (Frommer [2]). *Let $A = M - N$ be a splitting. If it is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leqslant \rho(\langle M \rangle^{-1}|N|) < 1$.*

2. A convergence theorem

Lemma 2.1. *Let A be a real matrix with unit diagonal elements. If there exists an integer $l > i$ such that $|a_{il}| > 0$ for each $i < n$, then $\sum_{j=1}^{n} \sum_{k=i+1}^{n} |a_{ik} a_{kj}| \neq 0$.*

Proof. If there exists an integer $l > i$ such that $|a_{il}| > 0$ for each $i < n$, then we have for some $l > i$ and each $i < n$

$$\sum_{j=1}^{n} \sum_{k=i+1}^{n} |a_{ik} a_{kj}| = |a_{il}| + |a_{il}| \sum_{j=1, j \neq l}^{n} |a_{lj}| + \sum_{\substack{k=i+1 \\ k \neq l}}^{n} |a_{ik}| \sum_{j=1}^{n} |a_{kj}| \neq 0. \qquad \square$$

Theorem 2.2. *Let A be an H-matrix with unit diagonal elements, $A_B = (I + BU)A = M_B - N_B$, $M_B = I - B\bar{D} - L - B\bar{E}$ and $N_B = U - BU + BU^2 + B\bar{F}$. Let $u = (u_1, \ldots, u_n)^T$ be a positive vector such that $\langle A \rangle u > 0$. Assume that there exists an integer $l > i$ such that $|a_{il}| > 0$ for each $i < n$ and put*

$$\beta_i' = \frac{u_i - \sum_{j=1}^{i-1} |a_{ij}| u_j + \sum_{j=i+1}^{n} |a_{ij}| u_j}{\sum_{j=1}^{n} \sum_{k=i+1}^{n} |a_{ik} a_{kj}| u_j}.$$

Then $\beta_i' > 1$, for $i < n$ and for $0 \leqslant \beta_i < \beta_i'$ $(i < n)$, the splitting $A_B = M_B - N_B$ is an H-splitting and $\rho(M_B^{-1}N_B) < 1$ so that the iteration (2) *converges to the solution of* (1).

Proof. By assumption, the vector $u > 0$ satisfies

$$u_i - \sum_{j=1, j\neq i}^{n} |a_{ij}|u_j > 0 \quad \text{for all } i.$$

Therefore, we have

$$u_i - \sum_{j=1}^{i-1} |a_{ij}|u_j + \sum_{j=i+1}^{n} |a_{ij}|u_j - \sum_{j=1}^{n}\sum_{k=i+1}^{n} |a_{ik}a_{kj}|u_j$$

$$= u_i - \sum_{j=1, j\neq i}^{n} |a_{ij}|u_j + \sum_{k=i+1}^{n} |a_{ik}| \left\{ u_k - \sum_{j=1, j\neq k}^{n} |a_{kj}|u_j \right\} > 0 \quad \text{for } i < n.$$

From Lemma 2.1, we obtain

$$u_i - \sum_{j=1}^{i-1} |a_{ij}|u_j + \sum_{j=i+1}^{n} |a_{ij}|u_j > \sum_{j=1}^{n}\sum_{k=i+1}^{n} |a_{ik}a_{kj}|u_j > 0 \quad \text{for } i < n.$$

This implies

$$\frac{u_i - \sum_{j=1}^{i-1} |a_{ij}|u_j + \sum_{j=i+1}^{n} |a_{ij}|u_j}{\sum_{j=1}^{n}\sum_{k=i+1}^{n} |a_{ik}a_{kj}|u_j} > 1 \quad \text{for } i < n.$$

Hence, $\beta_i' > 1$ for $i < n$.

Let $\{(\langle M_B\rangle - |N_B|)u\}_i$ be the ith element in the vector $(\langle M_B\rangle - |N_B|)u$ for $i < n$. Then we obtain for $i < n$

$$\{(\langle M_B\rangle - |N_B|u\}_i$$

$$= |1 - \beta_i \sum_{k=i+1}^{n} a_{ik}a_{ki}|u_i - \sum_{j=1}^{i-1} |a_{ij} - \beta_i \sum_{k=i+1}^{n} a_{ik}a_{kj}|u_j$$

$$- \sum_{j=i+1}^{n} |a_{ij} - \beta_i \sum_{k=i+1}^{n} a_{ik}a_{kj}|u_j$$

$$\geqslant u_i - \beta_i \sum_{k=i+1}^{n} |a_{ik}a_{ki}|u_i - \sum_{j=1}^{i-1} |a_{ij}|u_j - \beta_i \sum_{j=1}^{i-1}\sum_{k=i+1}^{n} |a_{ik}a_{kj}|u_j$$

$$- \sum_{j=i+1}^{n} |(1-\beta_i)a_{ij}|u_j - \beta_i \sum_{j=i+1}^{n}\sum_{k=i+1, k\neq j}^{n} |a_{ik}a_{kj}|u_j,$$

and

$$\{(\langle M_B\rangle - |N_B|)u\}_n = u_n - \sum_{j=1, j\neq i}^{n} |a_{nj}|u_j > 0.$$

If $0 \leqslant \beta_i \leqslant 1$ for $i < n$, then we have

$$\begin{aligned}
&\{(\langle M_B\rangle - |N_B|)u\}_i \\
&\geqslant u_i - \sum_{j=1, j\neq i}^{n} |a_{ij}|u_j + \beta_i \sum_{j=i+1}^{n} |a_{ij}|u_j - \beta_i \sum_{j=1}^{n} \sum_{k=i+1, k\neq j}^{n} |a_{ik}a_{kj}|u_j \\
&= u_i - \sum_{j=1, j\neq i}^{n} |a_{ij}|u_j + \beta_i \sum_{k=i+1}^{n} |a_{ik}| \left\{ u_k - \sum_{j=1, j\neq k}^{n} |a_{kj}|u_j \right\} > 0.
\end{aligned}$$

Furthermore, if $1 < \beta_i < \beta_i'$ for $i < n$, then we obtain

$$\{(\langle M_B\rangle - |N_B|u\}_i \geqslant u_i - \sum_{j=1}^{i-1} |a_{ij}|u_j + \sum_{j=i+1}^{n} |a_{ij}|u_j - \beta_i \sum_{j=1}^{n} \sum_{k=i+1}^{n} |a_{ik}a_{kj}|u_j > 0.$$

Therefore, by Theorem 1.4, $\langle M_B\rangle - |N_B|$ is an M-matrix for $0 \leqslant \beta_i < \beta_i'$ $(i < n)$. That is, $A_B = M_B - N_B$ is an H-splitting for $0 \leqslant \beta_i < \beta_i'$ $(i < n)$. Hence, an application of Theorem 1.5 yields $\rho(M_B^{-1}N_B) < 1$ for $0 \leqslant \beta_i < \beta_i'$ $(i < n)$. □

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