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On optimal orientations of cartesian products of graphs (I)

K.M. Koh*, E.G. Tay

Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260, Singapore

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Abstract

For a graph G, let $\mathscr{D}(G)$ be the family of strong orientations of G, and define $\vec{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. Let $G \times H$ denote the cartesian product of the graphs G and H. In this paper, we determine completely the values of $\vec{d}(K_m \times P_n)$, $\vec{d}(K_m \times K_n)$ and $\vec{d}(K_n \times C_{2k+1})$, except $\vec{d}(K_3 \times C_{2k+1})$, $k \geqslant 2$, where K_n, P_n and C_n denote the complete graph, path and cycle of order n, respectively. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Let G (resp., D) be a graph (resp., digraph) with vertex set $V(G)$ (resp., $V(D)$). For $v \in V(G)$, the *eccentricity* $e(v)$ of v is defined as $e(v) = \max\{d(v,x) \mid x \in V(G)\}$, where $d(v,x)$ denotes the distance from v to x. The notion $e(v)$ in D is similarly defined. The *diameter* of G (resp., D), denoted by $d(G)$ (resp., $d(D)$), is defined as $d(G) = \max\{e(v) \mid v \in V(G)\}$ (resp., $d(D) = \max\{e(v) \mid v \in V(D)\}$).

An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if every two vertices in D are mutually reachable in D. An edge e in a connected graph G is a *bridge* if $G - e$ is disconnected. Robbins' celebrated one-way street theorem [16] states that *a connected graph G has a strong orientation if and only if no edge of G is a bridge.* Efficient algorithms for finding a strong orientation for a bridgeless connected graph can be found in Roberts [17], Boesch and Tindell [1] and Chung et al. [2]. Boesch and Tindell [1] extended Robbins' result to mixed graphs where edges could be directed or undirected. Chung et al. [2] provided a linear-time algorithm for testing whether a mixed graph has a strong orientation and finding one if it does. As another possible way of extending Robbins' theorem, Boesch and Tindell [1] (see also [3]) introduced further the notion

* Corresponding author. E-mail: matkohkm@leonis.nus.sg.

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$\rho(G)$ given below. Given a connected graph G containing no bridges, let $\mathscr{D}(G)$ be the family of strong orientations of G. Define

$$\rho(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\} - d(G).$$

The first term on the right-hand side of the above equality is essential. Let us write

$$\vec{d}(G) = \min\{d(D) \mid (D \in \mathscr{D}(G)\}.$$

The problem of evaluating $\vec{d}(G)$ for an arbitrary connected graph G is very difficult. As a matter of fact, Chvátal and Thomassen [3] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

On the other hand, the parameter $\vec{d}(G)$ has been studied in various classes of graphs including complete graphs [1,12,15], complete bipartite graphs [1,22,4], complete k-partite graphs for $k \geqslant 3$ [5,7,8,14] and n-cubes [14,22,13]. Let $G \times H$ denote the cartesian product of two graphs G and H, and P_k the path of order k (i.e. of length $k-1$), C_k the cycle of order k (i.e. of length k) and K_n the complete graph of size n. Roberts and Xu [18–21], and independently Koh and Tan [6], evaluated the quantity $\vec{d}(P_m \times P_n)$. Very recently, Koh and Tay [9–11] evaluated the quantities $\vec{d}(C_{2n} \times P_k)$, $\vec{d}(C_{2m} \times C_{2n})$ and $\vec{d}(G_1 \times G_2 \times \cdots \times G_m)$, where $\{G_i \mid 1 \leqslant i \leqslant m\}$ is certain combination of paths and cycles.

In this paper, we shall focus on the products $K_m \times P_n$, $K_m \times K_n$ and $K_p \times C_{2k+1}$, where $m \geqslant 2$, $n \geqslant 2$, $p \geqslant 4$ and $k \geqslant 1$ and establish the following results:

Theorem 1. *For $m \geqslant 2$ and $n \geqslant 2$,*
(i)

$$\vec{d}(K_m \times P_n) = \begin{cases} n+2 & \text{if } (m,n) \in \{(2,3),(2,5),(3,2)\}, \\ n+1 & \text{otherwise;} \end{cases}$$

(ii)

$$\rho(K_m \times P_n) = \begin{cases} 2 & \text{if } (m,n) \in \{(2,3),(2,5),(3,2)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2. *For $m \geqslant 2$ and $n \geqslant 2$,*
(i)

$$\vec{d}(K_m \times K_n) = \begin{cases} 4 & \text{if } (m,n) = (3,2), \\ 3 & \text{otherwise;} \end{cases}$$

(ii)

$$\rho(K_m \times K_n) = \begin{cases} 2 & \text{if } (m,n) = (3,2), \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3. *For* $m \geqslant 4$ *and* $k \geqslant 1$,
(i) $\vec{d}(K_m \times C_{2k+1}) = k+2$;
(ii) $\rho(K_m \times C_{2k+1}) = 1$.

Note that the case when $m=3$ in Theorem 3 has not been settled yet. Let $H \in \{P_n, K_n, C_{2k+1}\}$. In showing that $\rho(K_m \times H) = 1$ for almost all the cases as shown above, K_4 always poses difficulties due to the fact that $\rho(K_4) = 2$ (while $\rho(K_m) = 1$ for $m \geqslant 3$, $m \neq 4$). We have, however, managed to show that $\rho(K_4 \times H) = 1$.

2. Notation and terminology

The *cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either '$u_1 = v_1$ and $u_2 v_2 \in E(G_2)$' or '$u_2 = v_2$ and $u_1 v_1 \in E(G_1)$'.

We write $V(K_m \times P_n) = V(K_m \times K_n) = V(K_m \times C_n) = \{(i,j) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. Thus, two distinct vertices (i,j) and (i',j') are adjacent in $K_m \times P_n$ iff either '$i = i'$ and $|j - j'| = 1$' or '$j = j'$'; adjacent in $K_m \times K_n$ iff either $i = i'$ or $j = j'$; and adjacent in $K_m \times C_n$ iff either '$i = i'$ and $j - j' \equiv \pm 1 \pmod{n}$' or '$j = j'$'.

Let $H \in \{P_n, K_n, C_n\}$, and let $F \in \mathscr{D}(K_m \times H))$ and A a subdigraph of F. The eccentricity, outdegree and indegree of a vertex (i,j) in A are denoted, respectively, by $e_A(i,j), s_A(i,j)$ and $s_A^-(i,j)$. The subscript A is omitted if $A = F$.

Let D be a digraph. A dipath (resp., dicycle) in D is simply called a path (resp., cycle) in D. For $X \subseteq V(D)$, the subdigraph of D induced by X is denoted by $D[X]$. Given $F \in \mathscr{D}(K_m \times H)$ and $1 \leqslant j \leqslant n$, let

$$F_j = F[V(K_m) \times \{j\}],$$

and for $1 \leqslant i \leqslant m$, let

$$F^i = F[\{i\} \times V(H)].$$

Let $A \in \mathscr{D}(K_m)$ and $B \in \mathscr{D}(K_n)$. We write $F_j \equiv A$ (resp., $F^i \equiv B$) if the mapping $\alpha : F_j \to A$ defined by $\alpha(i,j) = i$ (resp., $\beta : F^i \to B$ defined by $\beta(i,j) = j$) is an isomorphism of F_j onto A (resp., F^i onto B). We also write $F_j \equiv F_s$ if $F_j \equiv A$ and $F_s \equiv A$; and $F^i \equiv F^r$ if $F^i \equiv B$ and $F^r \equiv B$.

For $x, y \in V(D)$, we write '$x \to y$' or '$y \leftarrow x$' if x is adjacent to y in D. The *converse* of D, denoted by $\tilde{D}$, is the digraph obtained from D by reversing each arc in D.

3. The graphs $K_m \times P_n$ where $m \geqslant 4$

In this section, we shall show that $\vec{d}(K_m \times P_n) = n + 1$ for all $m \geqslant 4$ and $n \geqslant 2$. First of all, we have the following observation for the general case.

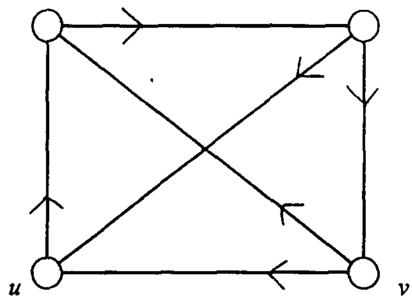

Fig. 1.

Lemma 1. $\vec{d}(K_m \times P_n) \geqslant n+1$ *for all* $m \geqslant 2$ *and* $n \geqslant 2$.

Proof. Let $F \in \mathscr{D}(K_m \times P_n)$. Clearly, $(i, n-1) \to (i, n)$ in F for some i, $1 \leqslant i \leqslant m$. But then $d((i,n),(i,1)) \geqslant n+1$ in F. $\square$

Proposition 1. $\vec{d}(K_m \times P_n) = n+1$ *for all* $m \geqslant 4$ *and* $n \geqslant 2$.

Proof. By Lemma 1, it suffices to provide an orientation of $K_m \times P_n$ of diameter $n+1$. It is known (see [1,12]) that

$$\vec{d}(K_m) = \begin{cases} 2 & \text{if } m \neq 4, \\ 3 & \text{if } m = 4. \end{cases}$$

We observe that $|\mathscr{D}(K_4)| = 1$, up to isomorphism. Also, for $A \in \mathscr{D}(K_4)$, there exists a unique pair of vertices u, v in A such that $d(u,v) = 3$ as shown in Fig. 1.

Let $A \in \mathscr{D}(K_{m-1})$ such that

$$d(A) = \begin{cases} 2 & \text{if } m \neq 5, \\ 3 & \text{if } m = 5. \end{cases}$$

and let $B \in \mathscr{D}(K_m)$ such that

$$d(B) = \begin{cases} 2 & \text{if } m \neq 4, \\ 3 & \text{if } m = 4, \end{cases}$$

and if $d(u,v) = 3$ in B, then $u \neq m$ and $v \neq m$. For convenience, let $V(A) = \{1, 2, \ldots, m-1\}$.

Now, define an orientation F of $K_m \times P_n$ as follows:

(i) $F[V(A) \times \{1\}] \equiv A$ and for $1 \leqslant i \leqslant m-1$, $(m,1) \to (i,1)$.

(ii) For $2 \leqslant j \leqslant n-1$, $F_j \equiv B$.

(iii) $F[V(A) \times \{n\}] \equiv \tilde{A}$ and for $1 \leqslant i \leqslant m-1$, $(i,n) \to (m,n)$.

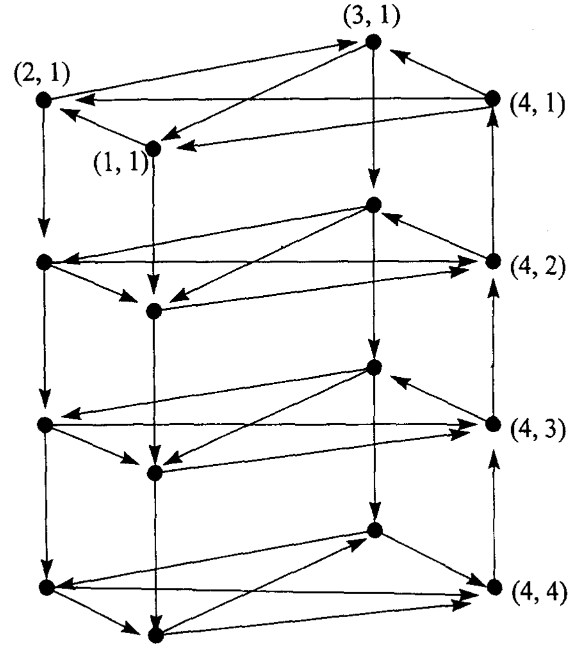

Fig. 2.

(iv) For $1 \leqslant i \leqslant m-1$, $(i,1) \to (i,2) \to \cdots \to (i,n)$.

(v) $(m,n) \to (m,n-1) \to \cdots \to (m,1)$.

Such an orientation F of $K_4 \times P_4$ is shown in Fig. 2.

We shall now prove that $d(F) = n+1$ by showing that $e(x) \leqslant n+1$ for each vertex x in F. There are six cases to consider.

Case 1(a): $x = (m,1)$.

1. Clearly, $d(x,y) \leqslant 1$ for all $y \in V(F_1)$.
2. For $2 \leqslant a \leqslant n-1$, the existence of the paths: $(m,1)(j,1)(j,2)\ldots(j,a)$, where $j = 1,2,\ldots,m-1$ in F and the fact that $d((j,a),(m,a)) \leqslant 2$ in B show that $d(x,y) \leqslant n+1$ for all $y \in V(F_a)$.
3. The existence of the paths $(m,1)(j,1)(j,2)\ldots(j,n)(m,n)$, $j = 1,2,\ldots,m-1$ in F shows that $d(x,y) \leqslant n+1$ for all $y \in V(F_n)$.

Case 1(b): $x = (i,1)$ *where* $1 \leqslant i \leqslant m-1$.

1. The fact that $d(A) \leqslant 3$ and that $d(x,(m,1)) \leqslant d(x,(i,2)) + d((i,2),(m,2)) + d((m,2),(m,1)) \leqslant 1+2+1$ if $n \geqslant 3$ and the existence of the path $(i,1)(i,2)(m,2)(m,1)$ if $n=2$ show that $d(x,y) \leqslant n+1$ for all $y \in V(F_1)$.
2. For $2 \leqslant a \leqslant n-1$ and $1 \leqslant j \leqslant m$, $d(x,(j,a)) \leqslant d(x,(i,a)) + d((i,a),(j,a)) \leqslant n-2+3 = n+1$. Thus, $d(x,y) \leqslant n+1$ for all $y \in V(F_a)$.
3. Let $y = (j,n) \in V(F_n)$. If $d(x,(j,1)) \leqslant 1$ in F_1, then $(i,1)(j,1)(j,2)\ldots(j,n)(m,n)$ is a path of length $n+1$ in F. If $(j,1) \to x$ in F_1, then as $F[V(A) \times \{n\}] \equiv \tilde{A}$,

$(i,1)(i,2)\dots(i,n)(j,n)(m,n)$ is a path of length $n+1$ in F. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_n)$.

Case 2(a): $x=(m,a)$ *where* $2\leqslant a\leqslant n-1$.

1. For $1\leqslant j\leqslant m$, $d(x,(j,1))\leqslant d(x,(m,1))+d((m,1),(j,1))\leqslant n-2+1=n-1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_1)$.
2. For $1\leqslant j\leqslant m$ and $2\leqslant b<a$, $d(x,(j,b))\leqslant d(x,(m,b))+d((m,b),(j,b))\leqslant n-3+2=n-1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_b)$.
3. Clearly, $d(x,y)\leqslant 2$ for all y in $V(F_a)$.
4. For $1\leqslant j\leqslant m-1$ and $a<b\leqslant n-1$, $d(x,(j,b))\leqslant d(x,(j,a))+d((j,a),(j,b))\leqslant 2+n-3=n-1$ and $d(x,(m,b))\leqslant d(x,(p,a))+d((p,a),(p,b))+d((p,b),(m,b))\leqslant 1+n-3+2=n$ for some $1\leqslant p\leqslant m-1$ such that $x\rightarrow(p,a)$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_b)$.
5. For $1\leqslant j\leqslant m-1$, $d(x,(j,n))\leqslant d(x,(j,a))+d((j,a),(j,n))\leqslant 2+n-2=n$. Also, $d(x,(m,n))\leqslant d(x,(p,a))+d((p,a),(p,n))+d((p,n),(m,n))\leqslant 1+n-2+1=n$ for some $1\leqslant p\leqslant m-1$ such that $x\rightarrow(p,a)$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_n)$.

Case 2(b): $x=(i,a)$ *where* $1\leqslant i\leqslant m-1$ *and* $2\leqslant a\leqslant n-1$.

1. For $1\leqslant j\leqslant m$, $d(x,(j,1))\leqslant d(x,(m,a))+d((m,a),(m,1))+d((m,1),(j,1))\leqslant 2+n-2+1=n+1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_1)$.
2. For $1\leqslant j\leqslant m$ and $2\leqslant b<a$, $d(x,(j,b))\leqslant d(x,(m,a))+d((m,a),(m,b))+d((m,b),(j,b))\leqslant 2+n-3+2=n+1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_b)$.
3. Clearly, $d(x,y)\leqslant 3$ for all y in $V(F_a)$.
4. For $1\leqslant j\leqslant m$ and $a<b\leqslant n-1$, $d(x,(j,b))\leqslant d(x,(i,b))+d((i,b),(j,b))\leqslant n-3+3=n$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_b)$.
5. For $1\leqslant j\leqslant m$, $d(x,(j,n))\leqslant d(x,(i,n))+d((i,n),(j,n))\leqslant n-2+3=n+1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_n)$.

Case 3(a): $x=(m,n)$.

1. For $1\leqslant j\leqslant m$ and $1\leqslant a\leqslant n-1$, $d(x,(j,a))\leqslant d(x,(m,a))+d((m,a),(j,a))\leqslant n-1+2=n+1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_a)$.
2. For $1\leqslant j\leqslant m-1$, $d(x,(j,n))\leqslant d(x,(m,n-1))+d((m,n-1),(j,n-1))+d((j,n-1),(j,n))\leqslant 1+2+1$ if $n\geqslant 3$ and the existence of the path $(m,n)(m,1)(j,1)(j,n)$ if $n=2$ show that $d(x,y)\leqslant n+1$ for all $y\in V(F_n)$.

Case 3(b): $x=(i,n)$ *where* $1\leqslant i\leqslant m-1$.

1. For $1\leqslant j\leqslant m$ and $2\leqslant a\leqslant n-1$, $d(x,(j,a))\leqslant d(x,(m,n))+d((m,n),(m,a))+d((m,a),(j,a))\leqslant 1+n-2+2=n+1$ and $d(x,(j,1))\leqslant d(x,(m,n))+d((m,n),(m,1))+d((m,1),(j,1))\leqslant 1+n-1+1=n+1$. Thus, $d(x,y)\leqslant n+1$ for all $y\in V(F_a)\cup V(F_1)$.
2. Clearly, $d(x,y)\leqslant 3$ for all y in $V(F_n)$.

The proof is now complete. □

4. The graphs $K_3\times P_n$

In this section, we shall show that $\vec{d}(K_3\times P_2)=4$ and $\vec{d}(K_3\times P_n)=n+1$ for all $n\geqslant 3$.

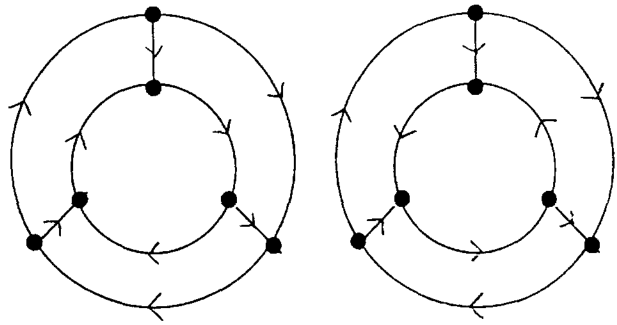

Fig. 3.

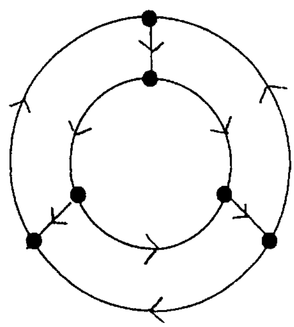

Fig. 4.

Proposition 2. $\vec{d}(K_3 \times P_2) = 4$.

Proof. It follows from Lemma 1 that $\vec{d}(K_3 \times P_2) \geqslant 3$.

Claim. *Let* $F \in \mathscr{D}(K_3 \times P_2)$*. If* $s^-_{F_i}(v) = 2$ *for some* $i = 1,2$ *and some* $v \in V(F_i)$*, then* $d(F) \geqslant 4$.

Suppose to the contrary that $d(F) = 3$. We may assume that $i = 1$ and $v = (1,1)$. This implies that $(1,1) \to (1,2)$ in F. As $d((1,1),(2,1)) \leqslant 3$ and $d((1,1),(3,1)) \leqslant 3$, we have $(1,2) \to (2,2) \to (2,1)$ and $(1,2) \to (3,2) \to (3,1)$ in F. As $d((2,1),(3,1)) \leqslant 3$, $(2,1) \to (3,1)$ in F. But then $d((3,1),(2,1)) \geqslant 4$, a contradiction.

Now suppose $\vec{d}(K_3 \times P_2) = 3$. By the above claim, we have only the two non-isomorphic orientations of $K_3 \times P_2$ of Fig. 3 to consider. However, both orientations have diameter 5. Hence $\vec{d}(K_3 \times P_2) \geqslant 4$.

It remains to provide an orientation of $K_3 \times P_2$ with diameter 4. Such an orientation is shown in Fig. 4.

The proof is thus complete. □

Proposition 3. $\vec{d}(K_3 \times P_n) = n + 1$ *for all* $n \geqslant 3$.

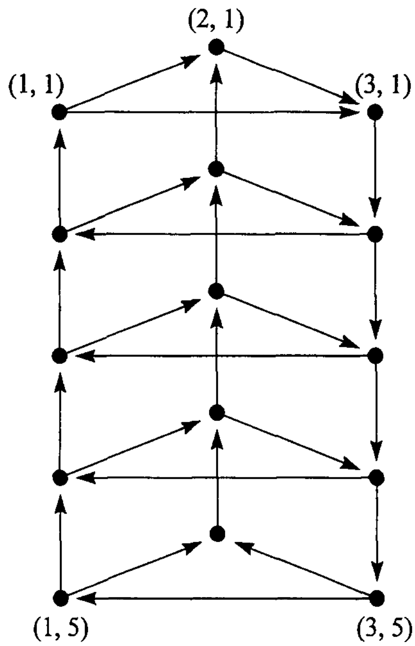

Fig. 5.

Proof. By Lemma 1, it suffices to provide an orientation of $K_3 \times P_n$ of diameter n + 1. Define an orientation F of $K_3 \times P_n$ as follows:

(i) $(1,1)\to(2,1)\to(3,1)$ and $(1,1)\to(3,1)$;

(ii) $(3,n)\to(1,n)\to(2,n)$ and $(3,n)\to(2,n)$;

(iii) For $2\leqslant j\leqslant n-1$, $(1,j)\to(2,j)\to(3,j)\to(1,j)$;

(iv) $(1,n)\to(1,n-1)\to\cdots\to(1,1)$, $(2,n)\to(2,n-1)\to\cdots\to(2,1)$, $(3,1)\to(3,2)\to\cdots\to(3,n)$.

Such an orientation of $K_3 \times P_5$ is shown in Fig. 5.

We shall now prove that $d(F) = n + 1$ by showing that $e(x) \leqslant n + 1$ for all $x \in V(F)$. We shall split our consideration into 9 cases.

Case 1(a): $x = (1,1)$. Consider the following paths in F:

1. $(1,1)(k,1)$ for $k=2,3$;
2. $(1,1)(3,1)(3,2)\ldots(3,n)(k,n)$, $k=1,2$;
3. For $2\leqslant j\leqslant n-1$, $(1,1)(3,1)(3,2)\ldots(3,j)(1,j)(2,j)$.

It can be checked that each of these paths is of length not exceeding $n + 1$ and the paths cover each vertex in F.

Case 1(b): $x=(2,1)$. Consider the following paths in F:

1. $(2,1)(3,1)(3,2)(1,2)(1,1)$;
2. $(2,1)(3,1)(3,2)\ldots(3,n)(k,n)$, $k=1,2$;
3. For $2\leqslant j\leqslant n-1$, $(2,1)(3,1)(3,2)\ldots(3,j)(1,j)(2,j)$.

It can be checked that each of these paths is of length not exceeding n + 1 and the paths cover each vertex in F.

Case 1(c): $x=(3,1)$. Consider the following paths in F:
1. $(3,1)(3,2)(1,2)(1,1)(2,1)$;
2. $(3,1)(3,2)\ldots(3,n)(k,n)$, $k=1,2$;
3. For $2\leqslant j\leqslant n-1$, $(3,1)(3,2)\ldots(3,j)(1,j)(2,j)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 2(a): $x=(1,n)$. Consider the following paths in F:
1. $(1,n)(2,n)$, $(1,n)(1,n-1)(2,n-1)(3,n-1)(3,n)$;
2. For $1\leqslant j\leqslant n-1$, $(1,n)(1,n-1)\ldots(1,j)(2,j)(3,j)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 2(b): $x=(2,n)$. Consider the following paths in F:
1. $(2,n)(2,n-1)(3,n-1)(3,n)(1,n)$;
2. $(2,n)(2,n-1)\ldots(2,1)(3,1)$, $(2,n)(2,n-1)(3,n-1)(1,n-1)(1,n-2)\ldots(1,1)$;
3. For $2\leqslant j\leqslant n-1$, $(2,n)(2,n-1)\ldots(2,j)(3,j)(1,j)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 2(c): $x=(3,n)$. Consider the following paths in F:
1. $(3,n)(k,n)$ for $k=1,2$;
2. $(3,n)(1,n)(1,n-1)\ldots(1,1)(k,1)$, $k=2,3$;
3. For $2\leqslant j\leqslant n-1$, $(3,n)(1,n)(1,n-1)\ldots(1,j)(2,j)(3,j)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 3(a): $x=(1,j)$, *where* $2\leqslant j\leqslant n-1$. Consider the following paths in F:
1. $(1,j)(2,j)(3,j)$;
2. $(1,j)(1,j-1)\ldots(1,1)(k,1)$, $k=2,3$;
3. $(1,j)(2,j)(3,j)(3,j+1)\ldots(3,n)(k,n)$, $k=1,2$;
4. For $j<a\leqslant n-1$, $(1,j)(2,j)(3,j)(3,j+1)\ldots(3,a)(1,a)(2,a)$; for $2\leqslant a<j$, $(1,j)(1,j-1)\ldots(1,a)(2,a)(3,a)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 3(b): $x=(2,j)$, *where* $2\leqslant j\leqslant n-1$. Consider the following paths in F:
1. $(2,j)(3,j)(1,j)$;
2. $(2,j)(2,j-1)\ldots(2,1)(3,1)$, $(2,j)(3,j)(1,j)(1,j-1)\ldots(1,1)$;
3. $(2,j)(3,j)(3,j+1)\ldots(3,n)(k,n), k=1,2$;
4. For $j<a\leqslant n-1$, $(2,j)(3,j)(3,j+1)\ldots(3,a)(1,a)(2,a)$; for $2\leqslant a<j$, $(2,j)(2,j-1)\ldots(2,a)(3,a)(1,a)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

Case 3(c): $x=(3,j)$, *where* $2\leqslant j\leqslant n-1$. Consider the following paths in F:
1. $(3,j)(1,j)(2,j)$;
2. $(3,j)(1,j)(1,j-1)\ldots(1,1)(k,1)$, $k=2,3$;
3. $(3,j)(3,j+1)\ldots(3,n)(k,n)$, $k=1,2$;

4. For $j<a\leqslant n-1$, $(3,j)(3,j+1)\dots(3,a)(1,a)(2,a)$; for $2\leqslant a<j$, $(3,j)(1,j)(1,j-1)\dots(1,a)(2,a)(3,a)$.

It can be checked that each of these paths is of length not exceeding $n+1$ and the paths cover each vertex in F.

The proof is now complete. □

Roberts and Xu [18] and, independently, Koh and Tan [6] have shown that

$$\vec{d}(K_2\times P_n)=\begin{cases} n+2 & \text{if } n\in\{3,5\},\\ n+1 & \text{otherwise.}\end{cases}$$

Combining this with Propositions 1–3 and noting that $d(K_m\times P_n)=n$, we have Theorem 1.

5. The graphs $K_m\times K_3$

In this section, we shall show that $\vec{d}(K_m\times K_3)=3$ for all $m\geqslant 3$. But first of all, we have the following inequality for the general case.

Lemma 2. *$\vec{d}(K_m\times K_n)\geqslant 3$ for all $m\geqslant 3$ and $n\geqslant 3$.*

Proof. Suppose to the contrary that there exists $F\in\mathscr{D}(K_m\times K_n)$ such that $d(F)=2$. We may assume $(i,2)\to(i,1)$ for some $i=1,2,\dots,m$ in F. Let $j=1,2,\dots,m, j\neq i$. As $d((i,1),(j,2))=2$ in $K_m\times K_n$, we must have $(i,1)\to(j,1)\to(j,2)$ in F. Let $k=1,2,\dots,m$, $k\neq i,j$. As $d((i,1),(k,2))=2$ in $K_m\times K_n$, we must have $(i,1)\to(k,1)\to(k,2)$ in F. But then $d((k,2),(j,1))\geqslant 3$ in F, a contradiction. The result thus follows. □

Proposition 4. *$\vec{d}(K_m\times K_3)=3$ for all $m\geqslant 3$.*

Proof. By Lemma 2, it suffices to provide an orientation of $K_m\times K_3$ of diameter 3.

For the case when $m=4$, the orientation of Fig. 6 is a desired one.

We now consider the case when $m\neq 4$. As $m\neq 4$, there exists $A\in\mathscr{D}(K_m)$ such that $d(A)=2$. Define an orientation F of $K_m\times K_3$ as follows:

(i) $F_1\equiv F_2\equiv A$ but $F_3\equiv\tilde{A}$;

(ii) For $i=1,2,\dots,m$, $(i,1)\to(i,2)\to(i,3)\to(i,1)$.

We shall now prove that $d(F)=3$ by showing that $e(x)\leqslant 3$ for each vertex x in F. There are three cases to consider.

Case 1: $x=(i,1)$, *where* $i=1,2,\dots,m$.

1. As $d(F_1)=d(A)=2$, it is clear that $d(x,y)\leqslant 2$ in F for all $y\in V(F_1)$.
2. As $d(F_1)=2$ and $(j,1)\to(j,2)$ for all $j=1,2,\dots,m$, it follows that $d(x,y)\leqslant 3$ for all $y\in V(F_2)$.

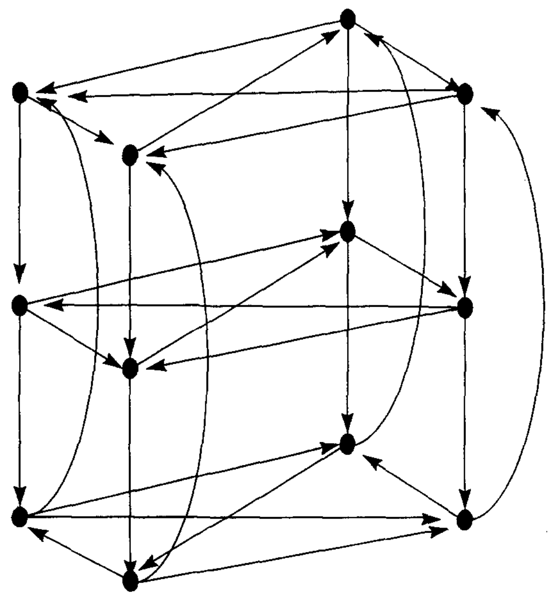

Fig. 6.

3. Let $y=(k,3)\in V(F_3)$. If $d(x,(k,1))\leqslant 1$ in F_1, then $(i,1)(k,1)(k,2)(k,3)$ is a x-y path of length at most 3 in F. If $d(x,(k,1))=2$ in F_1, then as $F_3\equiv\tilde{F}_1$, $(i,1)(i,2)$ $(i,3)(k,3)$ is a x-y path of length 3 in F. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_3)$.

Case 2: $x=(i,2)$, *where* $i=1,2,\ldots,m$.

1. As $d(F_2)=2$, $d(x,y)\leqslant 2$ in F for all $y\in V(F_2)$.
2. Let $y_j=(k,j)\in V(F_j)$, where $j=3,1$. If $d(x,(k,2))\leqslant 1$ in F_2, then $(i,2)(k,2)$ $(k,3)(k,1)$ is a x-y_1 path of length at most 3 in F which contains y_3. If $d(x,(k,2))=2$ in F_2, then $(i,2)(i,3)(k,3)(k,1)$ is a x-y_1 path of length 3 in F which contains y_3. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_3)\cup V(F_1)$.

Case 3: $x=(i,3)$, *where* $i=1,2,\ldots,m$.

1. As $d(F_3)=2, d(x,y)\leqslant 2$ for all $y\in V(F_3)$.
2. Let $y_j=(k,j)\in V(F_j)$, where $j=1,2$. If $d(x,(k,3))\leqslant 1$ in F_3, then $(i,3)(k,3)$ $(k,1)(k,2)$ is a x-y_2 path of length at most 3 in F which contains y_1. If $d(x,(k,3))=2$ in F_3, then $(i,3)(i,1)(k,1)(k,2)$ is a x-y_2 path of length 3 in F which contains y_1. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_1)\cup V(F_2)$.

The proof that $\vec{d}(K_m\times K_3)=3$ is now complete. □

6. The graphs $K_m\times K_4$

We shall proceed in this section to show that $\vec{d}(K_m\times K_4)=3$. As the result that $\vec{d}(K_3\times K_4)=3$ was established in Proposition 4, we shall assume in this section that $m\geqslant 4$.

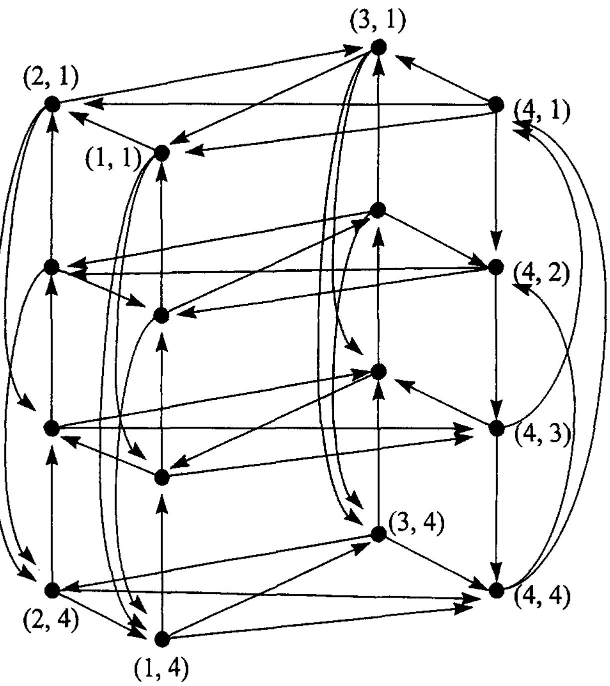

Fig. 7.

Proposition 5. $\vec{d}(K_m \times K_4) = 3$ *for all* $m \geq 4$.

Proof. Let $A \in \mathscr{D}(K_{m-1})$ such that $d(A) = 2$ if $m \neq 5$ and $d(A) = 3$ if $m = 5$. For convenience, let $V = V(A) = \{1, 2, \ldots, m-1\}$. Define an orientation F of $K_m \times K_4$ as follows:

(i) $F[V \times \{1\}] \equiv F[V \times \{3\}] \equiv A$ but $F[V \times \{2\}] \equiv F[V \times \{4\}] \equiv \tilde{A}$.

(ii) For $i = 1, 2, \ldots, m-1$, $(m,1) \to (i,1)$ and $(i,4) \to (m,4)$. For $i = 1, 2, \ldots, m-2$, $(m-1,2) \to (m,2) \to (i,2)$ and $(i,3) \to (m,3) \to (m-1,3)$.

(iii) For $i = 1, 2, \ldots, m-1$, $(i,1) \to (i,3)$, $(i,2) \to (i,4)$, $(i,1) \to (i,4) \to (i,3) \to (i,2) \to (i,1)$; $(m,1) \to (m,2) \to (m,3) \to (m,4) \to (m,1)$, $(m,3) \to (m,1)$, $(m,4) \to (m,2)$.

Such an orientation F of $K_4 \times K_4$ is shown in Fig. 7.

We shall now prove that $d(F) = 3$ by showing that $e(x) \leq 3$ for all $x \in V(F)$. We shall split our consideration into 8 cases.

Case 1(a): $x = (m, 1)$.

1. Clearly, $d(x, y) \leq 1$ for all $y \in V(F_1)$.
2. The existence of the paths $(m,1)(m,2)(m,3)$ and $(m,1)(j,1)(j,3)(j,2)$, $j = 1, 2, \ldots, m-1$ in F shows that $d(x, y) \leq 3$ for all $y \in V(F_2) \cup V(F_3)$.
3. The existence of the paths $(m,1)(j,1)(j,4)(m,4)$, $j = 1, 2, \ldots, m-1$ in F shows that $d(x, y) \leq 3$ for all $y \in V(F_4)$.

Case 1(b): $x=(i,1)$, *where* $1\leqslant i\leqslant m-1$.

1. Clearly, $d(x,y)\leqslant 3$ for all y in $V(F_1)\backslash\{(m,1)\}$. Observe that $(i,1)(i,4)\ (m,4)(m,1)$ is a path in F. Thus, $d(x,(m,1))\leqslant 3$.
2. Let $j=1,2,\ldots,m-1$. If $d((i,1),(j,1))\leqslant 1$ in $F_1[V\times\{1\}]$, then $(i,1)(j,1)(j,3)(j,2)$ is a path in F; otherwise, $(i,1)(i,3)(i,2)(j,2)$ is a path in F. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_2)\backslash\{(m,2)\}$. Note that $(i,1)(i,4)(m,4)(m,2)$ is a path in F. Thus, $d(x,(m,2))\leqslant 3$.
3. Let $j=1,2,\ldots,m-1$. If $d((i,1),(j,1))\leqslant 1$ in $F_1[V\times\{1\}]$, then $(i,1)(j,1)(j,3)$ is a path in F; otherwise, $(i,1)(i,4)(j,4)(j,3)$ is a path in F. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_3)\backslash\{(m,3)\}$. For $1\leqslant i\leqslant m-2$, $(i,1)(i,3)(m,3)$ is a path in F. As $s_{F_3[V\times\{3\}]}(m-1,3)>0$, $(m-1,3)\to(p,3)$ for some $p=1,2,\ldots,m-2$. Thus $(m-1,1)(m-1,3)(p,3)(m,3)$ is a path in F. Thus, $d(x,(m,3))\leqslant 3$.
4. Let $j=1,2,\ldots,m-1$. If $d((i,1),(j,1))\leqslant 1$ in $F_1[V\times\{1\}]$, then $(i,1)(j,1)(j,4)$ is a path in F; otherwise, $(i,1)(i,4)(j,4)$ is a path in F. Thus, $d(x,y)\leqslant 2$ for all $y\in V(F_4)\backslash\{(m,4)\}$. Note also that $(i,1)(i,4)(m,4)$ is a path in F. Thus, $d(x,(m,4))\leqslant 2$.

Case 2(a): $x=(m,2)$.

1. The existence of the paths $(m,2)(m,3)(m,1)(j,1)$, $j=1,2,\ldots,m-1$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_1)$.
2. Clearly, $d(x,(j,2))=1$ for all $j=1,2,\ldots,m-2$. As $s^-_{F_2[V\times\{2\}]}(m-1,2)>0$, $(p,2)\to(m-1,2)$ for some $p=1,2,\ldots,m-2$. Thus, $(m,2)(p,2)(m-1,2)$ is a path in F, and so $d(x,(m-1,2))=2$.
3. The existence of the paths $(m,2)(j,2)(j,4)(j,3)$, $j=1,2,\ldots,m-2$ and $(m,2)(m,3)(m-1,3)$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_3)$.
4. As $s^-_{F_4[V\times\{4\}]}(m-1,4)>0$, $(p,4)\to(m-1,4)$ for some $p=1,2,\ldots,m-2$. Thus $(m,2)(p,2)(p,4)(m-1,4)$ is a path in F. The existence of this path together with the paths $(m,2)(j,2)(j,4)(m,4)$, $1\leqslant j\leqslant m-2$, in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_4)$.

Case 2(b): $x=(i,2)$, *where* $1\leqslant i\leqslant m-1$.

1. Let $j=1,2,\ldots,m-1$. If $d((i,2),(j,2))\leqslant 1$ in $F_2[V\times\{2\}]$, then $(i,2)(j,2)(j,1)$ is a path in F; otherwise, $(i,2)(i,1)(j,1)$ is a path in F. The existence of these paths together with the path $(i,2)(i,4)(m,4)(m,1)$ shows that $d(x,y)\leqslant 3$ for all $y\in V(F_1)$.
2. The fact that $d(F_2[V\times\{2\}])\leqslant 3$ and the existence of the path $(i,2)(i,4)(m,4)(m,2)$ show that $d(x,y)\leqslant 3$ for all $y\in V(F_2)$.
3. Let $j=1,2,\ldots,m-1$. If $d((i,2),(j,2))\leqslant 1$ in $F_2[V\times\{2\}]$, then $(i,2)(j,2)(j,4)(j,3)$ is a path in F; otherwise, $(i,2)(i,4)(i,3)(j,3)$ is a path in F. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_3)\backslash\{(m,3)\}$. For $1\leqslant i\leqslant m-2$, $(i,2)(i,4)(i,3)(m,3)$ is a path in F; and for $i=m-1$, $(m-1,2)(m,2)(m,3)$ is a path in F. Thus, $d(x,(m,3))\leqslant 3$.
4. Let $j=1,2,\ldots,m-1$. If $d((i,2),(j,2))\leqslant 1$ in $F_2[V\times\{2\}]$, then $(i,2)(j,2)(j,4)$ is a path in F; otherwise, $(i,2)(i,1)(j,1)(j,4)$ is a path in F. The existence of these paths together with the path $(i,2)(i,4)(m,4)$ shows that $d(x,y)\leqslant 3$ for all $y\in V(F_4)$.

Case 3(a): $x=(m,3)$.

1. The existence of the paths $(m,3)(m,1)(j,1)$, $1\leqslant j\leqslant m-1$, in F shows that $d(x,y)\leqslant 2$ for all $y\in V(F_1)$.
2. The existence of the paths $(m,3)(m,4)(m,2)(j,2)$, $1\leqslant j\leqslant m-2$, and $(m,3)(m-1,3)(m-1,2)$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_2)$.
3. The existence of the paths $(m,3)(m,1)(j,1)(j,3)$, $1\leqslant j\leqslant m-1$, in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_3)$.
4. The existence of the paths $(m,3)(m,1)(j,1)(j,4)$, $1\leqslant j\leqslant m-1$, and $(m,3)(m,4)$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_4)$.

Case 3(b): $x=(i,3)$, *where* $1\leqslant i\leqslant m-1$.

1. Let $j=1,2,\ldots,m-1$. If $d((i,3),(j,3))\leqslant 1$ in $F_3[V\times\{3\}]$, then $(i,3)(j,3)(j,2)(j,1)$ is a path in F; otherwise, $(i,3)(i,2)(j,2)(j,1)$ is a path in F. Thus, $d(x,y)\leqslant 3$ for all $y\in V(F_1)\backslash\{(m,1)\}$. For $1\leqslant i\leqslant m-2$, $(i,3)(m,3)(m,1)$ is a path in F; and for $i=m-1$, there exists $p=1,2,\ldots,m-2$ such that $(m-1,3)(p,3)(m,3)(m,1)$ is a path in F. Thus, $d(x,(m,1))\leqslant 3$.
2. Let $j=1,2,\ldots,m-1$. If $d((i,3),(j,3))\leqslant 1$ in $F_3[V\times\{3\}]$, then $(i,3)(j,3)(j,2)$ is a path in F; otherwise, $(i,3)(i,2)(j,2)$ is a path in F. Thus, $d(x,y)\leqslant 2$ for all $y\in V(F_2)\backslash\{(m,2)\}$. For $1\leqslant i\leqslant m-2$, $(i,3)(m,3)(m,1)(m,2)$ is a path in F; and for $i=m-1$, $(m-1,3)(m-1,2)(m,2)$ is a path in F. Thus, $d(x,(m,2))\leqslant 3$.
3. Clearly, $d(x,y)\leqslant 3$ for all y in $V(F_3)\backslash\{(m,3)\}$. For $1\leqslant i\leqslant m-2$, $(i,3)(m,3)$ is a path in F; and for $i=m-1$, there exists $p=1,2,\ldots,m-2$ such that $(m-1,3)(p,3)(m,3)$ is a path in F. Thus, $d(x,(m,3))\leqslant 2$.
4. Let $j=1,2,\ldots,m-1$. If $d((i,3),(j,3))\leqslant 1$ in $F_3[V\times\{3\}]$, then $(i,3)(j,3)(j,2)(j,4)$ is a path in F; otherwise, $(i,3)(i,2)(i,4)(j,4)$ is a path in F. Thus, $d(x,y)\leq 3$ for all $y\in V(F_4)\backslash\{(m,4)\}$. For $1\leqslant i\leqslant m-2$, $(i,3)(m,3)(m,4)$ is a path in F; and for $i=m-1$, there exists $p=1,2,\ldots,m-2$ such that $(m-1,3)(p,3)(m,3)(m,4)$ is a path in F. Thus, $d(x,(m,4))\leqslant 3$.

Case 4(a): $x=(m,4)$.

1. The existence of the paths $(m,4)(m,1)(j,1)$, $1\leqslant j\leqslant m-1$, shows that $d(x,y)\leqslant 2$ for all y in $V(F_1)$.
2. The existence of the paths $(m,4)(m,2)(j,2)$, $1\leqslant j\leqslant m-2$, and $(m,4)(m,2)(p,2)(m-1,2)$ for some $p=1,2,\ldots,m-2$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_2)$.
3. The existence of the paths $(m,4)(m,1)(j,1)(j,3)$, $1\leqslant j\leq m-1$, and $(m,4)(m,2)(m,3)$ in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_3)$.
4. The existence of the paths $(m,4)(m,1)(j,1)(j,4)$, $1\leqslant j\leqslant m-1$, in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_4)$.

Case 4(b): $x=(i,4)$, *where* $1\leqslant i\leqslant m-1$.

1. The existence of the paths $(i,4)(m,4)(m,1)(j,1)$, $1\leqslant j\leqslant m-1$, in F shows that $d(x,y)\leqslant 3$ for all $y\in V(F_1)$.
2. Let $j=1,2,\ldots,m-1$. If $d((i,4),(j,4))\leqslant 1$ in $F_4[V\times\{4\}]$, then $(i,4)(j,4)(j,3)(j,2)$ is a path in F; otherwise, $(i,4)(i,3)(i,2)(j,2)$ is a path in F. Also $(i,4)(m,4)(m,2)$ is a path in F. It follows that $d(x,y)\leqslant 3$ for all y in $V(F_2)$.

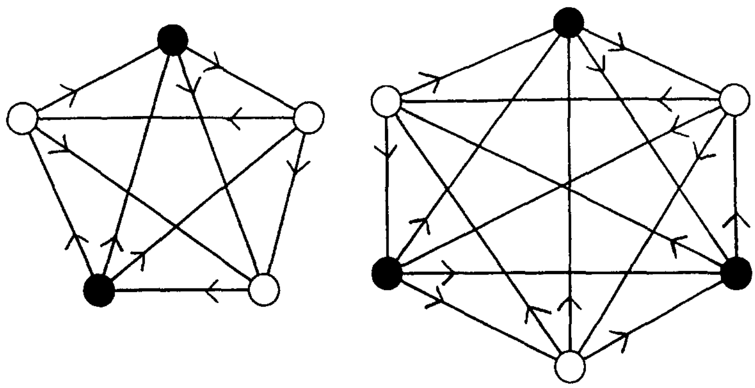

Fig. 8.

3. Let $j=1,2,\ldots,m-1$. If $d((i,4),(j,4))\leqslant 1$ in $F_4[V\times\{4\}]$, then $(i,4)(j,4)(j,3)$ is a path in F; otherwise, $(i,4)(i,3)(j,3)$ is a path in F. Also $(i,4)(m,4)(m,2)(m,3)$ is a path in F. It follows that $d(x,y)\leqslant 3$ for all y in $V(F_3)$.
4. The fact that $d(x,y)\leqslant 3$ for all y in $V(F_4)$ is obvious.

The proof is now complete. □

7. The graphs $K_m \times K_n$ where $m,n\geqslant 5$

In this section we shall prove that $\vec{d}(K_m\times K_n)=3$ for all $m\geqslant 5$ and $n\geqslant 5$.

A *2-colouring* of K_m, $m\geqslant 3$, is a mapping $\theta : V(K_m)\to\{\text{black(b), white(w)}\}$. Let $F\in\mathscr{D}(K_m)$ and θ a 2-colouring of K_m. A 3-cycle C in F is said to be *bichromatic* if $\theta(u)\neq\theta(v)$ for some u, v in $V(C)$. We begin with the following observation.

Lemma 3. *For $m\geqslant 5$, there exist $F\in\mathscr{D}(K_m)$ with $d(F)=2$ and a 2-colouring θ of K_m such that*
(i) *every 3-cycle in F is bichromatic*;
(ii) *if $u\to v$ and $\theta(u)=\theta(v)$, then there exists a $u-v$ path of length not exceeding 3 such that $\theta(x)\neq\theta(u)$ for some internal vertex x of the $u-v$ path.*

Proof. The statement is true for $m=5$ and $m=6$ as shown in Fig. 8.

Assume the statement is true for $m=p\geqslant 5$. Consider $m=p+2$. Let $F\in\mathscr{D}(K_p)$ and θ be a 2-colouring of K_p satisfying the hypothesis. Extend F on K_p to K_{p+2} by assigning $p+2\to p+1$, $p+1\to i$ and $i\to p+2$ for all $i=1,2,\ldots,p$. Extend θ on K_p to K_{p+2} by defining $\theta(p+1)=b$ and $\theta(p+2)=w$ (see Fig. 9). Let F' and θ' be the resulting extensions of F and θ, respectively.

It is straightforward to check that $d(F')=2$ and that both of F' and θ' satisfy condition (i).

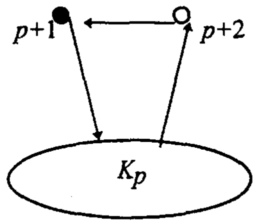

Fig. 9.

We shall now show that (ii) holds. Assume that $u \to v$ and $\theta'(u) = \theta'(v) = b$ (the case that $\theta'(u) = \theta'(v) = w$ can be handled dually). If $u \in V(F)$, then $v \in V(F)$, and thus the result follows by induction. Thus, suppose that $u = p + 1$, and so $v \in V(F)$. Trivially, there exists $x \in V(F)$ such that $\theta'(x) = w$. As $d(F) = 2$, there exists a $x - v$ path Q of length at most 2. Thus ux followed by Q is a required $u - v$ path. □

Remark. Part (i) of Lemma 3 will be used to prove Proposition 6 below, and both (i) and (ii) of Lemma 3 will be applied to establish Proposition 8 in the next section.

We are now in a position to establish the following result.

Proposition 6. $\vec{d}(K_m \times K_n) = 3$ *for all* $m \geqslant 5$ *and* $n \geqslant 5$.

Proof. Let $A \in \mathscr{D}(K_n)$ with $d(A) = 2$. Let $B \in \mathscr{D}(K_m)$ and θ be a 2-colouring of K_m satisfying the conditions stated in Lemma 3. Define an orientation H of $K_m \times K_n$ as follows:
(i) $H_j \equiv B$ for all $j = 1, 2, \ldots, n$;
(ii) For $i = 1, 2, \ldots, m$,

$$H^i \equiv \begin{cases} A & \text{if } \theta(i) = b, \\ \tilde{A} & \text{if } \theta(i) = w. \end{cases}$$

We shall now prove that $d(H) = 3$ by showing that $d(x, y) \leqslant 3$ for all x, y in $V(H)$.
Let $x = (i, j)$ and $y = (i', j')$, where $i, i' \in \{1, 2, \ldots, m\}$ and $j, j' \in \{1, 2, \ldots, n\}$. If $i = i'$, then $d(x, y) \leqslant 2$ as $d(A) = 2$. Thus assume that $i \neq i'$. As $d(B) = 2$, i and i' are contained in a 3-cycle C. By Lemma 3(i), C is bichromatic. By the definition of H given above, $H[V(C) \times V(K_n)] \cong F$, where $F \in \mathscr{D}(K_3 \times K_n)$ as introduced in the proof of Proposition 4. Since $d(F) = 3$, $d(x, y) \leqslant 3$ in H. The proof is thus complete. □

Now, combining Propositions 4–6 with Theorem 1 (for $n = 2$) and noting that $d(K_m \times K_n) = 2$, we arrive at Theorem 2.

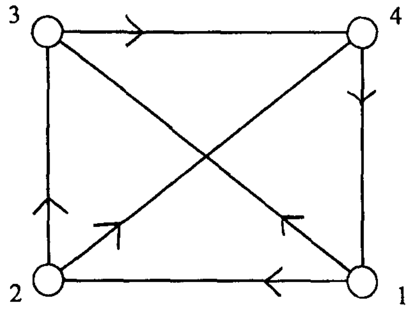

Fig. 10.

8. The graphs $K_m \times C_{2k+1}$ where $m \geqslant 4$ and $k \geqslant 1$

Our aim in this section is to show that $\vec{d}(K_m \times C_{2k+1}) = k + 2$ for all $m \geqslant 4$ and $k \geqslant 1$. First of all, we have the following result for the general case.

Lemma 4. *$\vec{d}(K_m \times C_{2k+1}) \geqslant k + 2$ for all $m \geqslant 2$ and $k \geqslant 1$.*

Proof. Suppose to the contrary that there exists $F \in \mathscr{D}(K_m \times C_{2k+1})$ such that $d(F) = k + 1$. We may assume $(2,1) \to (1,1)$.

As $d((1,1),(2,k+1)) = k+1$ in $K_m \times C_{2k+1}$, we must have $(1,1) \to (1,2)$ in F. As $d((2,k+1),(1,1)) = k+1$, $(2,k+1) \to (2,k) \to (2,k-1) \to \cdots \to (2,1)$. Hence, to ensure that $d((1,1),(2,k+1)) = k+1$, we must have further $(1,2) \to (1,3) \to \cdots \to (1,k+1) \to (2,k+1)$. As $d((2,k+2),(1,2)) \leqslant k+1$, $(2,k+2) \to (2,k+1)$. But then $d((2,k+1),(1,2k+1)) \geqslant k+2$, a contradiction.

The result thus follows. □

The fact that $\rho(K_4) = 2$ requires an ad hoc approach to proving the first result in this section.

Proposition 7. *$\vec{d}(K_4 \times C_{2k+1}) = k + 2$ for all $k \geqslant 1$.*

Proof. As the result that $\vec{d}(K_4 \times K_3) = 3$ was established in Proposition 4, we shall assume that $k \geqslant 2$.

By Lemma 4, it suffices to provide an orientation of $K_4 \times C_{2k+1}$ of diameter $k+2$. First, define $A \in \mathscr{D}(K_4)$ as follows (see Fig. 10):

(i) $1 \to 2 \to 3 \to 4 \to 1$;

(ii) $1 \to 3$ and $2 \to 4$.

Note that

(i) $d_A(3,2) = d_{\tilde{A}}(2,3) = 3$;

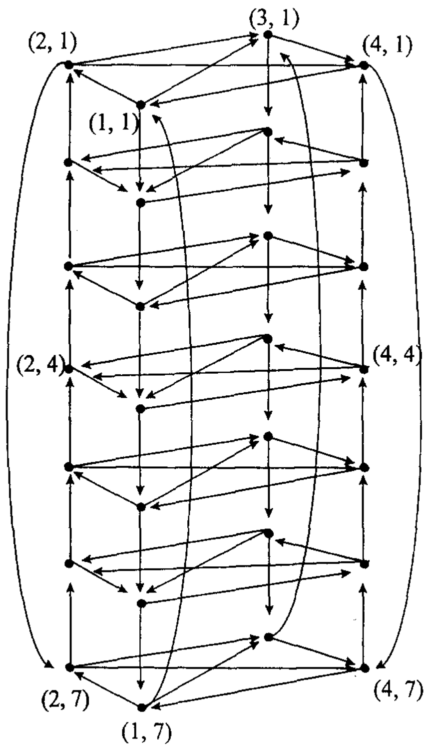

Fig. 11.

(ii) the mapping $f: V(A) \to V(\tilde{A})$ defined by $f(1)=4$, $f(2)=3$, $f(3)=2$ and $f(4)=1$ is an isomorphism from A onto $\tilde{A}$.

Now define $F \in \mathscr{D}(K_4 \times C_{2k+1})$ as follows:

(i) For $i=1,3$, $(i,1) \to (i,2) \to \cdots \to (i,2k+1) \to (i,1)$;

(ii) For $i=2,4$, $(i,2k+1) \to (i,2k) \to \cdots \to (i,1) \to (i,2k+1)$;

(iii) For $j \equiv 1 \pmod 2$, $1 \leqslant j \leqslant 2k+1$, $F_j \equiv A$;

(iv) For $j \equiv 0 \pmod 2$, $2 \leqslant j \leqslant 2k$, $F_j \equiv \tilde{A}$.

Such an orientation F of $K_4 \times C_7$ is shown in Fig. 11.

We shall now prove that $d(F)=k+2$ by showing that $d(x,y) \leqslant k+2$ for all x, y in $V(F)$.

Let $x=(i,j)$ and $y=(i',j')$, where $i,i' \in \{1,2,3,4\}$, $j,j' \in \{1,2,\ldots,2k+1\}$ and j,j' are taken modulo $2k+1$.

Case 1: $i=1$.

1_1. For $j=j'$ and $i'=2,3,4$, $d((1,j),(i',j')) \leqslant 2$.

1_2. For $j+1 \leqslant j' \leqslant j+k+2$, $(1,j)(1,j+1)\ldots(1,j')$ is a path of length not exceeding $k+2$.

1_3. For $j+k+3 \leqslant j' \leqslant j-1$,

- if $F_j \equiv A \equiv F_{j'}$, $(1,j)(2,j)(4,j)(4,j-1)\ldots(4,j')(1,j')$ is a path of length not exceeding $k+1$;

- if $F_j \equiv A$ and $F_{j'} \equiv \tilde{A}$, $(1,j)(2,j)(2,j-1)\dots(2,j')(1,j')$ is a path of length not exceeding k;
- if $F_j \equiv \tilde{A}$ and $F_{j'} \equiv A$, $(1,j)(4,j)(4,j-1)\dots(4,j')(1,j')$ is a path of length not exceeding k; and
- if $F_j \equiv \tilde{A} \equiv F_{j'}$, $(1,j)(4,j)(4,j-1)\dots(4,j')(3,j')(1,j')$ is a path of length not exceeding $k+1$.

1_4. For $j+1 \leqslant j' \leqslant j+k$ and $i'=2,3,4$, $d((1,j),(i',j')) \leqslant d((1,j),(1,j')) + d((1,j'),(i',j')) \leqslant k+2$.

1_5. For $j+k+1 \leqslant j' \leqslant j-1$,

- if $F_j \equiv A \equiv F_{j'}$, $(1,j)(2,j)(2,j-1)\dots(2,j')(i',j')$, where $i'=3,4$, is a path of length not exceeding $k+2$;
- if $F_j \equiv A$ and $F_{j'} \equiv \tilde{A}$, $(1,j)(2,j)(2,j-1)\dots(2,j')$, $(1,j)(3,j)(4,j)(4,j-1)\dots(4,j')$, $(1,j)(3,j)(4,j)(4,j-1)\dots(4,j')(3,j')$, for $j' \neq j+k+1$, and $(1,j) \times (3,j)(3,j+1)\dots(3,j+k+1)$ are paths of length not exceeding $k+2$;
- if $F_j \equiv \tilde{A}$ and $F_{j'} \equiv A$, $(1,j)(4,j)(4,j-1)\dots(4,j')$, $(1,j)(4,j)(4,j-1)\dots(4,j')(1,j')(i',j')$, where $i'=2,3$, $j' \neq j+k+1$, and $(1,j)(1,j+1)\dots(1,j+k+1)(i',j+k+1)$, where $i'=2,3$, are paths of length not exceeding $k+2$; and
- if $F_j \equiv \tilde{A} \equiv F_{j'}$, $(1,j)(4,j)(4,j-1)\dots(4,j')(i',j')$, where $i'=2,3$, are paths of length not exceeding $k+2$.

Case 2: $i=2$.

2_1. For $j=j'$ and $i'=1,3,4$, $d((2,j),(i',j')) \leqslant 3$.

2_2. For $j+k-1 \leqslant j' \leqslant j-1$, $(2,j)(2,j-1)\dots(2,j')$ is a path of length not exceeding $k+2$.

2_3. For $j+1 \leqslant j' \leqslant j+k-2$,

- if $F_j \equiv A \equiv F_{j'}$, $(2,j)(4,j)(1,j)(1,j+1)\dots(1,j')(2,j')$ is a path of length not exceeding $k+1$;
- if $F_j \equiv A$ and $F_{j'} \equiv \tilde{A}$, $(2,j)(3,j)(3,j+1)\dots(3,j')(2,j')$ is a path of length not exceeding k;
- if $F_j \equiv \tilde{A}$ and $F_{j'} \equiv A$, $(2,j)(1,j)(1,j+1)\dots(1,j')(2,j')$ is a path of length not exceeding k; and
- if $F_j \equiv \tilde{A} \equiv F_{j'}$, $(2,j)(1,j)(1,j+1)\dots(1,j')(4,j')(2,j')$ is a path of length not exceeding $k+1$.

2_4. For $j+k+1 \leqslant j' \leqslant j-1$,

- if $F_{j'} \equiv A$, then $d((2,j)(i',j')) \leqslant d((2,j),(2,j')) + d((2,j'),(i',j')) \leqslant k+2$ for $i'=1,3,4$;
- if $F_{j'} \equiv \tilde{A}$, then $d((2,j)(i',j')) \leqslant d((2,j),(2,j')) + d((2,j'),(i',j')) \leqslant k+2$ for $i'=1,4$.

2_5. For $j+k+2 \leqslant j' \leqslant j-1$,

- if $F_{j'} \equiv \tilde{A}$, then $d((2,j)(3,j')) \leqslant d((2,j),(2,j')) + d((2,j'),(3,j')) \leqslant k-1+3 = k+2$.

2_6. If $F_{j+k+1} \equiv \tilde{A}$ and $F_p \equiv A$, where $j+k+2 \leqslant p \leqslant j$, then $d((2,j),(3,j+k+1)) \leqslant d((2,j),(2,p)) + d((2,p),(4,p)) + d((4,p),(4,j+k+1)) + d((4,j+k+1),(3,j+$

$k+1)$). Observe that $d((2,p),(4,p)) = d((4,j+k+1),(3,j+k+1)) = 1$ and $d((2,j),(2,p)) + d((4,p),(4,j+k+1))$ is equal to the distance from $(2,j)$ to $(2,j+k+1)$ in F^2, which is k. Thus, $d((2,j),(3,j+k+1)) \leqslant k+2$.

2_7. For $j+1 \leqslant j' \leqslant j+k$ and $F_j \equiv A \equiv F_{j'}$, $(2,j)(3,j)(3,j+1)\ldots(3,j')(4,j')$ and $(2,j)(3,j)(3,j+1)\ldots(3,p)(1,p)(1,p+1)\ldots(1,j')$, where $j+1 \leqslant p \leqslant j+k-1$ and $F_p \equiv \tilde{A}$, are paths of length not exceeding $k+2$.

2_8. If $F_j \equiv A$ and $F_{j'} \equiv \tilde{A}$, then $(2,j)(3,j)(3,j+1)\ldots(3,j')(1,j')$ for $j+1 \leqslant j' \leqslant j+k$, $(2,j)(3,j)(3,j+1)\ldots(3,j')(1,j')(4,j')$ for $j+1 \leqslant j' \leqslant j+k-1$, and $(2,j)(4,j)(4,j-1)\ldots(4,j+k+1)(4,j+k)$ are paths of length not exceeding $k+2$.

2_9. If $F_j \equiv \tilde{A}$ and $F_{j'} \equiv A$, then $(2,j)(1,j)(1,j+1)\ldots(1,j')(3,j')$ for $j+1 \leqslant j' \leqslant j+k$, $(2,j)(1,j)(1,j+1)\ldots(1,j')(3,j')(4,j')$ for $j+1 \leqslant j' \leqslant j+k-1$, and $(2,j)(2,j-1)\ldots(2,j+k)(4,j+k)$ are paths of length not exceeding $k+2$.

2_{10}. For $j+1 \leqslant j' \leqslant j+k$ and $F_j \equiv \tilde{A} \equiv F_{j'}$, $(2,j)(1,j)(1,j+1)\ldots(1,j')(4,j')$ and $(2,j)(1,j)(1,j+1)\ldots(1,p)(3,p)(3,p+1)\ldots(3,j')$, where $j+1 \leqslant p \leqslant j+k-1$ and $F_p \equiv A$, are paths of length not exceeding $k+2$.

Case 3: $i=3$. The argument is similar to that of $i=2$ since 2 and 3 in $\tilde{A}$ are the isomorphic images of 3 and 2 in A under f, respectively (see Fig. 10).

Case 4: $i=4$. The argument is similar to that of $i=1$ since 1 and 4 in $\tilde{A}$ are the isomorphic images of 4 and 1 in A under f respectively (see Fig. 10).

The proof is now complete. □

Finally, we shall now apply Lemma 3 to determine $\vec{d}(K_m \times C_{2k+1})$, where $m \geqslant 5$.

Proposition 8. $\vec{d}(K_m \times C_{2k+1}) = k+2$ *for* $m \geqslant 5$ *and* $k \geqslant 1$.

Proof. Let $B \in \mathscr{D}(K_m)$ and θ be a 2-colouring of K_m satisfying the conditions stated in Lemma 3. Define an orientation H of $K_m \times C_{2k+1}$ as follows:

(i) $H_j \equiv B$ for all $j = 1,2,\ldots,2k+1$;

(ii) For $i = 1,2,\ldots,m$ and $\theta(i) = b$, $(i,1) \to (i,2) \to \cdots \to (i,2k+1) \to (i,1)$;

(iii) For $i = 1,2,\ldots,m$ and $\theta(i) = w$, $(i,2k+1) \to (i,2k) \to \cdots \to (i,1) \to (i,2k+1)$.

We shall now prove that $d(H) = k+2$ by showing that $d(x,y) \leqslant k+2$ for all x,y in $V(H)$.

Let $x=(i,j)$ and $y=(i',j')$ where $i,i' \in \{1,2,\ldots,m\}$ and $j,j' \in \{1,2,\ldots,2k+1\}$. Note that j and j' are taken modulo $2k+1$. Let $\theta(i) = b$. (The case that $\theta(i) = w$ can be handled similarly.) Suppose $i = i'$. Then $d(x,y) \leqslant k+2$ for $j+1 \leqslant j' \leqslant j+k+2$. So we consider $j+k+3 \leqslant j' \leqslant j-1$. As $d(B) = 2$, i is contained in a 3-cycle C. By Lemma 3, C is bichromatic. Hence, C contains a p such that $\theta(p) = w$. Now $d(x,y) \leqslant d(x,(p,j)) + d((p,j),(p,j')) + d((p,j'),y) = d((p,j),(p,j')) + d(x,(p,j)) + d((p,j'),y) \leqslant k-2+3 = k+1$.

Assume now that $i \neq i'$. Suppose we have

$$(*) \quad (i,j) \to (i',j) \quad \text{and} \quad \theta(i') = b.$$

By Lemma 3, there exists a $(i,j)-(i',j)$ path of length not exceeding 3 such that $\theta(p) = w$ for some internal vertex (p,j) of the $(i,j)-(i',j)$ path. Then for $j+k$

$+2 \leqslant j' \leqslant j-1$, $d(x,y) \leqslant d(x,(p,j)) + d((p,j),(p,j')) + d((p,j'),y) \leqslant d((p,j),(p,j')) + d(x,(p,j)) + d((p,j'),y) \leqslant k-1+3 = k+2$. For $j \leqslant j' \leqslant j+k+1$, $d(x,y) \leqslant d(x,(i,j')) + d((i,j'),y) \leqslant k+1+1 = k+2$.

Suppose $(*)$ does not hold. As $d(B)=2$, i and i' are contained in a 3-cycle C. By Lemma 3, C is bichromatic. Hence, C contains a p such that $\theta(p)=w$ and either $p=i'$ or $p \rightarrow i'$. Then for $j+k+1 \leqslant j' \leqslant j-1$, $d(x,y) \leqslant d(x,(p,j)) + d((p,j),(p,j')) + d((p,j'),y) \leqslant d((p,j),(p,j')) + d(x,(p,j)) + d((p,j'),y) \leqslant k+2$. For $j \leqslant j' \leqslant j+k$, $d(x,y) \leqslant d(x,(i,j')) + d((i,j'),y)) \leqslant k+2$.

The proof is thus complete. □

Now combining Propositions 7 and 8, and noting that $d(K_m \times C_{2k+1}) = k+1$, we arrive at Theorem 3.

Finally, we would like to point out that the problem of determining $\vec{d}(K_3 \times C_{2k+1})$ is not as easy as we may believe and has not been settled yet.

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