Inverse spectral problems for Bessel operators

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Abstract

We study the inverse spectral problem for a class of Bessel operators given in $L_2(0, 1)$ by the differential expression

$$-\left(\frac{d}{dx} - \frac{\kappa}{x} - v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} + v\right)$$

with $\kappa \in \mathbb{N}$ and a real-valued function $v \in L_p(0, 1)$, $p \in [1, \infty)$, subject to various boundary conditions. We describe completely the spectral data of these operators, i.e., the spectra and corresponding norming constants, and give the algorithm of reconstruction of $v$ from the spectral data.

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1. Introduction

The main aim of the paper is to solve the inverse spectral problem for a class of general Bessel operators on the interval \([0, 1]\) generated by the differential expression (1.3) below. Such operators naturally appear in the spectral analysis of the radial Schrödinger operators \(-\Delta + q(|x|)\) acting on a unit ball of \(\mathbb{R}^3\). Indeed, it is well known that the latter can be decomposed in spherical coordinates into the direct sum of operators \(T_\kappa\) parametrized by the angular momentum \(\kappa \in \mathbb{Z}_+\) and given by

\[
T_\kappa = -\frac{d^2}{dx^2} + \frac{\kappa(\kappa + 1)}{x^2} + q(x).
\]

For real-valued \(q \in L_2(0, 1)\) and \(\kappa \geq 1\), the operator \(T_\kappa\) becomes self-adjoint under the boundary condition

\[
y(1) \cos \theta = y'(1) \sin \theta, \quad \theta \in [0, \pi),
\]

and is then denoted by \(T(\theta, \kappa, q)\). This operator is usually called the Bessel operator since for \(q \equiv 0\) its eigenfunctions are expressed in terms of Bessel functions, see [32]. When \(\kappa = 0\), \(T(\theta, 0, q)\) is given by a regular Sturm–Liouville differential expression, and another boundary condition is needed to get a self-adjoint operator; for \(\eta \in [0, \pi)\), we denote by \(T_\eta(\theta, 0, q)\) the restriction of \(T(\theta, 0, q)\) by the boundary condition

\[
y(0) \cos \eta = y'(0) \sin \eta.
\]

The operator \(T(\theta, \kappa, q)\) is bounded below and has a simple discrete spectrum. It was shown in [7] that for \(q \in L_2(0, 1)\) the eigenvalues \(\lambda_n^2(\theta, \kappa, q)\) of \(T(\theta, \kappa, q)\), when arranged in increasing order, have the following asymptotics:

\[
\lambda_n^2(0, \kappa, q) = \pi^2 \left( n + \frac{\kappa}{2} \right)^2 + s_0 + r_n(0), \quad (1.1)
\]

\[
\lambda_n^2(\theta, \kappa, q) = \pi^2 \left( n + \frac{\kappa - 1}{2} \right)^2 + s_\theta + r_n(\theta), \quad \theta \neq 0, \quad (1.2)
\]

where \(s_\theta\) are real constants (in particular, \(s_0 = \int_0^1 q \, dx - \kappa(\kappa + 1)\)) and the sequences \((r_n(\theta))\) belong to \(\ell_2(\mathbb{N})\). We observe that the nonzero momentum \(\kappa\) shifts the main asymptotic term of \(\lambda_n\) by \(\kappa/2\), while the remainders are as for the case \(\kappa = 0\). Loosely speaking, the operator \(T(\theta, \kappa, q)\) has \(\kappa/2\) eigenvalues less than a regular Sturm–Liouville operator.

For the case \(\kappa = 0\), there exists a detailed inverse spectral theory, cf. [15,22–25]. In particular, Pöschel and Trubowitz in [25] studied in detail the mapping sending the potential \(q\) into the corresponding spectral data—the sequences of eigenvalues and the corresponding norming constants—and proved that this mapping is one-to-one and analytic.

In [19] the method of [25] was extended to the case \(\kappa = 1\), allowing a complete description of the possible spectra of the operators \(T(0, 1, q)\) when \(q\) runs through \(L_2(\mathbb{R}, 0, 1)\), the real space of real-valued functions in \(L_2(0, 1)\), and the structure of the corresponding isospectral manifold. Recently, Serier [29] extended this analysis to the case of an arbitrary \(\kappa \in \mathbb{N}\).
The spectra of the operators $T(\theta, \kappa, q)$ for an arbitrary $\kappa \in \mathbb{N}$ and $\theta \in [0, \pi)$ were also completely characterized in [7], see relations (1.1) and (1.2) above, using the Crum–Darboux, or “commutators of factors” transformation, which allowed to study the properties of the corresponding isospectral sets. Developing the ideas of [25], in [8] the author proved several results on the unique reconstruction of $T(\theta, \kappa, q)$ from the spectral data for all real $\kappa \geq -\frac{1}{2}$ (but without characterizing the spectral data). Some uniqueness results are also given in [31].

We observe that earlier Gasymov announced the solution of the inverse spectral problem for the Bessel operators $T(\theta, \kappa, q)$ with $\kappa \in \mathbb{N}$ and $q \in L_{2, \mathbb{R}}(0, 1)$ in the short communication [14], but has not given complete proofs afterwards. His approach was based on the double commutation method, allowing reduction of the inverse problem to the classical case $\kappa = 0$.

In quantum mechanics there are many physical models which lead to radial Schrödinger operators with potentials $q$ that are not integrable at the origin. For instance, the model for the hydrogen atom [6] uses potentials having the $1/|x|$-type Coulomb singularity. For such cases the above-mentioned analysis does not apply and there is little information about the spectral properties of the corresponding operators. Moreover, even the definition of a Bessel operator in this case requires some effort.

That was the motivation for us to consider the class of more general Bessel operators on the unit interval. Namely, for a real-valued $v \in L_p(0, 1)$, $p \in [1, \infty)$, and $\kappa \in \mathbb{Z}_+$ we consider a differential expression

$$\ell(\kappa, v)y := -\left(\frac{d}{dx} - \frac{\kappa}{x} - v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} + v\right)y$$

defined on functions $y$ that together with their quasi-derivatives $y^{[1]} := y' + (\frac{\kappa}{x} + v)y$ are absolutely continuous on $[\varepsilon, 1]$ for every $\varepsilon \in (0, 1)$ and, for $\theta \in [0, \pi)$, introduce an operator $S(\theta, \kappa, v)$ as the restriction of $\ell(\kappa, v)$ onto the domain

$$\text{dom } S(\theta, \kappa, v) := \left\{ y \in \text{dom } \ell(\kappa, v) \cap L_2(0, 1) \mid \ell(\kappa, v)y \in L_2(0, 1), \ y(1) \cos \theta = y^{[1]}(1) \sin \theta \right\}.$$

If $\kappa = 0$, then the point $x = 0$ is in the limit circle case, and a boundary condition is required to make $S(\theta, 0, v)$ self-adjoint. In this case we impose the additional boundary condition

$$y(0) \cos \eta = y^{[1]}(0) \sin \eta, \quad \eta \in [0, \pi).$$

and denote by $S_\eta(\theta, 0, v)$ the corresponding restriction of $S(\theta, 0, v)$.

If $v \in L_2(0, 1)$, then in the distributional sense $\ell(\kappa, v)$ is the Bessel differential expression

$$\ell(\kappa, v)y = -y''(x) + \frac{\kappa(\kappa + 1)}{x^2}y + qy$$

with $q := -v' + v^2 + 2kv/x$; in particular, for $v$ a nonzero constant we get a Coulomb-type singularity as in the hydrogen atom model. However, this $q$ need not be a distribution if $v \notin L_2(0, 1)$ and the above representation in the potential form is then useless. On the other hand, for any $\theta_0 \in [0, \pi)$ and any $q \in L_2(0, 1)$ the operator $T(\theta_0, \kappa, q)$ coincides with $S(\theta, \kappa, v) - cI$ for $I$ the identity operator and for some $\theta \in [0, \pi)$, $v \in L_2(0, 1)$, and $c \in \mathbb{R}$; see Corollary 2.10. Therefore the class of Bessel operators we treat in this paper is wider than the one which has
been studied before. Also our results generalize those known for the Bessel operators $T(\theta, \kappa, q)$ with $q \in L_2(0, 1)$, see Remark 2.4.

We observe that in the case $\kappa = 0$ the operator $S_\eta(\theta, 0, v)$ (recall that the subscript $\eta$ specifies the boundary condition (1.4) at $x = 0$ to make the operator self-adjoint) is unitarily equivalent to a Sturm–Liouville operator written in the impedance form. Such impedance Sturm–Liouville operators have been extensively studied for $v \in L^2(0, 1)$; in particular, a detailed direct and inverse spectral analysis was carried out in [4,5,10,27]; see also [3] for $v \in L^p(0, 1)$ and [12,13] for the case with a singular $v$.

In this paper we present a complete solution of the inverse spectral problem for the operator $S(\theta, \kappa, v)$ under the blanket hypothesis that the angular momentum $\kappa$ is a natural number and the impedance potential $v$ is a real-valued function belonging to $L^p(0, 1)$, $p \in [1, \infty)$. Namely, we give an explicit description of the spectral data and find the algorithm reconstructing $v$ from the spectrum and the corresponding norming constants. For natural $\kappa$, these results extend the uniqueness theorems of [8,29].

We define the norming constants $\alpha_n = \alpha_n(\theta, \kappa, v)$ corresponding to the eigenvalues $\lambda^2_n = \lambda^2_n(\theta, \kappa, v)$ of the operator $S(\theta, \kappa, v)$ as

$$\alpha_n := \|y(\cdot, \lambda_n)\|^2,$$

where $y(\cdot, \lambda)$ is a solution of the equation $\ell(\kappa, v)y = \lambda^2y$ satisfying the terminal conditions

$$y(1) = 0,$$

$$y^{[1]}(1) = \lambda, \quad \theta = 0,$$

$$y(1) = \sin \theta, \quad y^{[1]}(1) = \cos \theta, \quad \theta \neq 0.$$  

We notice that for $\theta = 0$ this definition gives $y(\cdot, 0) \equiv 0$; however, $\lambda^2 = 0$ will never be an eigenvalue for the Dirichlet boundary condition at $x = 1$ and we shall never use $y(\cdot, 0)$ in this case, so that this flaw in the definition is inessential; cf. Remark 3.3.

The asymptotics of $\lambda^2_n$ and $\alpha_n$ is given in the following theorem. We denote by $s_n(f)$ and $c_n(f)$ the $n$th sine and cosine Fourier coefficients of a function $f \in L^1(0, 1)$, viz.

$$s_n(f) := \int_0^1 f(t) \sin \pi nt \, dt, \quad c_n(f) := \int_0^1 f(t) \cos \pi nt \, dt.$$  

**Theorem 1.1.** Let $\theta \in [0, \pi)$, $\kappa \in \mathbb{N}$, and $v \in L_p(0, 1)$. Then there exist functions $g_1$ and $g_2$ in $L_p(0, 1)$ such that the eigenvalues $\lambda^2_n$ and the corresponding norming constants $\alpha_n$ of the operator $S(\theta, \kappa, v)$ satisfy the relations

$$\lambda_n = \pi \left(n + \frac{\kappa}{2}\right) + s_{2n+\kappa}(g_1), \quad \alpha_n = 2 + c_{2n+\kappa}(g_2), \quad \theta = 0,$$  

$$\lambda_n = \pi \left(n + \frac{\kappa - 1}{2}\right) + s_{2n+\kappa-1}(g_1), \quad \alpha_n = 2 + c_{2n+\kappa-1}(g_2), \quad \theta > 0.$$  

We next show that the above characterization of the spectral data is complete, i.e., that any sequences of $\lambda^2_n$ and $\alpha_n$ as in Theorem 1.1 are indeed sequences of eigenvalues and norming constants for some Bessel operator of the class considered. We note, however, that for $p > 1$ the
sets of even and odd sine (and cosine) Fourier coefficients of functions in $L_p(0, 1)$ coincide, so that, given the spectral data, it may not be possible to distinguish between the cases $\theta = 0$ with angular momentum $\kappa$ and $\theta > 0$ with angular momentum $\kappa + 1$. Therefore we have to specify the type of the boundary condition at $x = 1$, i.e., whether $\theta = 0$ or $\theta > 0$, if we want a Bessel operator with the specified spectral data to be unique.

Next, there is an intrinsic nonuniqueness in the representation of the Bessel operators $S(\theta, \kappa, v)$, known also in the regular case $\kappa = 0$, cf. [27]. Namely, for any $\theta_0 \in (0, \pi)$ and $v_0 \in L_p(0, 1)$ there exist an interval $(a, b) \subset (0, \pi)$ and a family of $v_0 \in L_p(0, 1)$ corresponding to $\theta \in (a, b)$ such that $v_{\theta_0} = v_0$ and all the operators $S(\theta, \kappa, v_0), \theta \in (a, b)$, coincide. However, such a representation is uniquely determined by specifying the value of $\theta > 0$, and we shall usually take the simplest case $\theta = \pi/2$, see Section 2. Observe that then the operator $S(\pi/2, \kappa, v)$ is positive, and thus such are its eigenvalues. As we shall show in Section 2, the assumption that the operator $S(\theta, \kappa, v)$ is positive is not restrictive and can be made without loss of generality.

A similar situation is for $\theta = 0$, i.e., for the Dirichlet boundary condition at $x = 1$. Here too one has a family of impedance potentials $v_m, m \in \mathbb{R}$, such that $\int_0^1 v_m \, dx = m$ and all $S(0, \kappa, v_0), m \in \mathbb{R}$, coincide. However, specification of the mean value of the impedance potential $v$ determines it uniquely among all equivalent representations.

**Theorem 1.2.** Assume that sequences $(\lambda_n^2)_{n \in \mathbb{N}}$ and $(\alpha_n)_{n \in \mathbb{N}}$ of positive numbers are such that $\lambda_n^2$ strictly increase with $n$ and (1.8) holds. Then there exists a unique real-valued $v \in L_2(0, 1)$ such that $\lambda_n^2$ and $\alpha_n$ are respectively the eigenvalues and the norming constants of the Bessel operator $S(\pi/2, \kappa, v)$.

If the asymptotics (1.7) holds instead of (1.8), then the sequences $(\lambda_n^2)$ and $(\alpha_n)$ form the spectral data for a unique Bessel operator $S(0, \kappa, v)$. Moreover, the potential $v$ is uniquely fixed by specifying its mean value.

The main idea behind our construction is, like in [7,9,14], to reduce the case $\kappa \in \mathbb{N}$ to the case $\kappa = 0$. However, while the above authors used the Crum–Darboux transformation (earlier suggested in, e.g., [1, Chapter VI.1], [11, Chapter 13], and [23, Problem 2.3.6]), we apply the double Crum–Darboux, or double commutation method of creating or removing bound states well known in the scattering theory [16,17]. The transformations of Theorems 4.1 and 4.3 change $\kappa$ by 2 and insert or remove one eigenvalue but affect neither the other eigenvalues nor the corresponding norming constants. This makes it possible to establish the asymptotics of the spectral data as well as to present the reconstruction algorithm.

We also mention another type of inverse spectral problem for Bessel operators (or for the corresponding Schrödinger operators), namely that of the determination of the potential $q$ from the spectra of $T(0, \kappa, q)$ for two different integer $\kappa$, say $\kappa_1$ and $\kappa_2$. In [30] the cases $\kappa_1 = 0$ and $\kappa_2 = 1$ were considered, and it was proved there that the intersection of the corresponding isospectral manifolds is locally of finite dimension. A similar statement was proved in [9] for a more general case when $\kappa_2 - \kappa_1$ is odd. In [28] the authors gave additional evidence for uniqueness when $q$ is small in $L_2(0, 1)$ and $\kappa_1, \kappa_2 = 0, 1, 2, 3$.

The paper is organized as follows. In Section 2 we study some properties of the differential expressions $\ell(\kappa, v)$ and the operators $S(\theta, \kappa, v)$. In particular, we explain that it suffices to restrict ourselves to $\theta = 0$ or $\theta = \pi/2$, and that adding to $\ell(\kappa, v)$ a real-valued potential $q \in L_1(0, 1)$ does not lead to more general differential expressions. The double commutation transformation is introduced in Section 3 and some of its properties are established therein. The effect of such transformations on the spectral data is explained in Section 4. Finally, in Section 5 we prove our
We say that a function $f$ is locally absolutely continuous on $(0, 1)$ (and write $f \in AC(0, 1)$) if $f$ is absolutely continuous on the interval $[\varepsilon, 1]$ for every $\varepsilon \in (0, 1)$. We shall write $f(x) \sim g(x)$ as $x \to 0$ if $f(x)/g(x)$ tends to a finite nonzero limit as $x \to 0$.

**Notations.** We say that a function $f$ is locally absolutely continuous on $(0, 1)$ (and write $f \in AC(0, 1)$) if $f$ is absolutely continuous on the interval $[\varepsilon, 1]$ for every $\varepsilon \in (0, 1)$. We shall write $f(x) \sim g(x)$ as $x \to 0$ if $f(x)/g(x)$ tends to a finite nonzero limit as $x \to 0$.

**2. Preliminaries**

2.1. Differential expressions

We recall that for an angular momentum $\kappa \in \mathbb{Z}_+$ and a real-valued function $\nu \in L_p(0, 1)$, $p \in [1, \infty)$, $\ell(\kappa, \nu)$ denotes the differential expression of (1.3) on domain $\ell(\kappa, \nu) = \{ f \in AC(0, 1) | f^{[1]} \in AC(0, 1) \}$. The definition of $\ell(\kappa, \nu)$ suggests that the differential equation $\ell(\kappa, \nu)y = \lambda^2y + f$ with $\lambda^2 \in \mathbb{C}$ and $f \in L_1(0, 1)$ should be interpreted as a first order system

\[
\frac{d}{dx} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{\kappa}{x} + v \\ \lambda^2 \\ -1 \\ -\kappa - v \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -f \end{pmatrix}
\]  

(2.1)

for $u_1 = y$ and $u_2 = y^{[1]} := y' + (\frac{\kappa}{x} + v)y$. For every complex numbers $c_1$ and $c_2$ this system possesses a unique solution on $(0, 1]$ satisfying the terminal conditions $u_1(1) = c_1$ and $u_2(1) = c_2$.

It follows that for every $f \in L_1(0, 1)$ and every $\lambda, c_1, c_2 \in \mathbb{C}$ the equation $\ell(\kappa, \nu)y = \lambda^2y + f$ has a unique solution satisfying the conditions $y(1) = c_1$ and $y^{[1]}(1) = c_2$; moreover, both $y$ and $y^{[1]}$ are locally absolutely continuous on $(0, 1]$ and $y \in W^1_p(\varepsilon, 1)$ for every $\varepsilon \in (0, 1)$.

Next we take a real-valued $q \in L_1(0, 1)$ and observe that the differential expression $\ell(\kappa, \nu) + q$ is well defined on domain $\ell(\kappa, \nu)$. Our aim is to show that under some conditions (that will turn out to be rather nonrestrictive) the differential expression $\ell(\kappa, \nu) + q$ coincides with $\ell(\kappa, \tilde{v})$ for some $\tilde{v} \in L_p(0, 1)$, so that adding $q$ to $\ell(\kappa, \nu)$ does not lead to a more general differential expression. We start with the following simple observation.

**Lemma 2.1.** Let $u$ be a solution of the equation $\ell(\kappa, \nu)y + qy = 0$ that is positive on $(0, 1]$. Then the differential expressions

\[
\ell(\kappa, \nu) + q \quad \text{and} \quad -\left( \frac{d}{dx} + \frac{u'}{u} \right) \left( \frac{d}{dx} - \frac{u'}{u} \right) := \tilde{\ell}
\]

coincide, i.e., dom $\ell(\kappa, \nu) = \text{dom} \tilde{\ell}$ and $\ell(\kappa, \nu)f + qf = \tilde{\ell}f$ for every $f \in \text{dom} \ell(\kappa, \nu)$.

We observe that, since $u \in W^1_p(\varepsilon, 1)$ for every $\varepsilon \in (0, 1)$, a natural domain for $\tilde{\ell}$ is

\[
\text{dom} \tilde{\ell} := \left\{ y \in AC(0, 1) \mid y' - \frac{u'}{u} y \in AC(0, 1) \right\}.
\]

**Proof.** Since $u$ is a solution of the equation $\ell(\kappa, \nu)y + qy = 0$, the functions $u$ and $u^{[1]} := u' + (\frac{\kappa}{x} + v)u$ are locally absolutely continuous on $(0, 1]$. The equality

\[
f^{[1]} = f' - \frac{u'}{u} f + \frac{u^{[1]}}{u} f
\]
shows that for a function $f \in AC(0, 1]$ the quasi-derivative $f^{[1]}$ and $f' - \frac{u'}{u} f$ are locally absolutely continuous on $(0, 1]$ or not simultaneously, and therefore the domains of both differential expressions under consideration coincide. Moreover, a straightforward calculation gives

$$\left(-\frac{d}{dx} + \frac{u'}{u}\right)\left(\frac{d}{dx} - \frac{u'}{u}\right) = -\left(\frac{d}{dx} - \frac{\kappa}{x} - v + \frac{u^{[1]}}{u}\right)\left(\frac{d}{dx} + \frac{\kappa}{x} + v - \frac{u^{[1]}}{u}\right)$$

$$= \ell(\kappa, v) - \frac{u^{[1]}}{u}\left(\frac{d}{dx} - \frac{u'}{u}\right) + \left(\frac{d}{dx} - \frac{\kappa}{x} - v\right)\frac{u^{[1]}}{u}$$

$$= \ell(\kappa, v) - \frac{1}{u}\ell(\kappa, v)u = \ell(\kappa, v) + q,$$

and the proof is complete. □

In fact, the differential expression $\tilde{\ell}$ equals $\ell(\kappa, \hat{v})$ for some $\hat{v} \in L_p(0, 1)$. To show this, we have to study the behavior of the solution $u$ of the above lemma at $x = 0$.

**Lemma 2.2.** Assume that $\kappa \in \mathbb{N}$, $v \in L_p(0, 1)$, and $q \in L_1(0, 1)$. Then the equation

$$\ell(\kappa, v)y + qy = 0 \quad (2.2)$$

has two solutions of the form $x^{-\kappa}u_1(x)$ and $x^{\kappa+1}u_2(x)$ respectively, with some functions $u_1, u_2 \in W^1_p(0, 1)$ that do not vanish at $x = 0$.

**Proof.** Rewriting the above equation as a first order system, we conclude that any solution of (2.2) belongs to $W^1_p(\varepsilon, 1)$ for every $\varepsilon \in (0, 1)$. Therefore it suffices to find two solutions of (2.2) that have the required representation in $(0, \varepsilon)$ for some $\varepsilon > 0$.

We shall prove below that there are $\varepsilon > 0$ and $\tilde{v} \in L_p(0, \varepsilon)$ such that the differential expressions

$$\ell(\kappa, v) + q \quad \text{and} \quad \ell(\kappa, \tilde{v})$$

coincide on $(0, \varepsilon)$. Straightforward calculations then show that the functions

$$y_1(x) := x^{-\kappa}\exp\left(-\int_0^x \tilde{v}(t) \, dt\right)$$

and

$$y_2(x) := y_1(x) \int_0^x \frac{dt}{y_1^2(t)}$$

are solutions of the equation $\ell(\kappa, \tilde{v})y = 0$. We clearly have $y_1(x) = x^{-\kappa}u_1(x)$ with $u_1 \in W^1_p(0, \varepsilon)$ and $u_1(0) = 1$; by Lemma A.3 also $y_2(x) = x^{\kappa+1}u_2(x)$ with $u_2 \in W^1_p(0, \varepsilon)$ and $u_2(0) = 1/(2\kappa + 1)$.
We seek for the function \( \tilde{v} \) in the form \( v + w \) with \( w \in AC[0, \varepsilon] \). Writing \( \ell(\kappa, v + w) \) as

\[
\ell(\kappa, v + w) = \ell(\kappa, v) + w \left( \frac{d}{dx} + \frac{\kappa}{x} + v + w \right) - \left( \frac{d}{dx} - \frac{\kappa}{x} - v \right) w
\]

we see that \( w \) should satisfy the equality

\[
-w' + 2 \left( \frac{\kappa}{x} + v \right) w + w^2 = q. \tag{2.3}
\]

Since the equation

\[
-y' + 2 \left( \frac{\kappa}{x} + v \right) y = r
\]

with \( r \in L_1(0, \varepsilon) \) has a solution

\[
y(x) = \int_x^\varepsilon \left( \frac{x}{t} \right)^{2\kappa} \exp \left( 2 \int_t^x v(s) \, ds \right) r(t) \, dt,
\]

it suffices to construct \( w \) as a solution of the equation

\[
y(x) = \int_x^\varepsilon \left( \frac{x}{t} \right)^{2\kappa} \exp \left( 2 \int_t^x v(s) \, ds \right) [q(t) - y^2(t)] \, dt. \tag{2.4}
\]

Considering the right-hand side of (2.4) as a nonlinear mapping \( R \) in the space \( L_\infty := L_\infty(0, \varepsilon) \), we see that

\[
\| R(0) \|_{L_\infty} \leq \exp \left\{ 2 \| v \|_{L_1(0, \varepsilon)} \right\} \| q \|_{L_1(0, \varepsilon)} \tag{2.5}
\]

and

\[
\| R(y_1) - R(y_2) \|_{L_\infty} \leq \varepsilon \exp \left\{ 2 \| v \|_{L_1(0, \varepsilon)} \right\} \left( \| y_1 \|_{L_\infty} + \| y_2 \|_{L_\infty} \right) \| y_1 - y_2 \|_{L_\infty}. \tag{2.6}
\]

Choosing \( \varepsilon \in (0, 1) \) so that

\[
\varepsilon \exp \left\{ 2 \| v \|_{L_1(0, \varepsilon)} \right\} < \frac{1}{4} \quad \text{and} \quad \exp \left\{ 2 \| v \|_{L_1(0, \varepsilon)} \right\} \| q \|_{L_1(0, \varepsilon)} < \frac{1}{2},
\]

we conclude from (2.5) and (2.6) that \( R \) is a contraction of the closed unit ball \( B_1 \) of \( L_\infty \). Therefore the equation \( y = R(y) \) has a unique solution in \( B_1 \), which we take as \( w \). By Lemma A.2 we have \( w(x)/x \in L_1(0, \varepsilon) \), whence (2.3) yields \( w' \in L_1(0, \varepsilon) \), i.e., \( w \in AC[0, \varepsilon] \). The proof is complete. \( \square \)
Corollary 2.3. Assume that \( \kappa \in \mathbb{N} \), \( v \in L_p(0,1) \), and \( q \in L_1(0,1) \), and let \( u \) be positive on \( (0,1] \) and satisfy the equality \( \ell(\kappa, v)u + qu = 0 \). Then there exists \( \hat{v} \in L_p(0,1) \) such that \( \ell(\kappa, \hat{v}) = \ell(\kappa, v) + q \).

Proof. According to Lemma 2.2, the solution \( u \) has either the form \( x^{\kappa+1}u_1 \) or \( x^{-\kappa}u_2 \) for some \( u_1, u_2 \in W^1_p(0,1) \) that are positive on \([0,1] \). In the latter case

\[
\frac{u'}{u} = -\frac{\kappa}{x} + \frac{u'_2}{u_2},
\]

and the claim follows from Lemma 2.1 with \( \hat{v} := -u'_2/u_2 \). In the former case we have

\[
\frac{u'}{u} = \frac{\kappa + 1}{x} - \hat{v}
\]

with \( \hat{v} := -u'_1/u_1 \in L_p(0,1) \), and \( \ell(\kappa, v) + q = \ell(-\kappa - 1, \hat{v}) \) by Lemma 2.1. It remains to observe that by Lemma A.1 there exists \( \hat{v} \in L_p(0,1) \) such that \( \ell(-\kappa - 1, \hat{v}) = \ell(\kappa, \hat{v}) \).

Remark 2.4. If \( v = 0 \) and \( q \in L_2(0,1) \), then the solution \( w \) of (2.3) constructed in the proof of Lemma 2.2 belongs to \( W^1_2(0,\varepsilon) \), and \( \ell(\kappa, 0) + q = \ell(\kappa, w) \). All the subsequent analysis can be adapted to this situation and will produce results that agree with those for the operator \( T(\theta, \kappa, q) \) obtained earlier in [7,8,14,19,29,31].

2.2. Nonuniqueness of the representation of \( \ell(\kappa, v) \)

We observe first that

\[ \ell(\kappa, v) = \ell(\kappa, \hat{v}) \]

if and only if \( x^{-\kappa} \exp(-\int_0^x \hat{v}(t)\,dt) \) is a solution of \( \ell(\kappa, v)y = 0 \). Therefore nonuniqueness of the representation of \( \ell(\kappa, v) \) depends on the existence of solutions of \( \ell(\kappa, v)y = 0 \) that are positive on \([0,1] \) and unbounded at \( x = 0 \).

All solutions of the equation \( \ell(\kappa, v)y = 0 \) are given by \( c_1y_1 + c_2y_2 \), with

\[
y_1(x) := x^{-\kappa} \exp\left(-\int_0^x v(t)\,dt\right) \quad \text{and} \quad y_2(x) := y_1(x) \int_0^x \frac{dt}{y_1^2(t)}, \quad (2.7)
\]

and \( c_1, c_2 \in \mathbb{C} \). Positive solutions must have \( c_1 > 0 \), so that, without loss of generality, we may take \( c_1 = 1 \), and then a generic positive solution that is unbounded at \( x = 0 \) equals \( y(x) = x^{-\kappa}u_c(x) \), with

\[
u_c(x) = \exp\left(-\int_0^x v(t)\,dt\right) \left(1 + c \int_0^x \frac{dt}{y_1^2(t)}\right), \quad c > c_0 := -\left(\int_0^x \frac{dt}{y_1^2(t)}\right)^{-1}. \quad (2.8)
\]
By Lemma 2.1 we have \( \ell(\kappa, v) = \ell(\kappa, -u'_c/u_c) \) for every \( c > c_0 \). The mean value of the new impedance potential \(-u'_c/u_c\) is equal to

\[
- \log u_c(1) = \int_0^1 v(t) \, dt - \log \left( 1 + c \int_0^1 \frac{dt}{y_1^2(t)} \right).
\]

It is clear that \( u_c(1) \) is a monotone function of \( c \) and that \( u_c(1) \) takes all positive values as \( c \) runs through the interval \((c_0, \infty)\). We thus get the following

**Proposition 2.5.** Fix \( \kappa \in \mathbb{N} \), a real-valued \( v \in L_p(0, 1) \), and \( m \in \mathbb{R} \). Then among all \( \hat{v} \in L_p(0, 1) \) such that \( \ell(\kappa, v) = \ell(\kappa, \hat{v}) \) there is a unique one satisfying

\[
\int_1^0 \hat{v} dx = m.
\]

This \( \hat{v} \) equals \(-u'_c/u_c\), where \( u_c \) is given by (2.8) and \( c \) is determined from the equation \( u_c(1) = e^{-m} \).

Assume that a function \( f \in \text{dom} \ell(\kappa, v) \) satisfies the boundary condition \( f[1](1) \sin \theta_0 = f(1) \cos \theta_0 \) with \( \theta_0 \in [0, \pi) \) and let \( u_c \) be a function of (2.8). Denote by

\[
f'^{(1)}_c := f' + \left( \frac{\kappa}{x} - \frac{u'_c}{u_c} \right)
\]

the quasi-derivative corresponding to the representation \( \ell(\kappa, -u'_c/u_c) \) of \( \ell(\kappa, v) \); in particular, \( f'^{(1)}_0 = f^{(1)} \). It follows from (2.8) that

\[
f'^{(1)}_c = f'^{(1)}_0 - \frac{c/y_2^2(x)}{1 + c \int_0^1 dt/y_2^2(t)} f,
\]

and \( f \) also satisfies the boundary condition

\[
f'^{(1)}_c(1) \sin \theta_c = f(1) \cos \theta_c,
\]

with \( \theta_c \in [0, \pi) \) determined from the relation

\[
\cot \theta_c = \cot \theta_0 - \frac{c/y_2^2(1)}{1 + c \int_0^1 dt/y_2^2(t)}.
\]

Thus, for every \( c > c_0 \) we have

\[
S(\theta_0, \kappa, v) = S(\theta_0, \kappa, -u'_0/u_0) = S(\theta_c, \kappa, -u'_c/u_c).
\]

Two cases are possible: either \( \theta_c = 0 \) for some \( c > c_0 \) or \( \theta_c > 0 \) for all \( c > c_0 \). In the first case we conclude from (2.9) that \( \theta_c = 0 \) for all \( c > c_0 \); thus

\[
S(0, \kappa, v) = S(0, \kappa, -u'_c/u_c),
\]

and the distributional potential \(-u'_c/u_c\) can be fixed by its mean value, see Proposition 2.5.

In the second case \( \cot \theta_c \) covers the interval \((\cot \theta_0 + c_0/y_2^2(1), \infty)\). Since \( c_0/y_2^2(1) = -y_2^{(1)}(1)/y_2(1) \), it follows from Lemma 2.8 below that this interval contains zero if and only
if the operator $S(\theta_0, \kappa, v)$ is positive. This positivity condition can be assumed without loss of generality (if $S(\theta_0, \kappa, v)$ is not positive, then add a suitable constant $C$ to make it positive and apply Lemma 2.9 and Corollary 2.3 to $q \equiv C$), so that there is a unique representation of $S(\theta_0, \kappa, v)$ as $S(\pi/2, \kappa, -u_c'/u_c)$ with $c > c_0$ found from the relation (2.9) by requiring that $\theta_c = \pi/2$.

In conclusion, we have shown that it suffices to study the operators $S(\theta, \kappa, v)$ only with the Dirichlet ($\theta = 0$) or Neumann ($\theta = \pi/2$) boundary conditions at $x = 1$.

### 2.3. Operators

Finally, we study the properties of the operators $S(\theta, \kappa, v)$ corresponding to the differential expressions $\ell(\kappa, v)$. We start with showing self-adjointness of these operators.

**Lemma 2.6.** The operator $S(\theta, \kappa, v)$, $\kappa \in \mathbb{N}$, is self-adjoint and has a simple discrete spectrum.

**Proof.** Given $f \in L^2(0, 1)$, we find from the explicit form of the solutions of system (2.1) that all solutions of the equation $\ell(\kappa, v)y = f$ that belong to $L^2(0, 1)$ are of the form $y = cy_2 + Gf$, where $c \in \mathbb{C},$

$$(Gf)(x) := y_1(x) \int_0^x f(s)y_2(s) \, ds + y_2(x) \int_x^1 f(s)y_1(s) \, ds$$

is a particular solutions to the nonhomogeneous equation, and $y_1$ and $y_2$ of (2.7) are solutions of the homogeneous equation.

Properties of the functions $y_1$ and $y_2$ and direct calculations show that the function $Gf$ and its quasi-derivative are absolutely continuous on $[0, 1]$ and $(Gf)(0) = 0$. It follows that every $y$ in the domain of $S(\theta, \kappa, v)$ vanishes at $x = 0$ and its quasi-derivative $y^{[1]}$ is absolutely continuous on $[0, 1]$. Integration by parts in the scalar product $\langle S(\theta, \kappa, v)y, y \rangle_{L^2(0, 1)}$ now shows that $S(\theta, \kappa, v)$ is a symmetric operator.

Assume first that $\theta = 0$; since a solution of $\ell(\kappa, v)y = 0$ that is in $L^2(0, 1)$ has the form $cy_2$ for some $c \in \mathbb{C}$ and $y_2(1) > 0$, it follows that $\lambda = 0$ is not an eigenvalue of $S(0, \kappa, v)$. Thus the unique solution $y$ of $\ell(\kappa, v)y = f$ belonging to $L^2(0, 1)$ and satisfying the boundary condition $y(1) = 0$ is given by

$$y = R_0f := Gf - \frac{(Gf)(1)}{y_2(1)}y_2,$$

so that the operator $R_0$ is a compact (in fact, a Hilbert–Schmidt) inverse of $S(0, \kappa, v)$.

Now we see that the operators $S(\theta, \kappa, v)$ are proper symmetric extensions of the symmetric operator with deficiency indices $(1, 1)$, which is the restriction of $S(0, \kappa, v)$ to the functions whose quasi-derivatives vanish at $x = 1$. Therefore the operators $S(\theta, \kappa, v)$ are self-adjoint for every $\theta \in [0, \pi)$. The Krein resolvent formula [21] (see also [2, Appendix A]) shows that all these operators have compact resolvents, i.e., discrete spectra.

Finally, if there were two linearly independent eigenfunctions of $S(\theta, \kappa, v)$ corresponding to an eigenvalue $\lambda_0^2$, then the equation $\ell(\kappa, v)y = \lambda_0^2y$ would have a nontrivial solution satisfying the terminal conditions $y(1) = y^{[1]}(1) = 0$. This would contradict the uniqueness of solutions for this equation, so that all eigenvalues of the operator $S(\theta, \kappa, v)$ are simple. □
Remark 2.7. Assume that the operator \( S = S(\theta, \kappa, v) \) is invertible and introduce the boundary functional \( b_\theta(y) := y^{[1]}(1) \sin \theta - y(1) \cos \theta \) on \( \text{dom} \ell(\kappa, v) \). Then \( b_\theta(y_2) \neq 0 \), and the unique solution \( y \) of \( \ell(\kappa, v)y = f \) that satisfies the boundary condition \( b_\theta(y) = 0 \) is given by

\[
y = R_\theta f := Gf - \frac{b_\theta(Gf)}{b_\theta(y_2)} y_2,
\]

(cf. (2.10)). In particular, \( b_{\pi/2}(Gf) = 0 \), so that the operator \( G = R_{\pi/2} \) is the inverse of \( S(\pi/2, \kappa, v) \).

Further dependence of \( S(\theta, \kappa, v) \) on \( \theta \in (0, \pi) \) can be derived from the properties of its quadratic form

\[
s_\theta[f] := \langle S(\theta, \kappa, v)f, f \rangle_{L^2(0,1)} = \frac{1}{1}\int_0^1 |f^{[1]}(t)|^2 dt - |f(1)|^2 \cot \theta.
\]

Consider first the case \( \theta = \pi/2 \), for which the term \( |f(1)|^2 \cot \theta \) disappears and assume that \( (f_n) \subset \text{dom} S(\pi/2, \kappa, v) \) is a Cauchy sequence with respect to the norm \( \| \cdot \|_1 \) given by \( \| f \|_1^2 := \| f \|_{L^2}^2 + s_{\pi/2}[f] \). Then there are \( f \) and \( g \) in \( L^2(0,1) \) such that \( f_n \to f \) and \( g_n := f_n^{[1]} \to g \) in \( L^2(0,1) \). Observe that \( f_n^{[1]} = g_n \) and \( f_n \in L^2(0,1) \) yield the relation

\[
f_n(x) = \int_0^x \left( \frac{t}{x} \right)^{\kappa} \exp \left( \int_y^x v(s) ds \right) g_n(t) dt := Hg_n(x).
\]

Since the operator \( H \) acts boundedly from \( L^2(0,1) \) into \( W^{1,2}_2(0,1) \) (cf. the proof of Lemma A.2), we conclude that \( f_n \to Hg \) in \( W^{1,2}_2(0,1) \), whence \( f = Hg \) and \( f^{[1]} = g \).

We observe that the set \( S_{\pi/2} := \{ y^{[1]} \mid y \in \text{dom} S(\pi/2, \kappa, v) \} \) is dense in \( L^2(0,1) \). Indeed, this is equivalent to saying that the range of the operator \( D_{\pi/2} \),

\[
D_{\pi/2}h := \left( \frac{d}{dx} + \frac{\kappa}{x} + v \right)(R_{\pi/2}h) = \frac{1}{y_1(x)} \int_x^1 h(t)y_1(t) dt,
\]

is dense in \( L^2(0,1) \), and the last statement follows from the fact that the nullspace of the operator \( D_{\pi/2}^* \) is trivial. Thus for every \( g \in L^2(0,1) \) there exists a sequence \( (g_n)_{n \in \mathbb{N}} \subset S_{\pi/2} \) tending to \( g \). Taking \( f_n := Hg_n \in \text{dom} S(\pi/2, \kappa, v) \), we see that the closure \( \bar{s}_{\pi/2} \) of \( s_{\pi/2} \) equals

\[
\bar{s}_{\pi/2}[f] = \int_0^1 |f^{[1]}(t)|^2 dt
\]
on \( \text{dom} \bar{s}_{\pi/2} \), which coincides with the range of \( H \) equipped with the norm \( \| \cdot \|_1 \) and is continuously embedded into \( W^{1,2}_2(0,1) \).
Next, since the point evaluation at \( x = 1 \) is a continuous functional on \( W^1_2(0, 1) \), the quadratic form \( |f(1)|^2 \cot \theta \) is relatively compact with respect to \( S_{\pi/2} \), and hence the quadratic form \( s_\theta \) is closable and its closure \( \tilde{s}_\theta \) has the same domain as \( S_{\pi/2} \) and acts therein via the right-hand side of (2.12).

Similar arguments applied to \( s_0 \) show that its closure \( \tilde{s}_0 \) coincides with the restriction of \( S_{\pi/2} \) to the functions that vanish at \( x = 1 \). Indeed, the set \( S_0 := \{ y^{[1]} | y \in \text{dom} S(0, \kappa, v) \} \) coincides with the range of the operator

\[
D_0 h := \left( \frac{d}{dx} + \frac{\kappa}{x} + v \right) (R_0 h) = \frac{1}{y_1(x)} \left[ \int_0^1 h(t)y_1(t) dt - \int_0^1 \frac{dt}{y_1^2(t)} \int_0^1 h(t)y_2(t) dt \right].
\]

Since the nullspace of \( D_0^* \) is spanned by the function \( 1/y_1 \), the range of \( D_0 \) is dense in \( L_2(0, 1) \oplus 1/y_1 \). It remains to observe that \( (Hg)(1) = 0 \) as soon as \( \int_0^1 g(t)/y_1(t) dt = 0 \), so that every function in \( \text{dom} \tilde{s}_0 \) vanishes at \( x = 1 \).

We also notice that since \( f^{[1]} = 0 \) implies \( f = cy_1 \) with \( c \in \mathbb{C} \) and \( y_1 \) of (2.7) and since the latter function is not in \( L_2(0, 1) \), the quadratic forms \( s_0 \) and \( s_{\pi/2} \) (and hence the operators \( S(0, \kappa, v) \) and \( S(\pi/2, \kappa, v) \)) are positive.

**Lemma 2.8.** The operator family \( S(\theta, \kappa, v) \) is monotone increasing with \( \theta \in (0, \pi) \). Determine \( \theta^* \in (0, \pi) \) from the relation \( \cot \theta^* = y_2^{[1]}(1)/y_2(1) \); then the operator \( S(\theta, \kappa, v) \) is positive for \( \theta \in (0, \theta^*) \), nonnegative with one-dimensional nullspace for \( \theta = \theta^* \), and is invertible and has exactly one negative eigenvalue for \( \theta \in (0, \theta^*) \).

**Proof.** Monotonicity is clear in view of the above explicit form of \( s_\theta \). The claims about nonpositive eigenvalues follow from variational principles if we observe that \( y \) is in the nullspace of \( S(\theta, \kappa, v) \) if and only if \( y = y_2 \) and \( \theta = \theta^* \).

Finally, we want to study perturbations of \( S(\theta, \kappa, v) \) by the operator \( Q \) of multiplication by a real-valued function \( q \in L_1(0, 1) \). Since

\[
|\langle qf, f \rangle_{L_2(0, 1)}| \leq \|f\|_{C[0,1]}^2 \|q\|_{L_1(0, 1)}
\]

and the embedding \( W^1_2(0, 1) \hookrightarrow C(0, 1) \) is compact by the Arzela–Ascoli theorem, the quadratic form \( q[f] := \langle qf, f \rangle_{L_2} \) is relatively compact with respect to \( \tilde{s}_\theta \), and by the KLMN theorem, see [26, Theorem X.17], the quadratic form \( \tilde{s}_\theta + q \) is closed on \( \text{dom} \tilde{s}_\theta \). We denote by \( S(\theta, \kappa, v) + Q \) the self-adjoint operator corresponding to \( \tilde{s}_\theta + q \); it follows that \( S(\theta, \kappa, v) + Q \) is bounded below and has discrete spectrum. We observe that if \( q \in L_2(0, 1) \), then it follows from (2.11) that the operator \( Q \) is relatively compact with respect to \( S(\theta, \kappa, v) \), and the sum \( S(\theta, \kappa, v) + Q \) is well defined and closed on \( \text{dom} S(\theta, \kappa, v) \) as the usual sum of two unbounded operators. With this at hand, we can now study positive solutions of the equation

\[
\ell(\kappa, v)y + qy = 0, \quad (2.13)
\]

cf. Section 2.1.
Lemma 2.9. Equation (2.13) has a positive solution on \((0, 1)\) if and only if the operator \(S(0, \kappa, v) + Q\) is positive in \(L_2(0, 1)\).

Proof. If \(u\) is a positive solution of (2.13) on \((0, 1)\), then by Corollary 2.3 there is \(\hat{\nu} \in L_p(0, 1)\) such that \(\ell(\kappa, v) + q = \ell(\kappa, \hat{\nu})\). Then \(S(0, \kappa, v) + Q = S(0, \kappa, \hat{\nu})\), and the latter operator is positive in \(L_2(0, 1)\).

Assume now that \(S(0, \kappa, v) + Q\) is positive and take a nontrivial solution \(u\) of Eq. (2.13) that vanishes at \(x = 0\). We claim that \(u\) vanishes nowhere on \((0, 1)\). Indeed, let there exist \(a \in (0, 1)\) such that \(u(a) = 0\). Then \(u^{[1]}(a) \neq 0\), and \(u\) is not an identical zero on \([0, a)\). We denote by \(u_a\) the nontrivial function that coincides with \(u\) on \([0, a)\) and equals zero on \([a, 1)\). Both \(u_a\) and \(u^{[1]}\) are in \(L_2(0, 1)\) and \(u_a(1) = 0\), so that \(u_0 \in \text{dom}\ 5_0\). Integration by parts (allowed since \(u^{[1]}\) is absolutely continuous on \([0, 1)\)) shows that \(5_0[u_a] + q[u_a] = 0\), which contradicts positivity of \(S(0, \kappa, v) + Q\). This shows that \(u\) does not vanish on \((0, 1)\) and thus some its scalar multiple is positive there. The lemma is proved. \(\square\)

Combining the results of this section, we derive Corollary 2.10 below, which implies, firstly, that the classical Bessel operators \(T(\theta_0, \kappa, q)\) coincide with Bessel operators of the form \(S(\theta, \kappa, v)\) up to additive constants and that it only suffices to treat the cases \(\theta = 0\) and \(\theta = \pi/2\) of the Dirichlet and (quasi-)Neumann boundary conditions at \(x = 1\).

Corollary 2.10. Let \(\theta \in [0, \pi)\), \(v \in L_p(0, 1)\) and \(q \in L_1(0, 1)\). Then there are \(c \in \mathbb{R}\) and \(v_1 \in L_p(0, 1)\) such that the operator \(S(\theta, \kappa, v) + Q + cI\) coincides with \(S(\theta_1, \kappa, v_1)\), with \(\theta_1 = 0\) if \(\theta = 0\) and \(\theta_1 = \pi/2\) otherwise.

3. Double commutation transformations

In this section we introduce transformations between various Bessel operators that appear in the double commutation method [16,17]. In the next section we shall show that, being properly chosen, such transformations lead to Bessel operators with different angular momenta and that they change the spectra of those Bessel operators almost isospectrally.

Assume that \(v\) is a function that is locally summable on \((0, 1)\) and let for every complex \(\lambda \in \mathbb{C}\) the function \(y(\cdot, \lambda)\) be a solution of the equation

\[-\left(\frac{d}{dx} - v\right)\left(\frac{d}{dx} + v\right)y = \lambda^2 y\]

satisfying the terminal conditions (1.5) or (1.6) according as \(\theta = 0\) or \(\theta = \pi/2\), with \(y^{[1]} := y' + v y\) therein.

The double commutation method rests on the following transformation. We choose \(\lambda_* \in \mathbb{R} \setminus \{0\}\) and \(\alpha_* \in \mathbb{R}\) so that the functions \(w_+(x) := 1 + \alpha_* \int_x^1 |y(t, \lambda_*)|^2 dt\) and \(w_-(x) := 1 + \alpha_* \int_x^1 \lambda_*^{-2} |y(1, \lambda_*)|^2 dt\) do not vanish on \((0, 1)\) and put

\[u(x, \lambda) := y(x, \lambda) - \frac{\alpha_* y(x, \lambda_*)}{w_+(x)} \int_x^1 y(t, \lambda_*) y(t, \lambda) dt\]  

(3.1)
and
\[
v_+(x) := \frac{w'_+(x)}{w_+(x)} - \frac{w'_-(x)}{w_-(x)}.
\]

**Theorem 3.1.** The function \( u(\cdot, \lambda) \) satisfies the equation
\[
- \left( \frac{d}{dx} - v - v_* \right) \left( \frac{d}{dx} + v + v_* \right) u = \lambda^2 u
\]
and the terminal conditions \( u(1) = y(1, \lambda) \) and \((u' + vu + v^* u)(1) = y^{[1]}(1, \lambda)\).

It is advantageous to reformulate the theorem in the Dirac equation setting. Indeed, the vector
\[
y(\cdot, \lambda) = \begin{pmatrix} y_+(\cdot, \lambda) \\ y_-(\cdot, \lambda) \end{pmatrix} := \begin{pmatrix} y(\cdot, \lambda) \\ \lambda^{-1} y^{[1]}(\cdot, \lambda) \end{pmatrix}
\]
satisfies the Dirac-type equation
\[
\left( B \frac{d}{dx} + V \right) y(\cdot, \lambda) = \lambda y(\cdot, \lambda),
\]
where \( V(x) = v(x) J \) and
\[
B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

We shall use below the projections \( y_{\pm}(x, \lambda) \) of the vector \( y(x, \lambda) \) onto the subspaces \( L_2(0, 1) \times \{0\} \) and \( \{0\} \times L_2(0, 1) \), respectively, i.e., \( y_+(x, \lambda) := (y_+(x, \lambda), 0)^\top \) and \( y_-(x, \lambda) := (0, y_-(x, \lambda))^\top \).

We now set \( K := \text{diag}\{k_+, k_-\} \) to be a diagonal matrix-valued function, with
\[
k_{\pm}(x, t) = -\frac{\alpha_* y_{\pm}(x, \lambda_\ast) y_{\pm}(t, \lambda_\ast)}{w_{\pm}(x)}, \quad \lambda = \lambda + \lambda_\ast
\]
and observe that \( k_{\pm}(x, x) = w'_\pm(x)/w_\pm(x) \) and that the matrix kernel \( K \) can be written in the form
\[
K(x, t) = -\frac{\alpha_* y_+(x, \lambda_\ast) y_+^\top(t, \lambda_\ast)}{w_+(x)} - \frac{\alpha_* y_-(x, \lambda_\ast) y_-^\top(t, \lambda_\ast)}{w_-(x)}.
\]

The vector analogue of the transformation (3.1) reads now
\[
\mathbf{u}(x, \lambda) := y(x, \lambda) + \int_x^1 K(x, t) y(t, \lambda) \, dt.
\]

It is easily seen that the first component \( u_+(\cdot, \lambda) \) of the vector \( \mathbf{u}(\cdot, \lambda) \) coincides with the function \( u(\cdot, \lambda) \) introduced above. Therefore it suffices to prove the following counterpart of Theorem 3.1.
Theorem 3.2. The function $u(\cdot, \lambda)$ solves the Dirac equation
\[
\left( B \frac{d}{dx} + V + V_\ast \right) u(\cdot, \lambda) = \lambda u(\cdot, \lambda),
\]
where the added potential $V_\ast$ equals
\[
V_\ast(x) = BK(x, x) - K(x, x)B = v_\ast(x)J.
\]

Proof. We start by establishing two auxiliary results. Firstly, we observe that the functions $y_{\pm}(\cdot, \lambda)$ satisfy the following relations:
\[
\left( B \frac{d}{dx} + V \right) y_{\pm}(\cdot, \lambda) = \lambda y_{\mp}(\cdot, \lambda). \tag{3.6}
\]
Secondly, since the matrix $J_1 := \text{diag}(1, -1)$ anticommutes with $B$ and $J$, it follows that the function $J_1 y(\cdot, \lambda)$ satisfies the equation
\[
\left( B \frac{d}{dx} + V \right) J_1 y(x, \lambda) = -\lambda J_1 y(x, \lambda),
\]
which implies for $\lambda \in \mathbb{R}$ the Lagrange-type identity
\[
\lambda \int_x^1 y^\top(t, \lambda)J_1 y(t, \lambda) \, dt = y_+(x, \lambda)y_-(x, \lambda). \tag{3.7}
\]

Now we turn to the proof and find first that
\[
\left( B \frac{d}{dx} + V \right) u(x, \lambda) = \left( B \frac{d}{dx} + V \right) y(x, \lambda) - BK(x, x)y(x, \lambda)
\]
\[
+ \int_x^1 \left( B \frac{d}{dx} + V \right) K(x, t)y(t, \lambda) \, dt.
\]
Using formulae (3.4) and (3.6), we obtain
\[
\left( B \frac{d}{dx} + V \right) K(x, t) = -\frac{\alpha_\ast \lambda_\ast y_-(x, \lambda_\ast)y_+^\top(t, \lambda_\ast)}{w_+(x)} - \frac{\alpha_\ast \lambda_\ast y_+(x, \lambda_\ast)y_-^\top(t, \lambda_\ast)}{w_-(x)}
\]
\[
+ B \frac{w_+^\prime(x)}{w_+(x)} \frac{\alpha_\ast y_+(x, \lambda_\ast)y_+^\top(t, \lambda_\ast)}{w_+(x)}
\]
\[
+ B \frac{w_-^\prime(x)}{w_-(x)} \frac{\alpha_\ast y_-(x, \lambda_\ast)y_-^\top(t, \lambda_\ast)}{w_-(x)}
\]
\[
= -\frac{\alpha_\ast \lambda_\ast y_-(x, \lambda_\ast)y_+^\top(t, \lambda_\ast)}{w_+(x)} - \frac{\alpha_\ast \lambda_\ast y_+(x, \lambda_\ast)y_-^\top(t, \lambda_\ast)}{w_-(x)}
\]
\[
- BK(x, x)K(x, t),
\]
so that

\[
\left( B \frac{d}{dx} + V \right) u(x, \lambda) = \lambda y(x, \lambda) - BK(x, x)u(x, \lambda)
\]

\[- \frac{\alpha_* \lambda_* y_-(x, \lambda_*)}{w_+(x)} \int_x^1 y_+(t, \lambda_*) y(t, \lambda) dt
\]

\[- \frac{\alpha_* \lambda_* y_+(x, \lambda_*)}{w_-(x)} \int_x^1 y_-(t, \lambda_*) y(t, \lambda) dt.
\]

Next,

\[
\lambda y(x, \lambda) = \lambda u(x, \lambda) - \lambda \int_x^1 K(x, t) y(t, \lambda) dt,
\]

and integration by parts in

\[
\lambda \int_x^1 K(x, t) y(t, \lambda) dt = \int_x^1 K(x, t) \left( B \frac{d}{dt} + V \right) y(t, \lambda) dt
\]

produces

\[
\lambda y(x, \lambda) = \lambda u(x, \lambda) + K(x, x)By(x, \lambda)
\]

\[+ \frac{\alpha_* \lambda_* y_+(x, \lambda_*)}{w_+(x)} \int_x^1 y_+(t, \lambda_*) y(t, \lambda) dt
\]

\[+ \frac{\alpha_* \lambda_* y_-(x, \lambda_*)}{w_-(x)} \int_x^1 y_-(t, \lambda_*) y(t, \lambda) dt.
\]

Observing that by (3.7)

\[
\frac{\lambda_*}{w_-(x)} - \frac{\lambda_*}{w_+(x)} = \frac{\lambda_* \alpha_*}{w_+(x)w_-(x)} \int_x^1 y(t, \lambda_*) J_1 y(t, \lambda_*) dt
\]

\[= \frac{\alpha_* y_+(x, \lambda_*) y_-(x, \lambda_*)}{w_+(x)w_-(x)},
\]

we conclude that

\[
\left( B \frac{d}{dx} + V \right) u(x, \lambda) = \lambda u(x, \lambda) + K(x, x)By(x, \lambda) - BK(x, x)u(x, \lambda)
\]

\[+ \frac{\alpha_*^2 y_+(x, \lambda_*) y_-(x, \lambda_*)}{w_+(x)w_-(x)} y_-(x, \lambda_*) \int_x^1 y_+(t, \lambda_*) y(t, \lambda) dt
\]
\[
\begin{align*}
-\frac{\alpha^2_y}{w_+(x)}y_+(x, \lambda_*)y_-(x, \lambda_*) & \int_0^1 y^\top_-(t, \lambda_*)y(t, \lambda) \, dt \\
= \lambda u(x, \lambda) + K(x, x)Bu(x, \lambda) & - BK(x, x)u(x, \lambda)
\end{align*}
\]

as required. The last equality follows from the calculation of the kernel \(K(x, x)BK(x, t)\); indeed, using the relations

\[
y_\pm \cdot y_\pm = 0, \quad y_\pm \cdot y_\mp = \mp y_+ (x, \lambda_*)y_-(x, \lambda_*),
\]

we find that

\[
K(x, x)BK(x, t) = -\frac{\alpha^2_y}{w_+(x)}y_+(x, \lambda_*)y_-(x, \lambda_*)y^\top_-(t, \lambda_*)
\]

\[
+ \frac{\alpha^2_y}{w_+(x)}y_-(x, \lambda_*)y_+(x, \lambda_*)y_+^\top(t, \lambda_*)
\]

The proof is complete. \(\square\)

**Remark 3.3.** Theorems 3.1 and 3.2 also hold for \(\lambda_* = 0\) provided \(\theta = \pi/2\). Indeed, in this case we have \(y_1^1(\cdot, 0) \equiv 0, y_-(\cdot, 0) \equiv 0,\) and \(w_- \equiv 1\), and all the above calculations remain true.

Before proceeding, we recall some other properties of the mapping \(y(\cdot, \lambda) \mapsto u(\cdot, \lambda)\) of (3.1). In fact, we can introduce an abstract analogue of this transformation in the following manner. Assume that \(f \in L^2_{\text{loc}}(0, 1)\) and that a nonzero \(\alpha \in \mathbb{R}\) is chosen so that

\[
\alpha \int_x^1 |f(s)|^2 \, ds > -1
\]

for all \(x \in (0, 1)\); then \(U = U(f, \alpha)\) is an operator in \(L^2_{\text{loc}}(0, 1)\) given by (cf. [17])

\[
(Ug)(x) = g(x) - \frac{\alpha f(x)}{1 + \alpha \int_x^1 |f(s)|^2 \, ds} \int_x^1 g(s)f(s) \, ds;
\]

in particular,

\[
(Uf)(x) := \frac{f(x)}{1 + \alpha \int_x^1 |f(s)|^2 \, ds}.
\]

It is clear that the mappings \(y_\pm(\cdot, \lambda) \mapsto u_\pm(\cdot, \lambda)\) of (3.5) are generated by the linear operators \(U(y_\pm(\cdot, \lambda_*), \alpha_*)\).

The inverse transformation \(U^{-1}\) is easily seen to be [17]

\[
(U^{-1}g)(x) = g(x) + \alpha f(x) \int_x^1 g(s)(Uf)(s) \, ds.
\]
Direct calculations (involving integration by parts and simple algebra) give

\[
\int_x^1 |(Ug)(s)|^2 \, ds = \int_x^1 |g(s)|^2 \, ds - \alpha \frac{\int_x^1 g(s) f(s) \, ds}{1 + \alpha \int_x^1 |f(s)|^2 \, ds};
\]

in particular,

\[
\int_x^1 |(Uf)(s)|^2 \, ds = \frac{1}{\alpha} \left[ 1 - \left( 1 + \alpha \int_x^1 |f(s)|^2 \, ds \right)^{-1} \right],
\]

so that \( Uf \in L^2(0, 1) \) if and only if either \( f \notin L^2(0, 1) \) or \( f \in L^2(0, 1) \) and \( 1 + \alpha \|f\| > 0 \).

Combining the above properties of the mapping \( U \), we conclude the following (see [17, Lemma 2.1]).

**Proposition 3.4.** The operator \( U \) performs a unitary equivalence of \( L^2(0, 1) \ominus f \) and \( L^2(0, 1) \ominus Uf \); here \( L^2(0, 1) \ominus f = L^2(0, 1) \) if \( f \) is not in \( L^2(0, 1) \), and similarly for \( L^2(0, 1) \ominus Uf \).

### 4. Spectral transformations of Bessel operators

In this section we show that the transformations of the previous section can be chosen so that they map between Bessel operators with different angular momenta almost isospectrally, i.e., they keep unchanged all but one eigenvalue and corresponding norming constant.

Assume that \( \kappa \in \mathbb{N} \) and that \( v \in L^p(0, 1) \), and denote by \( \lambda_n^2, n \in \mathbb{N} \), the eigenvalues of the operator \( S(\theta, \kappa, v) \), \( \theta = 0 \) or \( \theta = \pi/2 \). Denoting by \( y(\cdot, \lambda) \) a solution of the equation \( \ell(\kappa, v)y = \lambda^2 y \) satisfying the terminal conditions (1.5) or (1.6) according as \( \theta = 0 \) or \( \theta = \pi/2 \), we see that \( y(\cdot, \lambda_n) \) is an eigenvector corresponding to the eigenvalue \( \lambda_n^2 \). The number

\[
\alpha_n := \|y(\cdot, \lambda_n)\|^{-2} = \left( \int_0^1 |y(x, \lambda_n)|^2 \, dx \right)^{-1}
\]

is the norming constant corresponding to the eigenvalue \( \lambda_n^2 \). We recall (see Lemma 2.2) that the function \( y(\cdot, \lambda) \) has at \( x = 0 \) a zero of order \( \kappa + 1 \) if \( \lambda^2 \) is an eigenvalue and a pole of order \( \kappa \) otherwise; the orders of a zero and a pole at \( x = 0 \) for \( y^{[1]}(\cdot, \lambda) \) are \( \kappa \) and \( \kappa - 1 \), respectively.

#### 4.1. Removing an eigenvalue

Applying the transformation of (3.1) to the potential \( \psi_0(x) := \frac{\kappa}{x} + v(x) \) and the numbers \( \lambda_n := \lambda_k \) and \( \alpha_n := -\alpha_k \), we find by Theorem 3.1 that the function

\[
u(\cdot, \lambda) = U \left( y(\cdot, \lambda_k), -\alpha_k \right) y(\cdot, \lambda)
\]

satisfies the equation

\[
-\left( \frac{d}{dx} - \frac{\kappa}{x} - v - v_* \right) \left( \frac{d}{dx} + \frac{\kappa}{x} + v + v_* \right) u = \lambda^2 u
\]
where \( v_\ast = k_+(x, x) - k_-(x, x) \) and \( k_\pm \) are given by (3.3), with \( y_+(x, \lambda_\ast) = y(x, \lambda_k) \) and

\[
y_-(x, \lambda_\ast) = y^{[1]}(x, \lambda_k)/\lambda_k := \left[ y'(x, \lambda) + \left( \frac{\kappa}{x} + v(x) \right) y(x, \lambda) \right]/\lambda_k
\]

therein. We recall that \( k_\pm(x, x) = w'_\pm(x)/w_\pm(x) \) and observe that

\[
w_+(x) = 1 - \alpha_k \int_x^1 |y(t, \lambda_k)|^2 \, dt = \alpha_k \int_0^x |y(t, \lambda_k)|^2 \, dt
\]

is positive on \((0, 1]\) and has a zero of order \(2\kappa + 3\) at \(x = 0\). Relation (3.7) implies that

\[
\int_0^1 |y_-(t, \lambda_k)|^2 \, dt = \int_0^1 |y_+(t, \lambda_k)|^2 \, dt = \alpha_k^{-1}.
\]

so that \( w_-(x) = \alpha_k \lambda_k^{-2} \int_0^x |y^{[1]}(t, \lambda_k)|^2 \, dt \) is also positive on \((0, 1]\), with a zero of order \(2\kappa + 1\) at \(x = 0\). By Lemmata 2.2 and A.3 we conclude that \( w_+(x)/w_-(x) = x^2 \hat{w}(x) \) for some positive function \( \hat{w} \in W^1_p(0, 1) \), and thus

\[
v_\ast(x) = \frac{d}{dx} \log \frac{w_+(x)}{w_-(x)} = \frac{2}{x} + \frac{\hat{w}'(x)}{\hat{w}(x)},
\]

with \( \hat{w}'(x)/\hat{w}(x) \in L_p(0, 1) \). It follows that

\[
\frac{\kappa}{x} + v(x) + v_\ast(x) = \frac{\kappa + 2}{x} + \hat{v}(x)
\]

with \( \hat{v} := v + \hat{w}'/\hat{w} \in L_p(0, 1) \).

**Theorem 4.1.** The spectrum of the operator \( S(\theta, \kappa + 2, \hat{v}) \) consists of the eigenvalues \( \lambda_n^2, n \in \mathbb{N} \setminus \{ k \} \), the corresponding norming constants being \( \alpha_n \).

**Proof.** The function \( u(\cdot, \lambda) \) solves the equation \( \ell(\kappa + 2, \hat{v})u = \lambda^2 u \) and satisfies the terminal conditions \( u(1) = y(1, \lambda) \) and \( u^{[1]}(1) := (u' + (\kappa + 2)u/x + \hat{v}u)(1) = y^{[1]}(1, \lambda) \). It remains to prove that \( u(\cdot, \lambda) \) is in \( L_2(0, 1) \) if and only if \( \lambda^2 \) equals \( \lambda_n^2 \) for \( n \neq k \).

The considerations preceding Proposition 3.4 show that the function \( u(\cdot, \lambda_k) \) does not belong to \( L_2(0, 1) \). Since the operator \( U := U(y(\cdot, \lambda_k), -\alpha_k) \) performs the unitary equivalence of \( L_2(0, 1) \ominus y(\cdot, \lambda_k) \) and \( L_2(0, 1) \), it follows that \( u(\cdot, \lambda_n) \) is in \( L_2(0, 1) \) for every natural \( n \neq k \) and, moreover, \( \|u(\cdot, \lambda_n)\| = \|y(\cdot, \lambda_n)\| \) for such \( n \).

If there were \( \lambda^2 \in \mathbb{R} \setminus \{ \lambda_n^2 \}_{n \in \mathbb{N}} \) in the spectrum of \( S(\theta, \kappa + 2, \hat{v}) \), then \( u(\cdot, \lambda) \) would be the corresponding eigenfunction. Then \( y(\cdot, \lambda) = U^{-1} u(\cdot, \lambda) \) would belong to \( L_2(0, 1) \) and would be orthogonal to \( y(\cdot, \lambda_n) \) for all \( n \in \mathbb{N} \). Indeed, orthogonality to \( y(\cdot, \lambda_k) \) follows from the fact that every function in the range of \( U^{-1} \) has this property, and orthogonality to the other functions follows from the fact that \( U^{-1} \) is an isometry and that \( u(\cdot, \lambda) \) and \( u(\cdot, \lambda_n), n \in \mathbb{N} \setminus \{ k \} \), are orthogonal as eigenfunctions of \( S(\theta, \kappa + 2, \hat{v}) \). However, the system \( \{ y(\cdot, \lambda_n) \}_{n \in \mathbb{N}} \) is an orthogonal
basis of \( L_2(0, 1) \), which contradicts the above conclusion about \( y(\cdot, \lambda) \). Therefore \( S(\theta, \kappa + 2, \hat{v}) \)
has no other eigenvalues except \( \{ \lambda_n^2 \}_{n \in \mathbb{N} \setminus \{ k \}} \), and the proof is complete. \( \square \)

**Remark 4.2.** Similar results are also valid if we start with the regular case \( \kappa = 0 \). We consider separately three situations, when the initial operator is

(i) \( S_0(\theta, 0, v), \theta = 0, \pi/2 \); or
(ii) \( S_{\pi/2}(\pi/2, 0, v) \); or
(iii) \( S_{\pi/2}(0, 0, v) \).

We recall that \( S_\eta(\theta, 0, v) \) is the restriction of the operator \( S(\theta, 0, v) \) by the boundary condition (1.4) at \( x = 0 \).

The case of the operator \( S_0(\theta, 0, v) \) is similar to that of \( \kappa > 0 \): being applied to this operator, the above transformation produces \( S(\theta, 2, \hat{v}) \), and we can proceed as in the proof of Theorem 4.1.

If the initial operator was \( S_{\pi/2}(\pi/2, 0, v) \), then we have to remove the eigenvalue \( \lambda^* = 0 \) in order to get a positive operator. The corresponding eigenfunction \( y(x, 0) := \exp(\int_x^1 v(t) \, dt) \) satisfies the relation \( y^{[1]}(x, 0) \equiv 0 \), so that \( w_+ \) has a zero of order 1 at \( x = 0 \), while \( w_- \equiv 1 \). The new impedance potential has therefore the form

\[
v(x) + v_+(x) = \frac{1}{x} + \hat{v}(x)
\]

for some \( \hat{v} \in L_p(0, 1) \), and we get the operator \( S(\pi/2, 1, \hat{v}) \).

If we start with the operator \( S_{\pi/2}(0, 0, v) \), then the situation differs a little bit. Indeed, then the eigenfunction \( y(\cdot, \lambda_k) \) does not vanish at the origin, but its quasi-derivative \( y^{[1]}(\cdot, \lambda_k) \) does. As a result, the functions \( w_+ \) and \( w_- \) have at \( x = 0 \) zeros of orders 1 and 3, respectively. The new impedance potential has therefore the form

\[
v(x) + v_+(x) = -\frac{2}{x} + \tilde{v}(x)
\]

for some \( \tilde{v} \in L_p(0, 1) \). For every \( \lambda \in \mathbb{C} \), the function \( u(\cdot, \lambda) \) solves the equation \( \ell(-2, \tilde{v})u = \lambda^2 u \) and satisfies the terminal conditions \( u(1, \lambda) = y(1, \lambda) = 0 \) and

\[
\lambda u_-(1, \lambda) = [u'(x, \lambda) + (-2/x + \tilde{v}(x))u(x, \lambda)]|_{x=1} = y^{[1]}(1, \lambda) = \lambda.
\]

Applying Lemma A.1, we see that there is a function \( \hat{v} \in L_p(0, 1) \) such that the differential expressions \( \ell(-2, \tilde{v}) \) and \( \ell(1, \hat{v}) \) coincide and \( \hat{v} - \tilde{v} \) is a function that is continuous on \( (0, 1] \). Therefore

\[
u'(x, \lambda) + \left( \frac{1}{x} + \hat{v}(x) \right) u(x, \lambda) = u'(x, \lambda) + \left( -\frac{2}{x} + \tilde{v}(x) \right) u(x, \lambda) + \left( \hat{v} - \tilde{v} + \frac{3}{x} \right) u(x, \lambda),
\]

so that

\[
[u'(x, \lambda) + (1/x + \hat{v}(x))u(x, \lambda)]|_{x=1} = y^{[1]}(1, \lambda) = \lambda.
\]
It follows that $u(x, \lambda_n), n \in \mathbb{N} \setminus \{k\},$ are the properly normalized eigenfunctions of the operator $S(0, 1, \hat{v}).$

In all three cases we repeat the reasoning of the proof of Theorem 4.1 to show that the new Bessel operator has no other eigenvalues except $\{\lambda_n^2\}_{n \in \mathbb{N}} \setminus \{\lambda_k^2\}.$

4.2. Adding an eigenvalue

We assume first that $\kappa \geqslant 2,$ take a nonzero real $\lambda_\ast$ not in the spectrum of $S(\theta, \kappa, v)$ and a positive $\alpha_\ast > 0,$ and apply Theorem 3.1 to the impedance potential $v_0(x) := \frac{x}{x} + v(x).$ We find then that the function

$$u(\cdot, \lambda) = U(y(\cdot, \lambda_\ast), \alpha_\ast)$$

satisfies the equation

$$-\left( \frac{d}{dx} - \frac{\kappa}{x} - v - v_\ast \right) \left( \frac{d}{dx} + \frac{\kappa}{x} + v + v_\ast \right) u = \lambda^2 u$$

with $v_\ast$ of (3.2). Since $y(x, \lambda_\ast) \sim x^{-\kappa}$ and $y^{[1]}(x, \lambda_\ast) \sim x^{1-\kappa}$ as $x \to 0,$ we conclude from Lemmata 2.2 and A.2 that

$$w_+(x) = 1 + \alpha_\ast \int_x^1 |y(t, \lambda_\ast)|^2 dt = x^{-2\kappa+1} \hat{w}_+(x),$$

$$w_-(x) = 1 + \alpha_\ast \lambda_\ast^{-2} \int_x^1 |y^{[1]}(t, \lambda_\ast)|^2 dt = x^{-2\kappa+3} \hat{w}_-(x)$$

for some positive $\hat{w}_\pm$ belonging to $W^1_p(0, 1),$ and thus

$$v_0(x) + v_\ast(x) = \frac{\kappa - 2}{x} + \hat{v}(x)$$

with some $\hat{v} \in L_p(0, 1).$

**Theorem 4.3.** If $\kappa > 2,$ then the spectrum of the operator $S(\theta, \kappa - 2, \hat{v})$ consists of eigenvalues $\lambda_n^2, n \in \mathbb{N},$ and $\lambda_{\ast}^2,$ with the corresponding norming constants $\alpha_n$ and $\alpha_\ast,$ respectively. For $\kappa = 2$ the same is true about the operator $S_0(\theta, 0, \hat{v}).$

**Proof.** If $\kappa > 2,$ it suffices to prove that $u(\cdot, \lambda)$ belongs to $L_2(0, 1)$ if and only if $\lambda$ is either $\lambda_\ast$ or one of $\lambda_n, n \in \mathbb{N}.$ For $\kappa = 2,$ we have to show instead that the functions $u(\cdot, \lambda)$ satisfy the initial condition $u(0, \lambda) = 0$ only for such $\lambda.$

Using the properties of the mapping $U := U(y(\cdot, \lambda_\ast), \alpha_\ast),$ we conclude from relation (3.8) that the function $u(\cdot, \lambda_\ast) = Uy(\cdot, \lambda_\ast)$ belongs to $L_2(0, 1)$ and has norm $\alpha_\ast^{-1/2}.$ Since $U$ performs a unitary equivalence of $L_2(0, 1)$ and $L_2(0, 1) \ominus u(\cdot, \lambda_\ast),$ the function $u(\cdot, \lambda_\ast)$ belongs to $L_2(0, 1) \ominus u(\cdot, \lambda_\ast)$ for every $n \in \mathbb{N}$ and $\|u(\cdot, \lambda_n)\| = \|y(\cdot, \lambda_n)\| = \alpha_n^{-1/2}.$
If the operator $S(\theta, \kappa - 2, \hat{v})$ (or $S_0(\theta, 0, \hat{v})$ if $\kappa = 2$) had an eigenvalue $\lambda^2 \in \mathbb{R}$ different from $\lambda^2_n$ and all $\lambda^2_n$, $n \in \mathbb{N}$, then $u(\cdot, \lambda)$ would be an eigenfunction orthogonal to $u(\cdot, \lambda_n)$. However, then $y(\cdot, \lambda)$ would belong to $L_2(0, 1)$ and $\lambda^2$ would be an eigenvalue of the operator $S(\theta, \kappa, v)$, which is impossible.

For $\kappa > 2$ this proves that the spectrum of the operator $S(\theta, \kappa - 2, \hat{v})$ coincides with the set $\{\lambda^2_n\}_{n \in \mathbb{N}} \cup \{\lambda^2\}$ and that the norming constants are as stated. For $\kappa = 2$, we have to verify that the functions $u(\cdot, \lambda)$ for $\lambda = \lambda^*$ and $\lambda = \lambda_n$, $n \in \mathbb{N}$, vanish at the origin.

Indeed, for $\kappa = 2$ we have $\int_1^1 |y(s, \lambda_n)|^2 ds \sim x^{-3}$ as $x \to +0$, and therefore the limit

$$u(0, \lambda_n) = \lim_{x \to 0} \frac{y(x, \lambda_n)}{1 + \alpha_s \int_1^1 |y(s, \lambda_n)|^2 ds}$$

exists and is equal to zero. As the expression $y(x, \lambda_n)y(x, \lambda_n)$ is integrable at the origin, we see by (3.5) that the function

$$u(x, \lambda_n) = y(x, \lambda_n) - \alpha_s u(x, \lambda_n) \int_x^1 y(s, \lambda_n)y(s, \lambda_n) ds$$

vanishes at $x = 0$ for all $n \in \mathbb{N}$. Thus $\lambda^2_n$ and $\lambda^2$ are all the eigenvalues of the operator $S_0(\theta, 0, \hat{v})$. □

**Remark 4.4.** For $\kappa = 1$ the above reasonings must slightly be modified. If $\theta = \pi/2$, we take $\lambda_n = 0$; then the solution $y(\cdot, 0)$ coincides with the function $y_1$ of (2.7), and $y^{[1]} \equiv 0$. As a result, we see that the modified impedance potential has then the form

$$v_0(x) + v_n(x) = \hat{v}(x)$$

for some $\hat{v} \in L_p(0, 1)$. We shall show next that the functions $u(\cdot, \lambda)$ for $\lambda = 0$ and $\lambda = \lambda_n$ satisfy the initial condition $u^{[1]}(0) := (u' + \hat{v}u)(1) = 0$. Taking the second component of the vectorial relation (3.5) at $\lambda = 0$ and then at $\lambda = \lambda_n$, $n \in \mathbb{N}$, we conclude that $u^{[1]}(\cdot, 0) \equiv 0$ and then that $u^{[1]}(0, \lambda) = 0$ for all $\lambda = \lambda_n$. Therefore the conclusions of Theorem 4.3 hold in this case for the operator $S_{\pi/2}(\pi/2, 0, \hat{v})$.

If $\theta = 0$, then we take $\lambda_n > 0$, and the new impedance potential $v_0 + v_n$ has the form $\hat{v} - 1/x$ with some $\hat{v} \in L_p(0, 1)$. By Lemma A.1 we conclude that there is $\tilde{v} \in L_p(0, 1)$ such that the differential expressions $\ell(-1, \tilde{v})$ and $\ell(0, \hat{v})$ coincide and the function $\hat{v} - \tilde{v}$ is continuous on $(0, 1]$. Therefore the function $u(\cdot, \lambda)$ solves for every $\lambda \in \mathbb{C}$ the equation $\ell(0, \hat{v})u = \lambda^2 u$ and satisfies the terminal condition $u(1, \lambda) = y(1, \lambda) = 0$; moreover, since

$$u'(x, \lambda) + \hat{v}(x)u(x, \lambda) = u'(x, \lambda) + \left(-\frac{1}{x} + \tilde{v}(x)\right)u(x, \lambda) + \left(\hat{v} - \tilde{v} + \frac{1}{x}\right)u(x, \lambda),$$

we conclude that

$$\left[u'(x, \lambda) + \hat{v}(x)u(x, \lambda)\right]|_{x=1} = y^{[1]}(1, \lambda) = \lambda.$$
The properties of the mapping \( U(y(\cdot, \lambda_*), \alpha_*), n \in \mathbb{N} \), are pairwise orthogonal. The Lagrange-type formula for the differential expression \( \ell(0, \hat{v}) \) gives
\[
\int_0^1 u(x, \lambda)u(x, \mu) \, dt = \left[ u'(x, \lambda) + \hat{v}(x)u(x, \lambda) \right]_{x=0}^1 u(0, \mu) - u(0, \lambda) \left[ u'(x, \mu) + \hat{v}(x)u(x, \mu) \right]_{x=0}^1.
\]
Take \( \lambda \neq \mu \) in the set \( \{ \lambda_* \} \cup \{ \lambda_n \} \), then the left-hand side of the above equality vanishes, and thus so does the right-hand side. Simple arguments then show that either \( u(0, \lambda_*) = u(0, \lambda_n) = 0 \) for all \( n \in \mathbb{N} \), or none of these numbers vanishes. Relation (4.1) implies that \( u(0, \lambda_*) \neq 0 \), and we see that
\[
\left[ u'(x, \lambda_n) + \hat{v}(x)u(x, \lambda_n) \right]_{x=0}^1 = \left[ u'(x, \lambda_*) + \hat{v}(x)u(x, \lambda_*) \right]_{x=0}^1 := \cot \eta.
\]
It follows that the numbers \( \lambda_*^2 \) and \( \lambda_n^2, n \in \mathbb{N} \), are eigenvalues of the operator \( S(\theta, \kappa, v) \). Since the operator \( S(\theta, 0, \hat{v}) \) is positive by construction, analysis similar to that of Sections 2.2 and 2.3 shows that there is an equivalent representation of this operator as \( S_{\pi/2}(0, 0, \hat{v}) \) with some impedance potential \( \hat{v} \) in \( L^p(0, 1) \). As in Theorem 4.3, we conclude that the spectrum of this operator consists of eigenvalues \( \lambda_*^2 \) and \( \lambda_n^2, n \in \mathbb{N} \), with the corresponding norming constants \( \alpha_* \) and \( \alpha_n, n \in \mathbb{N} \).

5. Direct and inverse spectral analysis

5.1. Spectral data

Starting with the operator \( S(\theta, \kappa, v) \) with \( \kappa \in \mathbb{N} \) and \( v \in L^p(0, 1) \), we apply \( v := [(\kappa + 1)/2] \) times the transformations of Section 4.2 to arrive at the operator \( S(\theta, 0, v_0) \) with \( v_0 \in L^p(0, 1) \) and \( \eta = 0 \) for even \( \kappa \) and \( \eta = \pi/2 \) for odd \( \kappa \), whose spectrum has \( \nu \) extra eigenvalues, and the norming constants for the common eigenvalues coincide.

Recalling the description of the spectral data for regular Sturm–Liouville operators in impedance form given in [3], we derive the asymptotics of the spectral data for the operator \( S(\theta, \kappa, v) \) of Theorem 1.1.

5.2. Existence and the reconstruction algorithm

Now we demonstrate that any two sequences \( \{ \lambda_n^2 \}_{n \in \mathbb{N}} \) and \( \{ \alpha_n \}_{n \in \mathbb{N}} \) of positive numbers satisfying the asymptotics of Theorem 1.1, with \( \lambda_n \) strictly increasing, are indeed spectral data for some Bessel operator. We prove Theorem 1.2 by presenting the reconstruction algorithm and then justifying that it gives the operator searched for.

The reconstruction algorithm.

1. Choose \( v := [(\kappa + 1)/2] \) pairwise distinct nonnegative points \( \mu_1, \ldots, \mu_\nu \), not belonging to the set \( \{ \lambda_n^2 \}_{n \in \mathbb{N}} \) and \( \nu \) positive numbers \( \beta_1, \ldots, \beta_\nu \). We take \( \mu_1 = 0 \) if and only if \( \theta = \pi/2 \) and \( \kappa \) is odd.
2. Find a unique regular Sturm–Liouville operator $S_\eta(\theta, 0, v_0)$ in the impedance form, with $v_0 \in L_p(0, 1)$ and $\eta = 0$ for even $\kappa$ and $\eta = \pi/2$ for odd $\kappa$, for which the sets $\{\mu_1, \ldots, \mu_v\} \cup \{\lambda_n^2\}_{n \in \mathbb{N}}$ and $\{\beta_1, \ldots, \beta_v\} \cup \{\alpha_n\}_{n \in \mathbb{N}}$ are sets of eigenvalues and norming constants.

3. Remove consecutively $\nu$ eigenvalues $\mu_1, \ldots, \mu_\nu$ from the spectrum of the Bessel operator $S_\eta(\theta, 0, v_0)$ applying the procedure of Section 4.1. We end up with a Bessel operator $S(\theta, \kappa, v_\kappa)$, with some $v_\kappa \in L_p(0, 1)$.

That the sets $\{\mu_1, \ldots, \mu_\nu\} \cup \{\lambda_n^2\}_{n \in \mathbb{N}}$ and $\{\beta_1, \ldots, \beta_\nu\} \cup \{\alpha_n\}_{n \in \mathbb{N}}$ form spectral data for some regular impedance Sturm–Liouville operator $S_\eta(\theta, 0, v_0)$ as claimed in Step 2 follows from the properties of $\lambda_n$ and $\alpha_n$ stated in Theorem 1.1. The procedure for finding this regular Sturm–Liouville operator is given in, e.g., [3].

5.3. Uniqueness

Formally, the above reconstruction algorithm depends on the choice of the added eigenvalues $\mu_1, \ldots, \mu_v$ and their norming constants—e.g., the auxiliary potential $v_0$ clearly depends on these quantities. We prove next that the reconstructed operator $S(\theta, \kappa, v_\kappa)$ is unique.

**Theorem 5.1.** The radial Bessel operator is uniquely determined by the given spectral data.

**Proof.** The resolution of identity for $S(\theta, \kappa, v_\kappa)$ reads

$$I = \lim_{N \to \infty} \sum_{n=1}^{N} \alpha_n \langle \cdot, y_n \rangle_{L^2(0,1)} y_n,$$

$y_n := y(\cdot, \lambda_n)$ being the corresponding eigenfunctions and $\lim$ denoting limit in the strong operator topology. Restricting this equality to the interval $(\varepsilon, 1)$, we get a similar relation

$$I_\varepsilon = \lim_{N \to \infty} \sum_{n=1}^{N} \alpha_n \langle \cdot, y_n \rangle_{\varepsilon} y_n,$$

$I_\varepsilon$ denoting the identity operator in $L^2(\varepsilon, 1)$ and $\langle \cdot, \cdot \rangle_\varepsilon$ the scalar product therein.

Since the total impedance potential $v_\kappa + (\kappa/x)$ belongs to $L_p(\varepsilon, 1)$, there exists a transformation operator $I_\varepsilon + K_\varepsilon$ that maps the functions $s(\cdot, \lambda) := \sin(\lambda x - \lambda + \theta)$ on $(\varepsilon, 1)$ into the solution $y(\cdot, \lambda)$ of the equation $\ell(\kappa, v_\kappa) y = \lambda y$ on the interval $(\varepsilon, 1)$, see [3]. The operator $K_\varepsilon$ is an integral operator with an upper-triangular kernel and belongs to the algebra $\mathcal{G}_p(\varepsilon, 1)$ consisting of all integral operators in $L^2(\varepsilon, 1)$ whose kernels $k$ have the property that the mappings

$$x \mapsto k(x, \cdot), \quad x \mapsto k(\cdot, x)$$

are continuous from $[\varepsilon, 1]$ to $L_p(\varepsilon, 1)$. Therefore the above identity can be recast in the form

$$I_\varepsilon = (I_\varepsilon + K_\varepsilon) \left[ \lim_{N \to \infty} \sum_{n=1}^{N} \alpha_n \langle \cdot, s_n \rangle_{\varepsilon} s_n \right] (I_\varepsilon + K_\varepsilon^*),$$

where $s_n := s(\cdot, \lambda_n)$.
where \( s_n := s(\cdot, \lambda_n) \). In other words, the operator

\[
T_\varepsilon := \text{s-lim}_{N \to \infty} \sum_{n=1}^{N} \alpha_n (\cdot, s_n) \varepsilon s_n
\]

can be factorized \[18, \text{Chapter IV}\] as

\[
T_\varepsilon = (I_\varepsilon + K_\varepsilon)^{-1} (I_\varepsilon + K_\varepsilon^*)^{-1}.
\] (5.1)

We notice that the operator \( T_\varepsilon \), being a compression to \( L_2(\varepsilon, 1) \) of the operator \( I + F \) with an integral operator \( F \) belonging to \( \mathcal{G}_p(0, 1) \), see \[3\], differs from \( I_\varepsilon \) by an operator from \( \mathcal{G}_p(\varepsilon, 1) \). Moreover, \( T_\varepsilon \) is positive since \( \{s(\cdot, \lambda_n)\}_{n \in \mathbb{N}} \) is a complete set in \( L_2(\varepsilon, 1) \), for every \( \varepsilon > 0 \). It follows that \( T_\varepsilon \) can be (uniquely) factorized in \( \mathcal{G}_p(\varepsilon, 1) \), and comparison with (5.1) shows that \( (I_\varepsilon + K_\varepsilon)^{-1} \) and its adjoint are the corresponding factors.

Therefore the transformation operator \( I_\varepsilon + K_\varepsilon \) on \( (\varepsilon, 1) \) is uniquely determined by \( T_\varepsilon \), i.e., uniquely determined by the eigenvalues \( (\lambda_n^2)_{n \in \mathbb{N}} \) and the norming constants \( (\alpha_n)_{n \in \mathbb{N}} \). Now the eigenfunctions \( y(\cdot, \lambda_n) \) of the operator \( S(\theta, \kappa, v_\kappa) \) corresponding to the eigenvalue \( \lambda_n^2 \) is equal to

\[
y(x, \lambda_n) = s_n(x) + \int_x^1 k_\varepsilon(x, t)s_n(t) \, dt
\]

for \( x > \varepsilon \). Since \( \varepsilon \) was arbitrary, these eigenfunctions are uniquely determined on the whole interval \( (0, 1) \) by the spectral data, and so is the Bessel operator \( S(\theta, \kappa, v_\kappa) \).

**Remark 5.2.** If \( \theta = \pi/2 \), the corresponding impedance potential \( v_\kappa \) is unique by the results of Section 2.2 and it can be explicitly given as follows. For every \( \varepsilon > 0 \), the above operator \( K_\varepsilon \) uniquely determines the integral operator \( R_\varepsilon \) (that solves the related Krein equation) and the kernel \( r_\varepsilon \) of \( R_\varepsilon \) determines the potential \( v_\kappa(x) + \kappa/x \) on \( (\varepsilon, 1) \) via

\[
v_\kappa(x) + (\kappa/x) = r_\varepsilon(x, 0),
\]

see \[3\]. Since \( \varepsilon \) is arbitrary, we conclude that the spectral data determine the potential \( v_\kappa(x) + \kappa/x \) uniquely.

In the Dirichlet case \( \theta = 0 \), such a unique reconstruction of \( v_\kappa \) via the Krein equation is not possible. Moreover, by Proposition 2.5 we have a family of equivalent representations of the operator \( S(0, \kappa, v_{\kappa, m}) \) parametrized by the mean value \( m \in \mathbb{R} \) of the impedance potential \( v_{\kappa, m} \).

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Appendix A. Around Hardy operators

We collect here several auxiliary results whose proof is based on the general form of the Hardy inequality [20, Section 9.9].

Lemma A.1. Assume that \( v \in L^p(0, 1) \) and \( \kappa \in \mathbb{N} \). Then there exists \( \hat{v} \in L^p(0, 1) \) such that the differential expressions \( \ell(-\kappa, v) \) and \( \ell(\kappa - 1, \hat{v}) \) coincide. Moreover, for every \( d > 1 - 2\kappa \) the function \( \hat{v} \) can be chosen so that \( \hat{v} - v \) is continuous and \( (\hat{v} - v)(1) = d \).

Proof. We shall find a solution \( z \) of the equation \( \ell(-\kappa, v)z = 0 \) that has the form \( z(x) = x^{1-\kappa} \hat{z}(x) \) for a suitable function \( \hat{z} \in W^1_p(0, 1) \) that is positive on \([0, 1] \). Then

\[
-\left( \frac{d}{dx} + \frac{\kappa}{x} - v \right) \left( \frac{d}{dx} - \frac{\kappa}{x} + v \right) = -\left( \frac{d}{dx} + \frac{z'}{z} \right) \left( \frac{d}{dx} - \frac{z'}{z} \right)
\]

by Lemma 2.1, and it remains to observe that

\[
-\frac{z'}{z} = \frac{\kappa - 1}{x} + \hat{v}
\]

with \( \hat{v} := -\frac{z'}{\hat{z}} \in L^p(0, 1) \).

The solution \( z \) is explicitly given as

\[
z(x) = \frac{x^\kappa}{w(x)} \left[ 1 + c \int_x^1 \frac{w^2(t)}{t^{2\kappa}} \, dt \right]
\]

with \( w(x) = \exp\left[-\int_x^1 v(t) \, dt\right] \in W^1_p(0, 1) \) and \( c > 0 \), so that

\[
\hat{z}(x) = \frac{x^{2\kappa - 1}}{w(x)} + \frac{cx^{2\kappa - 1}}{w(x)} \int_x^1 \frac{w^2(t)}{t^{2\kappa}} \, dt.
\]

By Lemma A.2 we have \( \hat{z} \in W^1_p(0, 1) \); moreover, \( \hat{z}(0) = cw(0)/(2\kappa - 1) > 0 \) and hence \( \hat{z} \) is positive on \([0, 1] \).

Next we see that

\[
z'(x) = \left( \frac{\kappa}{x} - v \right) z(x) - cx^{-\kappa} w(x),
\]

so that

\[
\frac{\kappa - 1}{x} + \hat{v}(x) = -\frac{z'(x)}{z(x)} = -\frac{\kappa}{x} + v + \frac{cx^{-\kappa} w(x)}{z(x)}.
\]

As a result, we obtain that \( \hat{v} - v \) is continuous on \((0, 1] \) and \( (\hat{v} - v)(1) = 1 - 2\kappa + c \). The lemma is proved. \( \square \)
Lemma A.2. For every $p \in [1, \infty)$ and every $\alpha > 2 - 1/p$ the operator $T$ given by

$$(Tf)(x) := x^{\alpha-1} \int_{x}^{1} \frac{f(t)}{t^\alpha} \, dt$$

is continuous in $W^1_p(0, 1)$; moreover, $(Tf)(0) = f(0)/(\alpha - 1)$.

Proof. The value of $Tf$ at $x = 0$ can be calculated using de L’Hospital’s rule. Writing $f(t)$ as $f(1) - \int_{t}^{1} f'(s) \, ds$ and then integrating by parts, we obtain

$$(Tf)(x) = \frac{f(x)}{\alpha - 1} - \frac{x^{\alpha-1} f(1)}{\alpha - 1} + \frac{x^{\alpha-1}}{\alpha - 1} \int_{x}^{1} \frac{f'(s)}{s^\alpha} \, ds.$$ 

Since the first two summands are linear transformations that are bounded in $W^1_p(0, 1)$, it remains to prove that the third one is also bounded in $W^1_p(0, 1)$. Indeed, the function

$$g(x) := \frac{x^{\alpha-1}}{\alpha - 1} \int_{x}^{1} \frac{f'(s)}{s^\alpha} \, ds$$

is locally absolutely continuous and

$$g'(x) = x^{\alpha-2} \int_{x}^{1} \frac{f'(s)}{s^\alpha} \, ds - \frac{f'(x)}{\alpha - 1}.$$ 

Since the Hardy operator

$$h \mapsto x^{\alpha-2} \int_{x}^{1} \frac{h(s)}{s^\alpha} \, ds$$

is continuous in $L_p(0, 1)$ [20, Section 9.9], we see that $g'$ depends continuously in $L_p(0, 1)$ on $f'$. This shows that $Tf \in W^1_p(0, 1)$ and that $T$ is continuous in $W^1_p(0, 1)$. \qed

An analogue of the above lemma for the adjoint operator $T^*$ reads:

Lemma A.3. For every $p \in [1, \infty)$ and every $\alpha > 1 + 1/p$ the operator $T^*$ given by

$$(T^*f)(x) := \int_{0}^{x} \frac{t^{\alpha-1} f(t)}{x^\alpha} \, dt$$

is continuous in $W^1_p(0, 1)$; moreover, $(T^*f)(0) = f(0)/\alpha$. 


References


