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Prime graphs and exponential composition of species

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Abstract

In this paper, we enumerate prime graphs with respect to the Cartesian multiplication of graphs. We use the unique factorization of a connected graph into the product of prime graphs given by Sabidussi to find explicit formulas for labeled and unlabeled prime graphs. In the case of species, we construct the exponential composition of species based on the arithmetic product of species of Maia and Méndez, and express the species of connected graphs as the exponential composition of the species of prime graphs.

Keywords: Species; Cartesian product; Prime graphs; Exponential composition; Arithmetic product

1. Introduction

Under the well-known concept of Cartesian product of graphs (Definition 2.1), prime graphs (Definition 2.2) are non-trivial connected graphs that are indecomposable with respect to the Cartesian multiplication. Hence any connected graph can be decomposed into a product of prime graphs, and this decomposition is shown by Sabidussi [14] to be unique.

To count labeled prime graphs, we express the Dirichlet exponential generating series (Definition 2.4) of connected graphs as the exponential of the Dirichlet exponential generating series of prime graphs. To count unlabeled prime graphs, we see that the set of unlabeled connected graphs has a free commutative monoid structure with its prime set being the set of unlabeled prime graphs. This free commutative monoid structure enables us to count unlabeled prime graphs in terms of unlabeled connected graphs.

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Eventually, we aim at finding the cycle index of the species of prime graphs. To be more precise, we want to find the relation between the species of connected graphs and the species of prime graphs. To start with, we observe that the species associated to a graph is isomorphic to the molecular species corresponding to the automorphism group of this graph. This observation leads to a relation (Proposition 3.6) between the arithmetic product of species (Definition 3.2), studied by Maia and Méndez [9], and the Cartesian product of graphs. Moreover, a theorem (whose simplified but equivalent version is given by Proposition 2.3) of Sabidussi about the automorphism groups of connected graphs in terms of the automorphism groups of their prime factors plays an important role. We define a new operation, the exponential composition of species (Definition 3.10), which corresponds to the exponentiation group (Definition 1.3) in the case of molecular species and is related to the arithmetic product of species as the composition of species is related to the multiplication of species. We get a formula (Theorem 3.19) expressing the species of connected graphs as the exponential composition of the species of prime graphs. The enumeration of the species of prime graphs is therefore completed by applying the enumeration theorem (Theorem 3.15) for the exponential composition of species, which is a generalization of an enumeration theorem by Palmer and Robinson [11] on the cycle index polynomial of the exponentiation group.

An explicit formula for the inverse of the exponential composition would be nice to find, but that problem remains open.

1.1. Introduction to species and group actions

The combinatorial theory of species was initiated by Joyal [6,7]. For detailed definitions and descriptions about species, readers are referred to [2].

In short, species are classes of "labeled structures." More formally, a *species* (*of structures*) is a functor from the category of finite sets with bijections $\mathbb{B}$ to itself. A species F generates for each finite set U a finite set $F[U]$, which is called the set of F-*structures* on U, and for each bijection $\sigma: U \to V$ a bijection $F[\sigma]: F[U] \to F[V]$, which is called the *transport of F-structures along σ*. The symmetric group $\mathfrak{S}_n$ acts on the set $F[n] = F[\{1, 2, \ldots, n\}]$ by transport of structures. The $\mathfrak{S}_n$-orbits under this action are called *unlabeled* F-structures of order n.

Each species F is associated with three generating series, the *exponential generating series* $F(x) = \sum_{n\geqslant 0} |F[n]| x^n/n!$, the *type generating series* $\widetilde{F}(x) = \sum_{n\geqslant 0} f_n x^n$, where f_n is the number of unlabeled F-structures of order n, and the *cycle index*

$$Z_F = Z_F(p_1, p_2, \ldots) = \sum_{n\geqslant 0} \left(\sum_{\lambda \vdash n} \operatorname{fix} F[\lambda] \frac{p_\lambda}{z_\lambda} \right),$$

where $\operatorname{fix} F[\lambda]$ denotes the number of F-structures on $[n]$ fixed by $F[\sigma]$, σ is a permutation of $[n]$ with cycle type λ, p_λ is the power sum symmetric function (see Stanley [15, p. 297]) indexed by the partitions λ of n, and z_λ is the number of permutations in $\mathfrak{S}_n$ that commute with a permutation of cycle type λ.

The following identities (see Bergeron, Labelle and Leroux [2, p. 18]) illustrate the importance of the cycle index in the theory of species.

$$F(x) = Z_F(x, 0, 0, \ldots),$$
$$\widetilde{F}(x) = Z_F\big(x, x^2, x^3, \ldots\big).$$

For example, let $\mathscr{G}$ be the species of graphs. Note that by graphs we mean simple graphs, that is, graphs without loops or multiple edges. The cycle index of $\mathscr{G}$ was given in [2, p. 76]:

$$Z_{\mathscr{G}} = \sum_{n\geqslant 0}\left(\sum_{\lambda\vdash n} \operatorname{fix}\mathscr{G}[\lambda]\frac{p_\lambda}{z_\lambda}\right),$$

where

$$\operatorname{fix}\mathscr{G}[\lambda] = 2^{\frac{1}{2}\sum_{i,j\geqslant 1}\gcd(i,j)c_i(\lambda)c_j(\lambda)-\frac{1}{2}\sum_{k\geqslant 1}(k \bmod 2)c_k(\lambda)},$$

in which $c_i(\lambda)$ denotes the number of parts of length i in λ. Let $\mathscr{G}^c$ be the species of connected graphs, and $\mathscr{E}$ the species of sets. The observation that every graph is a set of connected graphs gives rise to the following species identity:

$$\mathscr{G} = \mathscr{E}\big(\mathscr{G}^c\big),$$

which can be read as "a graph is a set of connected graphs," and gives rise to the identities

$$\mathscr{G}^c(x) = \log\big(\mathscr{G}(x)\big),$$

$$\widetilde{\mathscr{G}^c}(x) = \sum_{k\geqslant 1}\frac{\mu(k)}{k}\log\big(\widetilde{\mathscr{G}}\big(x^k\big)\big),$$

$$Z_{\mathscr{G}^c} = \sum_{k\geqslant 1}\frac{\mu(k)}{k}\log(Z_{\mathscr{G}}\circ p_k), \tag{1.1}$$

where the operator $\circ$ on the right-hand side of (1.1) denotes the operation of plethysm on symmetric functions (see Stanley [15, p. 447]). For example, we can compute the first several terms of the cycle index of the species of connected graphs $\mathscr{G}^c$ using Maple:

$$\begin{aligned} Z_{\mathscr{G}^c} = {} & p_1 + \left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2\right) + \left(\frac{1}{3}p_3 + \frac{2}{3}p_1^3 + p_1p_2\right) \\ & + \left(\frac{19}{12}p_1^4 + 2p_1^2p_2 + \frac{5}{4}p_2^2 + \frac{2}{3}p_1p_3 + \frac{1}{2}p_4\right) \\ & + \left(193p_1^3p_2 + \frac{2}{3}p_2p_3 + \frac{91}{15}p_1^5 + 5p_1p_2^2 + \frac{4}{3}p_1^2p_3 + \frac{3}{5}p_5 + p_1p_4\right) \\ & + \left(\frac{1669}{45}p_1^6 + \frac{91}{3}p_1^4p_2 + \frac{38}{9}p_1^3p_3 + \frac{43}{2}p_1^2p_2^2 + 2p_1^2p_4 + \frac{8}{3}p_1p_2p_3 \right. \\ & \left. + \frac{4}{5}p_1p_5 + \frac{26}{3}p_2^3 + \frac{5}{2}p_2p_4 + \frac{25}{18}p_3^2 + \frac{5}{6}p_6\right) + \cdots. \end{aligned} \tag{1.2}$$

For operations of species, readers are referred to [2, pp. 1–58] for more detailed definitions of the sum $F_1 + F_2$, the product $F_1F_2 = F_1 \cdot F_2$, and the composition $F_1(F_2) = F_1 \circ F_2$ of arbitrary species F_1 and F_2.

The *quotient species* (see [2, p. 159]) is defined based on group actions. It appeared in [5] and [3] as an important tool in combinatorial enumeration. Suppose that a group A acts naturally (see [2, p. 393]) on a species F. The quotient species of F by A, denoted F/A, is defined to be such that for each finite set U, F/A-structures on U is the set of A-orbits of F-structures on U, and for each bijection $\sigma : U \to V$, the transport of structures $(F/A)[\sigma] : F[U]/A \to F[V]/A$ is induced from the bijection $F[\sigma]$ that sends each A-orbit of the set $F[U]$ to an A-orbit of the set $F[V]$.

The notion of *molecular species* plays an important role in the analysis of species. Roughly speaking, a molecular species is one that is indecomposable under addition. More precisely, a species M is molecular [16,17] if there is only one isomorphism class of M-structures, i.e., if any two arbitrary M-structures are isomorphic.

If M is molecular, then M is concentrated on n for some positive integer n, i.e., $M[U] \neq \emptyset$ if and only if $|U| = n$. If this is the case, then there is a subgroup A of $\mathfrak{S}_n$ such that M is isomorphic t to the quotient species of X^n, the species of linear orders on an n-element set, by A, i.e., $M = X^n/A$. Furthermore, for A and B two subgroups of $\mathfrak{S}_n$ for some n, the molecular species X^n/A is isomorphic to the molecular species X^n/B if and only if A and B are conjugate subgroups of $\mathfrak{S}_n$. In other words, for each positive integer n, we get a bijection δ_n from the set of conjugate classes of subgroups of the symmetric group of order n to the set of molecular species concentrated on the cardinality n. A formal construction for the molecular species X^n/A for a given subgroup A of $\mathfrak{S}_n$ is given by Bergeron, Labelle and Leroux [2, p. 144].

Pólya's *cycle index polynomial* [12, pp. 64–65] of a subgroup A of $\mathfrak{S}_n$ is defined to be

$$Z(A) = Z(A; p_1, p_2, \dots, p_n) = \frac{1}{|A|} \sum_{\sigma \in A} \prod_{k=1}^{n} p_k^{c_k(\sigma)},$$

where $c_k(\sigma)$ denotes the number of k-cycles in the permutation σ.

An application of Cauchy–Frobenius Theorem [13] (Lemma 3.14) gives that the cycle index polynomial of A is the same as the cycle index of the molecular species X^n/A (see [8, Example 7.4, p. 117]):

$$Z(A) = Z_{X^n/A}.$$

This formula illustrates that the cycle index series of species is a generalization of Pólya's cycle index polynomial.

Definition 1.1. An example of molecular species is the *species associated to a graph*. For each graph G we assign a species $\mathscr{O}_G$ to it such that for any finite set U, the set $\mathscr{O}_G[U]$ is the set of graphs isomorphic to G with vertex set U. The species $\mathscr{O}_G$ is the molecular species corresponding to the automorphism group of G as a subgroup of the symmetric group on the vertex set of G. We write $Z(G)$ for the cycle index of the species associated to the graph G, which is the same as the cycle index polynomial of the automorphism group of G. In other words,

$$Z(G) = Z_{\mathscr{O}_G} = Z\big(\operatorname{aut}(G)\big).$$

The fact that molecular species are indecomposable under addition leads to a *molecular decomposition* of any species [2, p. 141]. That is, every species of structures F is the sum of its molecular subspecies:

$$F = \sum_{\substack{M \subseteq F \\ M \text{ molecular}}} M.$$

Let A be a subgroup of $\mathfrak{S}_m$, and let B be a subgroup of $\mathfrak{S}_n$. We can construct new groups based on A and B.

Definition 1.2. The *product group* whose elements are of the form (a, b), where $a \in A$ and $b \in B$, and whose group operation is given by $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2)$, where a_1 and a_2

are elements of A, and b_1 and b_2 are elements of B, has two group representations, denoted by $A \circledast B$ and $A \times B$, where the group $A \circledast B$ acts on the set $[m+n]$ by

$$(a,b)(i) = \begin{cases} a(i), & \text{if } i \in \{1,2,\ldots,m\}, \\ b(i-m)+m, & \text{if } i \in \{m+1, m+2, \ldots, m+n\}, \end{cases} \tag{1.3}$$

and the group $A \times B$ acts on the set $[m] \times [n]$ by $(a,b)(i,j) = (a(i), b(j))$, for all $i \in [m]$ and $j \in [n]$.

Therefore, we can identify the group $A \circledast B$ with a subgroup of $\mathfrak{S}_{m+n}$, and the group $A \times B$ with a subgroup of $\mathfrak{S}_{mn}$.

Definition 1.3. The *wreath product* of A and B has group elements of the form (α, τ), where α is a permutation in A and τ is a function from $[m]$ to B. The composition of two elements (α, τ) and (β, η) of $B \wr A$ is given by

$$(\alpha,\tau)(\beta,\eta) = \big(\alpha\beta, (\tau \circ \beta)\eta\big),$$

where $\beta \in A$ is viewed as a function from $[m]$ to $[m]$, and $(\tau \circ \beta)\eta$ denotes the pointwise multiplication of $\tau \circ \beta$ and η, both functions from $[m]$ to B.

We introduce two group representations of the wreath product of A and B, denoted $B \wr A$ and B^A, which were studied in full detail by Palmer and Robinson [11].

First, the group $B \wr A$ acts on the set $[m] \times [n]$ by letting $(\alpha,\tau)(i,j) = (\alpha i, \tau(i)j)$, for all $i \in [m]$ and $j \in [n]$. Hence the group $B \wr A$ can be identified with a subgroup of $\mathfrak{S}_{mn}$.

Second, the group B^A acts on the set of functions from $[m]$ to $[n]$ by letting $(\alpha,\tau)(f) = g$ for $f : [m] \to [n]$, where $g : [m] \to [n]$ is defined by

$$\big((\alpha,\tau)f\big)(i) = g(i) = \tau(i)\big(f\big(\alpha^{-1}i\big)\big),$$

for any $i \in [m]$. We observe that the group B^A can be identified with a subgroup of $\mathfrak{S}_{n^m}$.

Yeh [16,17] proved the following species identities:

$$\frac{X^m}{A}\frac{X^n}{B} = \frac{X^{m+n}}{A \circledast B}, \qquad \frac{X^m}{A}\left(\frac{X^n}{B}\right) = \frac{X^{mn}}{B \wr A}.$$

Note that these results agree with Pólya's Theorems [12] for the cycle index polynomials of $A \circledast B$ and $B \wr A$. In this paper, we will study the molecular species $X^{mn}/(A \times B)$ (Section 3.1) and the molecular species X^{n^m}/B^A (Section 3.2).

2. Labeled and unlabeled prime graphs

2.1. Cartesian product of graphs

For any graph G, we let $V(G)$ be the vertex set of G, $E(G)$ the edge set of G, and $l(G) = |V(G)|$ the number of vertices in G. Two graphs G and H with the same number of vertices are said to be *isomorphic*, denoted $G \cong H$, if there exists a bijection from $V(G)$ to $V(H)$ that preserves adjacency. Such a bijection is called an *isomorphism* from G to H. In the case when G and H are identical, this bijection is called an *automorphism* of G. The collection of all automorphisms of G, denoted $\mathrm{aut}(G)$, constitutes a group called the *automorphism group* of G. We set $L(G)$ to be the number of graphs isomorphic to G with vertex set $V(G)$. It is easy to see that $L(G) = l(G)!/|\mathrm{aut}(G)|$. We use the notation $\sum_{i=1}^n G_i = G_1 + G_2 + \cdots + G_n$ to mean the disjoint union of a set of graphs $\{G_i\}_{i=1,\ldots,n}$.

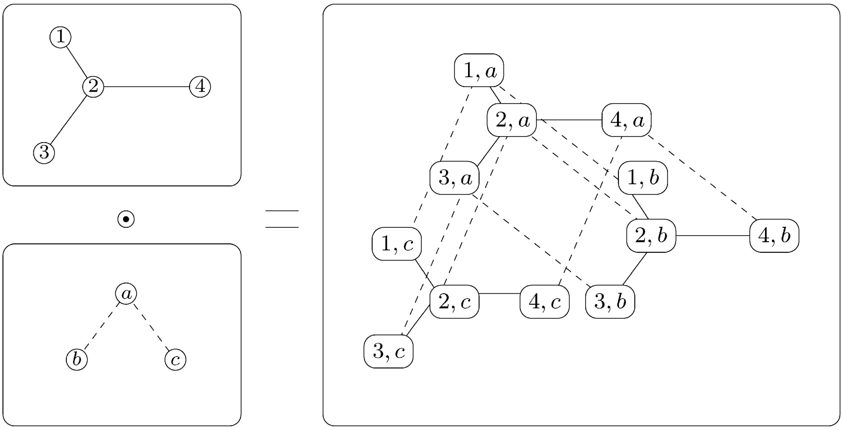


Fig. 1. The Cartesian product of a graph with vertex set $\{1, 2, 3, 4\}$ and a graph with vertex set $\{a, b, c\}$ is a graph with vertex set $\{(i, j)\}$, where $i \in \{1, 2, 3, 4\}$ and $j \in \{a, b, c\}$.

Definition 2.1. The *Cartesian product* of graphs G_1 and G_2, denoted $G_1 \odot G_2$, as defined by Sabidussi [14] under the name the *weak Cartesian product*, is the graph whose vertex set is $V(G_1 \odot G_2) = V(G_1) \times V(G_2) = \{(u, v)\colon u \in V(G_1), v \in V(G_2)\}$, in which (u, v) is adjacent to (w, z) if either $u = w$ and $\{v, z\} \in E(G_2)$ or $v = z$ and $\{u, w\} \in E(G_1)$.

An example of the Cartesian product of two graphs is given in Fig. 1.

For simplicity and without ambiguity, we call $G_1 \odot G_2$ the *product* of G_1 and G_2.

It can be verified straightforwardly that the Cartesian multiplication is commutative and associative up to isomorphism. We denote by G^n the Cartesian product of n copies of G.

Definition 2.2. A graph G is *prime* with respect to Cartesian multiplication if G is a connected graph with more than one vertex such that $G \cong H_1 \odot H_2$ implies that either H_1 or H_2 is a singleton vertex.

Two graphs G and H are called *relatively prime* with respect to Cartesian multiplication, if and only if $G \cong G_1 \odot J$ and $H \cong H_1 \odot J$ imply that J is a singleton vertex.

We denote by $\mathscr{P}$ the species of prime graphs. We see from Definition 2.2 that any non-trivial connected graph can be decomposed into a product of prime graphs. Sabidussi [14] proved that such a prime decomposition is unique up to isomorphism.

The automorphism groups of the Cartesian product of a set of graphs was studied by Sabidussi [14] and Palmer [10]. For example, Sabidussi proved that the automorphism group of the disjoint union of a set of graphs is isomorphic to the automorphism group of the Cartesian product of these graphs. Sabidussi also showed that the automorphism group of the Cartesian product of the disjoint union of two relatively prime graphs is the product of the automorphism group of these two graphs.

2.2. *Labeled prime graphs*

In this section all graphs considered are connected.

Sabidussi gave an important formula about the automorphism group of a connected graph using its prime factorization: If G is a connected graph with prime factorization

$$G \cong P_1^{s_1} \odot P_2^{s_2} \odot \cdots \odot P_k^{s_k},$$

where for $r = 1, 2, \ldots, k$, all P_r are distinct prime graphs, and all s_r are positive integers, then

$$\operatorname{aut}(G) \cong \prod_{r=1}^{k} \operatorname{aut}\bigl(P_r^{s_r}\bigr) \cong \prod_{r=1}^{k} \operatorname{aut}(P_r)^{\mathfrak{S}_{s_r}}.$$

Note that the $P_r^{s_r}$, for $r = 1, 2, \ldots, k$, are pairwise relatively prime. Since the automorphism group of the Cartesian product of the disjoint union of two relatively prime graphs is the product of the automorphism groups of the graphs, we see that Sabidussi's formula reduces equivalently to the following proposition:

Proposition 2.3. *(See Sabidussi [14].) Let P be a prime graph, and let k be a nonnegative integer. Then the automorphism group of P^k is the exponentiation group* $\operatorname{aut}(P)^{\mathfrak{S}_k}$*, i.e.,*

$$\operatorname{aut}\bigl(P^k\bigr) = \operatorname{aut}(P)^{\mathfrak{S}_k}.$$

In particular,

$$\bigl|\operatorname{aut}\bigl(P^k\bigr)\bigr| = \bigl|\operatorname{aut}(P)^{\mathfrak{S}_k}\bigr| = k! \cdot \bigl|\operatorname{aut}(P)\bigr|^k.$$

Definition 2.4. The *Dirichlet exponential generating series* for a sequence of numbers $\{a_n\}_{n\in\mathbf{N}}$ is defined by $\sum_{n\geqslant 1} a_n/(n!n^s)$.

Multiplication of Dirichlet exponential generating series is given by

$$\biggl(\sum_{n\geqslant 1} \frac{a_n}{n!n^s}\biggr)\biggl(\sum_{n\geqslant 1} \frac{b_n}{n!n^s}\biggr) = \sum_{n\geqslant 1} \frac{c_n}{n!n^s},$$

where

$$c_n = \sum_{k|n} \begin{Bmatrix} n \\ k \end{Bmatrix} a_k b_{n/k} = \sum_{k|n} \frac{n!}{k!(n/k)!} a_k b_{n/k}.$$

The *Dirichlet exponential generating function for a species F* with the restriction $F[\emptyset] = \emptyset$ is defined by

$$\mathfrak{D}(F) = \sum_{n\geqslant 1} \frac{|F[n]|}{n!n^s}.$$

The *Dirichlet exponential generating function for a graph G* is defined by

$$\mathfrak{D}(G) = \frac{L(G)}{l(G)! \cdot l(G)^s},$$

where $L(G)$ is the number of graphs isomorphic to G with vertex set $V(G)$, and $l(G)$ is the number of vertices of G. In other words,

$$\mathfrak{D}(G) = \mathfrak{D}(\mathscr{O}_G),$$

where $\mathscr{O}_G$ is the species associated to a graph defined by Definition 1.1. Recall that

$$L(G)=\frac{l(G)!}{|\mathrm{aut}(G)|}.$$

Therefore,

$$\mathfrak{D}(G)=\frac{1}{|\mathrm{aut}(G)|\cdot l(G)^s}.$$

Example 2.5. Let $\mathscr{P}$ be the species of prime graphs, let $\mathscr{G}^c$ be the species of connected graphs, let $\mathbb{C}$ be the set of unlabeled connected graphs, and let $\mathbb{P}$ be the set of unlabeled prime graphs. Then $\mathfrak{D}(\mathscr{G}^c)$ and $\mathfrak{D}(\mathscr{P})$ are the Dirichlet exponential generating functions for these two species, respectively:

$$\mathfrak{D}\big(\mathscr{G}^c\big)=\sum_{n\geqslant 1}\frac{|\mathscr{G}^c[n]|}{n!n^s}=\sum_{G\in\mathbb{C}}\mathfrak{D}(G),\qquad \mathfrak{D}(\mathscr{P})=\sum_{n\geqslant 1}\frac{|\mathscr{P}[n]|}{n!n^s}=\sum_{P\in\mathbb{P}}\mathfrak{D}(P).$$

Propositions 3.4 and 3.6 lead straightforwardly to the following lemma.

Lemma 2.6. *Let G_1 and G_2 be relatively prime graphs. Then*

$$\mathfrak{D}(G_1\odot G_2)=\mathfrak{D}(G_1)\mathfrak{D}(G_2). \tag{2.1}$$

Lemma 2.7. *Let P be any prime graph. Let T be the set of all nonnegative integer powers of P, i.e., $T=\bigcup_{k\geqslant 0}P^k$. Then the Dirichlet exponential generating functions for T and P are related by*

$$\mathfrak{D}(T)=\exp\big(\mathfrak{D}(P)\big). \tag{2.2}$$

Proof. We start with

$$\mathfrak{D}(P)=\frac{L(P)}{l(P)!\cdot l(P)^s}=\frac{1}{|\mathrm{aut}(P)|\cdot l(P)^s}.$$

It follows from Proposition 2.3 that

$$L\big(P^k\big)=\frac{l(P^k)!}{|\mathrm{aut}(P^k)|}=\frac{l(P^k)!}{k!\cdot|\mathrm{aut}(P)|^k},$$

and that

$$\mathfrak{D}\big(P^k\big)=\frac{L(P^k)}{l(P^k)!\cdot l(P^k)^s}=\frac{1}{k!\cdot|\mathrm{aut}(P)|^k\cdot l(P)^{ks}}=\frac{\mathfrak{D}(P)^k}{k!}.$$

Summing up on k, we get

$$\mathfrak{D}(T)=\sum_{k\geqslant 0}\frac{\mathfrak{D}(P)^k}{k!}=\exp\big(\mathfrak{D}(P)\big).\qquad\square$$

Theorem 2.8. *For $\mathfrak{D}(\mathscr{G}^c)$ and $\mathfrak{D}(\mathscr{P})$, we have*

$$\mathfrak{D}\big(\mathscr{G}^c\big)=\exp\big(\mathfrak{D}(\mathscr{P})\big).$$

Proof. Lemma 2.6 gives that the Dirichlet exponential generating function of a product of two relatively prime graphs is the product of the Dirichlet exponential generating functions of the two graphs. Since the operation of Cartesian product on graphs is associative up to isomorphism, it follows that if we have a set of pairwise relatively prime graphs $\{G_i\}_{i=1,2,\ldots,r}$, and let $G = \bigodot_{i=1}^{r} G_i$, then

$$\mathfrak{D}(G) = \prod_{i=1}^{r} \mathfrak{D}(G_i). \tag{2.3}$$

Now according to the definition of the Dirichlet exponential generating function for graphs, we get

$$\mathfrak{D}\big(\mathscr{G}^c\big) = \sum_{G \in \mathbb{C}} \mathfrak{D}(G) = \prod_{P \in \mathbb{P}} \mathfrak{D}\bigg(\sum_{k \geqslant 0} P^k\bigg) = \prod_{P \in \mathbb{P}} \exp\big(\mathfrak{D}(P)\big) = \exp\bigg(\sum_{P \in \mathbb{P}} \mathfrak{D}(P)\bigg)$$
$$= \exp\big(\mathfrak{D}(\mathscr{P})\big). \qquad \square$$

It is well known that the exponential generating series of the species of connected graphs $\mathscr{G}^c$ is

$$\mathscr{G}^c(x) = \sum_{n \geqslant 1} \big|\mathscr{G}^c[n]\big| \frac{x^n}{n!} = \log\bigg(\sum_{n \geqslant 1} 2^{\binom{n}{2}} \frac{x^n}{n!}\bigg)$$
$$= \frac{x}{1!} + \frac{x^2}{2!} + 4\frac{x^3}{3!} + 38\frac{x^4}{4!} + 728\frac{x^5}{5!} + 26704\frac{x^6}{6!} + 1866256\frac{x^7}{7!}$$
$$+ 251548592\frac{x^8}{8!} + 66296291072\frac{x^9}{9!} + \cdots.$$

We obtain $\mathfrak{D}(\mathscr{G}^c)$ by replacing x^n with n^{-s} for each n in the above expression:

$$\mathfrak{D}\big(\mathscr{G}^c\big) = \sum_{n \geqslant 1} \big|\mathscr{G}^c[n]\big| \frac{1}{n!n^s}$$
$$= \frac{1}{1!1^s} + \frac{1}{2!2^s} + 4\frac{1}{3!3^s} + 38\frac{1}{4!4^s} + 728\frac{1}{5!5^s} + 26704\frac{1}{6!6^s} + 1866256\frac{1}{7!7^s}$$
$$+ 251548592\frac{1}{8!8^s} + 66296291072\frac{1}{9!9^s} + \cdots.$$

Theorem 2.8 gives a way of counting labeled prime graphs by writing

$$\mathfrak{D}(\mathscr{P}) = \log \mathfrak{D}\big(\mathscr{G}^c\big).$$

For example, we write down the first terms of $\mathfrak{D}(\mathscr{P})$ as follows:

$$\mathfrak{D}(\mathscr{P}) = \frac{1}{2!2^s} + 4\frac{1}{3!3^s} + 35\frac{1}{4!4^s} + 728\frac{1}{5!5^s} + 26464\frac{1}{6!6^s} + 1866256\frac{1}{7!7^s}$$
$$+ 251518352\frac{1}{8!8^s} + 66296210432\frac{1}{9!9^s} + \cdots.$$

2.3. *Unlabeled prime graphs*

In this section all graphs considered are unlabeled and connected.

Definition 2.9. The (*formal*) *Dirichlet series* of a sequence $\{a_n\}_{n=1,2,\ldots,\infty}$ is defined to be $\sum_{n=1}^{\infty} a_n/n^s$.

The multiplication of Dirichlet series is given by

$$\sum_{n\geqslant 1}\frac{a_n}{n^s}\cdot\sum_{m\geqslant 1}\frac{b_n}{n^s}=\sum_{n\geqslant 1}\biggl(\sum_{k|n}a_k b_{n/k}\biggr)\frac{1}{n^s}.$$

Definition 2.10. A *monoid* is a semigroup with a unit. A *free commutative monoid* is a commutative monoid M with a set of primes $P\subseteq M$ such that each element $m\in M$ can be uniquely decomposed into a product of elements in P up to rearrangement. Let M be a free commutative monoid. We get a *monoid algebra* $\mathbf{C}M$, in which the elements are all formal sums $\sum_{m\in M} c_m m$, where $c_m\in\mathbf{C}$, with addition and multiplication defined naturally. For each $m\in M$, we associate a *length* $l(m)$ that is compatible with the multiplication in M. That is, for any $m_1,m_2\in M$, we have $l(m_1)l(m_2)=l(m_1m_2)$.

Let M be a free commutative monoid with prime set P. The following identity holds in the monoid algebra $\mathbf{C}M$:

$$\sum_{m\in M} m=\prod_{p\in P}\frac{1}{1-p}.$$

Furthermore, we can define a homomorphism from M to the ring of Dirichlet series under which each $m\in M$ is sent to $1/l(m)^s$, where l is a length function of M. Therefore,

$$\sum_{m\in M}\frac{1}{l(m)^s}=\prod_{p\in P}\frac{1}{1-l(p)^{-s}}.$$

Recall that $\mathbb{C}$ is the set of unlabeled connected graphs under the operation of Cartesian product. The unique factorization theorem of Sabidussi gives $\mathbb{C}$ the structure of a commutative free monoid with a set of primes $\mathbb{P}$, where $\mathbb{P}$ is the set of unlabeled prime graphs. This is saying that every element of $\mathbb{C}$ has a unique factorization of the form $b_1^{e_1}b_2^{e_2}\cdots b_k^{e_k}$, where the b_i are distinct primes in $\mathbb{P}$. Let $l(G)$, the number of vertices in G, be a length function for $\mathbb{C}$. We have the following proposition.

Proposition 2.11. *For $\mathbb{C}$ and $\mathbb{P}$, we have*

$$\sum_{G\in\mathbb{C}}\frac{1}{l(G)^s}=\prod_{P\in\mathbb{P}}\frac{1}{1-l(P)^{-s}}.$$

The enumeration of prime graphs was studied by Raphaël Bellec [1]. We use Dirichlet series to count unlabeled connected prime graphs.

Theorem 2.12. *Let $\widetilde{c}_n$ be the number of unlabeled connected graphs on n vertices, and let b_m be the number of unlabeled prime graphs on m vertices. Then we have*

$$\sum_{n\geqslant 1}\frac{\widetilde{c}_n}{n^s}=\prod_{m\geqslant 2}\frac{1}{(1-m^{-s})^{b_m}}. \tag{2.4}$$

Furthermore, if we define numbers d_n for positive integers n by

$$\sum_{n\geqslant 1}\frac{d_n}{n^s}=\log\sum_{n\geqslant 1}\frac{\widetilde{c}_n}{n^s}, \tag{2.5}$$

then

$$d_n=\sum_{m^l=n}\frac{b_m}{l}, \tag{2.6}$$

where the sum is over all pairs (m,l) of positive integers with $m^l=n$.

The proof of Theorem 2.12 follows Remark 2.13 and Proposition 2.14 below.

Remark 2.13. In what follows, we introduce an interesting recursive formula for computing d_n. To start with, we differentiate both sides of Eq. (2.5) with respect to s and simplify. We get that

$$\sum_{n\geqslant 2}\log n\frac{\widetilde{c}_n}{n^s}=\left(\sum_{n\geqslant 1}\frac{\widetilde{c}_n}{n^s}\right)\left(\sum_{n\geqslant 2}\log n\frac{d_n}{n^s}\right),$$

which gives

$$\widetilde{c}_n\log n=\sum_{ml=n}\widetilde{c}_m d_l\log l. \tag{2.7}$$

Since $\widetilde{c}_1$ is the number of connected graphs on 1 vertex, $\widetilde{c}_1=1$. It follows easily from Eq. (2.7) that $d_p=\widetilde{c}_p$ when p is a prime number. Therefore, if p is a prime number, $b_p=d_p=c_p$. This fact can be seen directly, since a connected graph with a prime number of vertices is a prime graph.

Raphaël Bellec used Eq. (2.7) to find formulae for d_n where n is a product of two different primes or a product of three different primes:

If $n=pq$ where $p\neq q$,

$$d_n=\widetilde{c}_n-\widetilde{c}_p\widetilde{c}_q. \tag{2.8}$$

If $n=pqr$ where p,q and r are distinct primes,

$$d_n=\widetilde{c}_n+2\widetilde{c}_p\widetilde{c}_q\widetilde{c}_r-\widetilde{c}_p\widetilde{c}_{qr}-\widetilde{c}_q\widetilde{c}_{pr}-\widetilde{c}_r\widetilde{c}_{pq}. \tag{2.9}$$

In fact, Eqs. (2.8) and (2.9) are special cases of the following proposition.

Proposition 2.14. *Let $d_n,\widetilde{c}_n$ be defined as above. Then we have*

$$d_n=\widetilde{c}_n-\frac{1}{2}\sum_{n_1n_2=n}\widetilde{c}_{n_1}\widetilde{c}_{n_2}+\frac{1}{3}\sum_{n_1n_2n_3=n}\widetilde{c}_{n_1}\widetilde{c}_{n_2}\widetilde{c}_{n_3}-\cdots. \tag{2.10}$$

Proof. We can use the identity

$$\log(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

to compute from Eq. (2.5) that

$$\sum_{n\geqslant 1}\frac{d_n}{n^s}=\log\left(1+\sum_{n\geqslant 2}\frac{\widetilde{c}_n}{n^s}\right)=\sum_{n\geqslant 2}\frac{\widetilde{c}_n}{n^s}-\frac{1}{2}\left(\sum_{n\geqslant 2}\frac{\widetilde{c}_n}{n^s}\right)^2+\frac{1}{3}\left(\sum_{n\geqslant 2}\frac{\widetilde{c}_n}{n^s}\right)^3-\cdots.$$

Equating coefficients of n^{-s} on both sides, we get Eq. (2.10) as a result. □

Proof of Theorem 2.12. We start with

$$\sum_m \frac{1}{l(m)^s} = \prod_p \frac{1}{1 - l(p)^{-s}}, \tag{2.11}$$

where the left-hand side is multiplied over all connected graphs, and the right-hand side is summed over all prime graphs. Regrouping the summands on the left-hand side with respect to the number of vertices in m, we get the left-hand side of Eq. (2.4). Regrouping the factors on the right-hand side with respect to the number of vertices in p, we get the right-hand side of Eq. (2.4).

Taking the logarithm of both sides of Eq. (2.4), we get

$$\begin{aligned}\log \sum_{n\geqslant 1} \frac{\widetilde{c}_n}{n^s} &= \log \prod_{m\geqslant 2} \frac{1}{(1-m^{-s})^{b_m}} = \sum_{m\geqslant 2} b_m \log \frac{1}{1-m^{-s}} \\ &= \sum_{m\geqslant 2}\left(b_m \sum_{l\geqslant 1} \frac{m^{-sl}}{l}\right) = \sum_{m\geqslant 2,\ l\geqslant 1} \frac{b_m}{l\, m^{sl}},\end{aligned}$$

and Eq. (2.6) follows immediately. □

Next, we will compute the numbers b_n in terms of the numbers d_n using the following lemma.

Lemma 2.15. *Let $\{D_i\}_{i=1,\ldots}$ and $\{J_i\}_{i=1,\ldots}$ be sequences of numbers satisfying*

$$D_k = \sum_{l|k} \frac{J_{k/l}}{l}, \tag{2.12}$$

and let μ be the Möbius function. Then we have

$$J_k = \frac{1}{k} \sum_{l|k} \mu\left(\frac{k}{l}\right) l D_l.$$

Proof. Multiplying by k on both sides of Eq. (2.12), we get

$$k D_k = \sum_{l|k} \frac{k}{l} J_{k/l} = \sum_{l|k} l J_l.$$

Applying the Möbius inversion formula, we get

$$k J_k = \sum_{l|k} \mu\left(\frac{k}{l}\right) l D_l.$$

Therefore,

$$J_k = \frac{1}{k} \sum_{l|k} \mu\left(\frac{k}{l}\right) l D_l. \qquad \square$$

Given any natural number n, let e be the largest number such that $n = r^e$ for some r. Note that r is not a power of a smaller integer. We let $D_k = d_{r^k}$, $J_k = b_{r^k}$. It follows that Eq. (2.6) is equivalent to Eq. (2.12).

Table 1
Numbers of labeled and unlabeled prime graphs on n vertices, denoted p_n^l and p_n^u, respectively, for $n \leqslant 16$

n	p_n^l	p_n^u
1	0	0
2	1	1
3	4	2
4	35	5
5	728	21
6	26464	110
7	1866256	853
8	251518352	11111
9	66296210432	261077
10	34496477587456	11716550
11	35641657548953344	1006700565
12	73354596197458024448	164059830354
13	301272202649664088951808	50335907869219
14	2471648811030427594714599424	29003487462847208
15	40527680937730480229320939012096	31397381142761241918
16	1328578958335783200943054119287117312	63969560113225176165143

Theorem 2.16. *For any natural number n, let e, r be as described in above. Then we have*

$$b_n = \frac{1}{e} \sum_{l|e} \mu\left(\frac{e}{l}\right) l d_{r^e}.$$

Proof. The result follows straightforwardly from Lemma 2.15. □

Table 1 gives the numbers of labeled and unlabeled prime graphs with no more than 16 vertices.

3. Exponential composition of species

3.1. Arithmetic product of species

The arithmetic product was studied by Maia and Méndez [9]. The *arithmetic product* of two molecular species X^m/A and X^n/B, where A is a subgroup of $\mathfrak{S}_m$ and B is a subgroup of $\mathfrak{S}_n$, can be defined to be the molecular species $X^{mn}/(A \times B)$, where $A \times B$ is the group representation of the product group of A and B acting on the set $[m] \times [n]$ (Definition 1.2).

In order to define the arithmetic product of general species, Maia and Méndez developed a decomposition of a set, called a *rectangle*.

Definition 3.1. Let U be a finite set. A *rectangle* on U of *height a* is a pair (π_1, π_2) such that π_1 is a partition of U with a blocks, each of size b, where $|U| = ab$, and π_2 is a partition of U with b blocks, each of size a, and if B is a block of π_1 and B' is a block of π_2, then $|B \cap B'| = 1$.

A *k-rectangle* on U is a k-tuple of partitions $(\pi_1, \pi_2, \dots, \pi_k)$ such that

(i) for each $i \in [k]$, π_i has a_i blocks, each of size $|U|/a_i$, where $|U| = \prod_{i=1}^{k} a_i$;

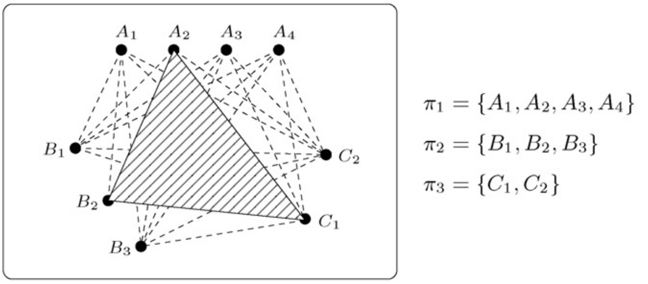


Fig. 2. A 3-rectangle (π_1, π_2, π_3), represented by a 3-partite graph, and labeled on the triangles.

(ii) for any k-tuple $(B_1, B_2, \dots, B_k)$, where B_i is a block of π_i for each $i \in [k]$, we have $|B_1 \cap B_2 \cap \cdots \cap B_k| = 1$. See Fig. 2 for a 3-rectangle (π_1, π_2, π_3) represented by a 3-partite graph.

We denote by $\mathscr{N}$ the species of rectangles, and by $\mathscr{N}^{(k)}$ the species of k-rectangles.

Let $n = \prod_{i=1}^{k} a_i$, and let Δ be the set of bijections of the form

$$\delta : [a_1] \times [a_2] \times \cdots \times [a_k] \to [n].$$

Note that the cardinality of the set Δ is $n!$. The group

$$\prod_{i=1}^{k} \mathfrak{S}_{a_i} = \{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)\colon\ \sigma_i \in \mathfrak{S}_{a_i}\}$$

acts on the set Δ by setting

$$(\sigma \cdot \delta)(i_1, i_2, \dots, i_k) = \delta\big(\sigma_1(i_1), \sigma_2(i_2), \dots, \sigma_k(i_k)\big),$$

for each $(i_1, i_2, \dots, i_k) \in [a_1] \times [a_2] \times \cdots \times [a_k]$. We observe that this group action result in a set of $\prod_{i=1}^{k} \mathfrak{S}_{a_i}$-orbits, and that each orbit consists of exactly $a_1!a_2!\cdots a_k!$ elements of Δ. Observe further that there is a one-to-one correspondence between the set of $\prod_{i=1}^{k} \mathfrak{S}_{a_i}$-orbits on the set Δ and the set of k-rectangles of the form $(\pi_1, \dots, \pi_k)$, where each π_i has a_i blocks. Therefore, the number of such k-rectangles is

$$\left\{ \begin{matrix} n \\ a_1, a_2, \dots, a_k \end{matrix} \right\} := \frac{n!}{a_1!a_2!\cdots a_k!}.$$

Definition 3.2. Let F_1 and F_2 be species of structures with $F_1[\emptyset] = F_2[\emptyset] = \emptyset$. The *arithmetic product* of F_1 and F_2, denoted $F_1 \boxdot F_2$, is defined by setting for each finite set U,

$$(F_1 \boxdot F_2)[U] = \sum_{(\pi_1 \pi_2) \in \mathscr{N}[U]} F_1[\pi_1] \times F_2[\pi_2],$$

where the sum represents the disjoint union (see Fig. 3).

In other words, an $F_1 \boxdot F_2$-structure on a finite set U is a tuple of the form $((\pi_1, f_1), (\pi_2, f_2))$, where (π_1, π_2) is a rectangle on U and f_i is an F_i-structure on the blocks of π_i for each i. A bijection $\sigma : U \to V$ sends a partition π of U to a partition π' of V, namely, $\sigma(\pi) = \pi' = \{\sigma(B)\colon\ B \text{ is a block of } \pi\}$. Thus σ induces a bijection $\sigma_\pi : \pi \to \pi'$, sending each block of π to a block of π'. The transport of structures for any bijection $\sigma : U \to V$ is defined by

$$(F_1 \boxdot F_2)[\sigma]\big((\pi_1, f_1), (\pi_2, f_2)\big) = \big(\big(\pi_1', F_1[\sigma_{\pi_1}](f_1)\big), \big(\pi_2', F_2[\sigma_{\pi_2}](f_2)\big)\big).$$

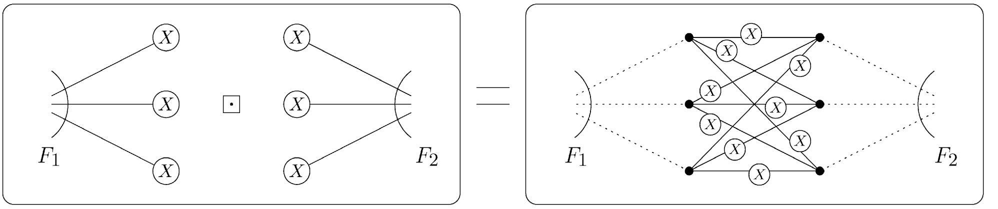


Fig. 3. Arithmetic product $F_1 \boxdot F_2$.

Maia and Méndez showed that the arithmetic product of species is commutative, associative, distributive, and with a unit X, the species of singleton sets:

$$F_1 \boxdot X = X \boxdot F_1 = F_1.$$

Definition 3.3. The *arithmetic product* of species $F_1, F_2, \dots, F_k$ with $F_i(\emptyset) = \emptyset$ for all i is defined by setting $\boxdot_{i=1}^k F_i = F_1 \boxdot F_2 \boxdot \cdots \boxdot F_k$, which sends each finite set U to the set

$$\mathop{\boxdot}_{i=1}^{k} F_i[U] = \sum F_1[\pi_1] \times F_2[\pi_2] \times \cdots \times F_k[\pi_k],$$

where the sum is taken over all k-rectangles $(\pi_1, \pi_2, \dots, \pi_k)$ of U, and represents the disjoint union. We denote by $F^{\boxdot k}$ the arithmetic product of k copies of F.

For each bijection $\sigma : U \to V$, the transport of structures of $\boxdot_{i=1}^k F_i$ along σ sends an $\boxdot_{i=1}^k F_i$-structure on U of the form

$$\big((\pi_1, f_1), (\pi_2, f_2), \dots, (\pi_k, f_k)\big)$$

to an $\boxdot_{i=1}^k F_i$-structure on V of the form

$$\big(\big(\pi_1', F_1[\sigma_{\pi_1}]f_1\big), \big(\pi_2', F_2[\sigma_{\pi_2}]f_2\big), \dots, \big(\pi_k', F_k[\sigma_{\pi_k}]f_k\big)\big),$$

where σ_{π_i} is the bijection induced by σ sending blocks of π_i to blocks of π_i'.

Maia and Méndez proved the following proposition which illustrates that the Dirichlet exponential generating functions are useful for enumeration involving the arithmetic product of species.

Proposition 3.4 *(Maia and Méndez). Let F_1 and F_2 be species with $F_i[\emptyset] = \emptyset$ for $i = 1, 2$. Then*

$$\mathfrak{D}(F_1 \boxdot F_2) = \mathfrak{D}(F_1)\mathfrak{D}(F_2). \tag{3.1}$$

Theorem 3.5 *(Maia and Méndez). Let species F_1 and F_2 satisfy $F_1[\emptyset] = F_2[\emptyset] = \emptyset$. Then we have*

$$Z_{F_1 \boxdot F_2} = Z_{F_1} \boxtimes Z_{F_2}, \tag{3.2}$$

where the operation $\boxtimes$ on the right-hand side of the equation is a bilinear operation on symmetric functions defined by setting

$$p_\nu := p_\lambda \boxtimes p_\mu,$$

where

$$c_k(\nu)=\sum_{\operatorname{lcm}(i,j)=k}\gcd(i,j)c_i(\lambda)c_j(\mu),$$

in which $\operatorname{lcm}(i,j)$ *denotes the* least common multiple *of* i *and* j, *and* $\gcd(i,j)$ *denotes the* greatest common divisor *of* i *and* j.

Furthermore, the arithmetic product of molecular species and the Cartesian product of graphs are closely related, as shown in the following proposition.

Proposition 3.6. *Let G_1 and G_2 be two graphs that are relatively prime to each other. Then the species associated to the Cartesian product of G_1 and G_2 is equivalent to the arithmetic product of the species associated to G_1 and the species associated to G_2. That is,*

$$\mathscr{O}_{G_1\odot G_2}=\mathscr{O}_{G_1}\boxdot\mathscr{O}_{G_2}. \tag{3.3}$$

Proof. Let $l(G_1)=m$ and $l(G_2)=n$. Then $l(G_1\odot G_2)=mn$.

Since G_1 and G_2 are relatively prime, we get

$$\operatorname{aut}(G_1\odot G_2)=\operatorname{aut}(G_1)\times\operatorname{aut}(G_2).$$

Therefore,

$$\begin{aligned}\mathscr{O}_{G_1\odot G_2}&=\frac{X^{l(G_1\odot G_2)}}{\operatorname{aut}(G_1\odot G_2)}=\frac{X^{mn}}{\operatorname{aut}(G_1)\times\operatorname{aut}(G_2)}=\frac{X^m}{\operatorname{aut}(G_1)}\boxdot\frac{X^n}{\operatorname{aut}(G_2)}\\&=\mathscr{O}_{G_1}\boxdot\mathscr{O}_{G_2}.\qquad\square\end{aligned}$$

Note that if G_1 and G_2 are not relatively prime to each other, then the species associated to the Cartesian product of G_1 and G_2 is generally different from the arithmetic product of $\mathscr{O}_{G_1}$ and $\mathscr{O}_{G_2}$. This is because the automorphism group of the product of the graphs is no longer the product of the automorphism groups of the graphs.

3.2. *Exponential composition of species*

Let A be a subgroup of $\mathfrak{S}_m$, and let B be a subgroup of $\mathfrak{S}_n$. The group B^A defined by Definition 1.3 acts on the set of functions from $[m]$ to $[n]$, and hence can be identified with a subgroup of $\mathfrak{S}_{n^m}$. This gives rise to a molecular species X^{n^m}/B^A, which is defined to be the *exponential composition* of species. A more general definition is given in the following.

Let F be a species of structures with $F[\emptyset]=\emptyset$, let k be a positive integer, and let A be a subgroup of $\mathfrak{S}_k$. Recall that an $F^{\boxdot k}$-structure on a finite set U is a tuple of the form

$$\big((\pi_1,f_1),(\pi_2,f_2),\dots,(\pi_k,f_k)\big),$$

where $(\pi_1,\pi_2,\dots,\pi_k)$ is a k-rectangle on U, and each f_i is an F-structure on the blocks of π_i. The group A acts on the set of $F^{\boxdot k}$-structures by permuting the subscripts of π_i and f_i, i.e.,

$$\alpha\big((\pi_1,f_1),\dots,(\pi_k,f_k)\big)=\big((\pi_{\alpha(1)},f_{\alpha(1)}),\dots,(\pi_{\alpha(k)},f_{\alpha(k)})\big),$$

where α is an element of A, $(\pi_{\alpha_1},\pi_{\alpha_2},\dots,\pi_{\alpha_k})$ is a k-rectangle on U, and each f_{α_i} is an F-structure on the blocks of π_{α_i}. It is easy to check that this action of A on $F^{\boxdot k}$-structures is natural, that is, it commutes with any bijection $\sigma:U\to V$. Hence we get a quotient species under this group action.

Definition 3.7 *(Exponential composition with a molecular species).* Let F be a species with $F[\emptyset]=\emptyset$. We define the *exponential composition* of F with the molecular species X^k/A to be the quotient species, denoted $(X^k/A)\langle F\rangle$, under the group action described in above. That is,

$$\big(X^k/A\big)\langle F\rangle := F^{\boxdot k}/A.$$

Theorem 3.8. *Let A and B be subgroups of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, respectively, and let B^A be the exponentiation group of A with B. Then we have*

$$\frac{X^m}{A}\left\langle\frac{X^n}{B}\right\rangle = \frac{X^{n^m}}{B^A}.$$

As a consequence, we have

$$Z_{(X^m/A)\langle X^n/B\rangle} = Z\big(B^A\big).$$

Proof. Since the arithmetic product is associative, we have

$$\left(\frac{X^n}{B}\right)^{\boxdot m} = \frac{X^N}{B^m},$$

where B^m is the product of m copies of B, acting on the set

$$\underbrace{[n]\times[n]\times\cdots\times[n]}_{m\text{ copies}}$$

piecewisely, and hence viewed as a subgroup of $\mathfrak{S}_{n^m}$. Therefore, the set of $(X^n/B)^{\boxdot m}$-structures on $[N]$ can be identified with the set of B^m-orbits of linear orders on $[N]$.

The group A acts on these B^m-orbits of linear orders by permuting the subscripts. This action results in the quotient species

$$\frac{X^m}{A}\left\langle\frac{X^n}{B}\right\rangle = \left(\frac{X^n}{B}\right)^{\boxdot m}/A = \left(\frac{X^N}{B^m}\right)/A.$$

We observe that an A-orbit of B^m-orbits of linear orders on $[N]$ admits an automorphism group isomorphic to the exponentiation group B^A, hence the quotient species $(X^N/B^m)/A$ is the same as the molecular species X^N/B^A. Fig. 4 illustrates a group action of A on a set of $(X^n/B)^{\boxdot m}$-structures. □

Definition 3.9. Let k be a positive integer, and F a species with $F[\emptyset]=\emptyset$. We define the *exponential composition* of F of *order k* to be the species

$$\mathscr{E}_k\langle F\rangle = F^{\boxdot k}/\mathfrak{S}_k.$$

We set $\mathscr{E}_0\langle F\rangle = X$.

Definition 3.10 *(Exponential composition of species).* Let F be a species with $F[\emptyset]=F[1]=\emptyset$. We define the *exponential composition* of F, denoted $\mathscr{E}\langle F\rangle$, to be the sum of $\mathscr{E}_k\langle F\rangle$ on all nonnegative integers k, i.e.,

$$\mathscr{E}\langle F\rangle = \sum_{k\geqslant 0}\mathscr{E}_k\langle F\rangle.$$

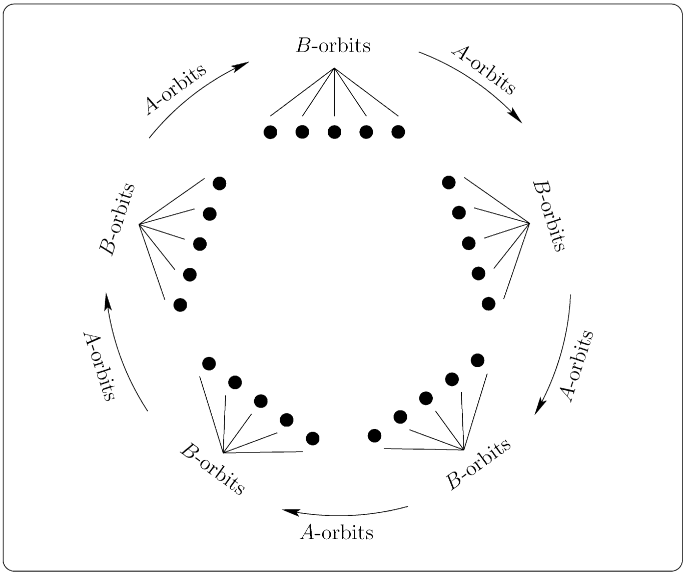


Fig. 4. $((X^n/B)^{\boxdot m})/A = X^{n^m}/(B^A)$.

The exponential composition of species has properties listed in the following theorems. Theorem 3.11 gives a connection between the exponential composition and the Dirichlet exponential generating function of species. Theorem 3.12 lists further properties of the exponential composition of the sum of two species.

Theorem 3.11. *Let F be a species with $F[\emptyset] = F[1] = \emptyset$. Then*

$$\mathfrak{D}\big(\mathscr{E}\langle F\rangle\big) = \exp\big(\mathfrak{D}(F)\big).$$

Proof. Each $F^{\boxdot k}/\mathfrak{S}_k$-structure on a finite set U is an $\mathfrak{S}_k$-orbit of $F^{\boxdot k}$-structures on U, where the action is taken by permuting the subscripts of the $F^{\boxdot k}$-structures. We observe that there are $k!$ $F^{\boxdot k}$-structures in each of the $\mathfrak{S}_k$-orbits. Therefore,

$$\left|\frac{F^{\boxdot k}}{\mathfrak{S}_k}[n]\right| = \frac{|F^{\boxdot k}[n]|}{k!},$$

and

$$\mathfrak{D}\big(\mathscr{E}_k\langle F\rangle\big) = \mathfrak{D}\big(F^{\boxdot k}/\mathfrak{S}_k\big) = \frac{\mathfrak{D}(F^{\boxdot k})}{k!} = \frac{\mathfrak{D}(F)^k}{k!}.$$

It follows that

$$\mathfrak{D}\big(\mathscr{E}\langle F\rangle\big) = \mathfrak{D}\bigg(\sum_{k\geqslant 0}\mathscr{E}_k\langle F\rangle\bigg) = \sum_{k\geqslant 0}\mathfrak{D}\big(\mathscr{E}_k\langle F\rangle\big) = \sum_{k\geqslant 0}\frac{\mathfrak{D}(F)^k}{k!} = \exp\big(\mathfrak{D}(F)\big). \qquad \square \tag{3.4}$$

Theorem 3.12 *(Properties of the exponential composition). Let F_1 and F_2 be species with $F_1[\emptyset] = F_2[\emptyset] = F_1[1] = F_2[1] = \emptyset$, and let k be any nonnegative integer. Then*

$$\mathscr{E}_k\langle F_1+F_2\rangle=\sum_{i=0}^{k}\mathscr{E}_i\langle F_1\rangle\boxdot\mathscr{E}_{k-i}\langle F_2\rangle,\qquad \mathscr{E}\langle F_1+F_2\rangle=\mathscr{E}\langle F_1\rangle\boxdot\mathscr{E}\langle F_2\rangle. \tag{3.5}$$

We observe that an $(F_1+F_2)^{\boxdot k}$-structure on a finite set U is a rectangle on U with each partition in the rectangle enriched with either an F_1 or an F_2-structure. Taking the $\mathfrak{S}_k$-orbits of these $(F_1+F_2)^{\boxdot k}$-structures on U means basically making every partition of the rectangle "indistinguishable." Hence in each $\mathfrak{S}_k$-orbit, all partitions enriched with an F_1-structure are grouped together to give an $\mathfrak{S}_{k_1}$-orbit of $F_1^{\boxdot k_1}$-structures, and the remaining partitions are grouped together to give an $\mathfrak{S}_{k_2}$-orbit of $F_2^{\boxdot k_2}$-structures, where k_1 and k_2 are nonnegative integers whose sum is equal to k.

Proof of Theorem 3.12. First, we prove that for any nonnegative integer k,

$$\mathscr{E}_k\langle F_1+F_2\rangle=\sum_{i=0}^{k}\mathscr{E}_i\langle F_1\rangle\boxdot\mathscr{E}_{k-i}\langle F_2\rangle.$$

The case when $k=0$ is trivial. Let us consider k to be a positive integer. Let s and t be nonnegative integers whose sum equals k. Let U be a finite set. To get an $\mathscr{E}_s\langle F_1\rangle\boxdot\mathscr{E}_t\langle F_2\rangle$-structure on U, we first take a rectangle (ρ,τ) on U, and then take an ordered pair (a,b), where a is an $\mathscr{E}_s\langle F_1\rangle$-structure on the blocks of ρ, and b is an $\mathscr{E}_t\langle F_2\rangle$-structure on the blocks of τ. That is,

$$a=\big\{(\rho_1,f_1),\dots,(\rho_s,f_s)\big\},\qquad b=\big\{(\tau_1,g_1),\dots,(\tau_t,g_t)\big\},$$

where $(\rho_1,\dots,\rho_s)$ is a rectangle on the blocks of ρ, $(\tau_1,\dots,\tau_t)$ is a rectangle on the blocks of τ, f_i is an F_1-structure on the blocks of ρ_i, and g_j is an F_2-structure on the blocks of τ_j.

As pointed out by Maia and Méndez [9], for any nonnegative integers i,j, the species of $(i+j)$-rectangles is isomorphic to the arithmetic product of the species of i-rectangles and the species of j-rectangles:

$$\mathscr{N}^{(i+j)}=\mathscr{N}^{(i)}\boxdot\mathscr{N}^{(j)}.$$

It follows that $(\rho_1,\dots,\rho_s,\tau_1,\dots,\tau_t)$ is a rectangle on U.

On the other hand, let x be an $\mathscr{E}_k\langle F_1+F_2\rangle$-structure on U. We can write x as a set of the form

$$x=\big\{(\pi_1,f_1),\dots,(\pi_r,f_r),(\pi_{r+1},g_{r+1}),\dots,(\pi_k,g_k)\big\},$$

where $(\pi_1,\pi_2,\dots,\pi_k)$ is a k-rectangle on U, r is a nonnegative integer between 0 and k, each f_i is an F_1-structure on π_i for $i=1,\dots,r$, and each g_j is an F_2-structure on π_j for $j=r+1,\dots,k$.

We then write $x=(x_1,x_2)$, where

$$x_1=\big\{(\pi_1,f_1),\dots,(\pi_r,f_r)\big\},\qquad x_2=\big\{(\pi_{r+1},g_{r+1}),\dots,(\pi_k,g_k)\big\}.$$

Hence running through values of s and t, we get that the set of $\mathscr{E}_s\langle F_1\rangle\boxdot\mathscr{E}_t\langle F_2\rangle$-structures on U, written in the form of the pairs (a,b) whose construction we described in above, corresponds naturally to the set of $\mathscr{E}_k\langle F_1+F_2\rangle$-structures on U.

The proof of

$$\mathscr{E}\langle F_1+F_2\rangle=\mathscr{E}\langle F_1\rangle\boxdot\mathscr{E}\langle F_2\rangle.$$

is straightforward using the properties of the arithmetic product, namely, the commutativity, associativity and distributivity:

$$\mathscr{E}\langle F_1+F_2\rangle=\sum_{k\geqslant 0}\mathscr{E}_k\langle F_1+F_2\rangle=\sum_{k\geqslant 0}\sum_{\substack{i+j=k\\ i,j\geqslant 0}}\mathscr{E}_i\langle F_1\rangle\boxdot\mathscr{E}_j\langle F_2\rangle$$
$$=\Bigl(\sum_{i\geqslant 0}\mathscr{E}_i\langle F_1\rangle\Bigr)\boxdot\Bigl(\sum_{j\geqslant 0}\mathscr{E}_j\langle F_2\rangle\Bigr)=\mathscr{E}\langle F_1\rangle\boxdot\mathscr{E}\langle F_2\rangle. \qquad \square$$

Note that identity (3.5) is analogous to the identity about the composition of a sum of species with the species of sets $\mathscr{E}$:

$$\mathscr{E}(F_1+F_2)=\mathscr{E}(F_1)\mathscr{E}(F_2).$$

What is more, (3.5) illustrates a kind of distributivity of the exponential composition. In fact, if a species of structures F has its molecular decomposition written in the form

$$F=\sum_{\substack{M\subseteq F\\ M\text{ molecular}}}M,$$

then the exponential composition of F can be written as

$$\mathscr{E}\langle F\rangle=\mathop{\boxdot}_{\substack{M\subseteq F\\ M\text{ molecular}}}\mathscr{E}\langle M\rangle.$$

3.3. Cycle index of exponential composition

The cycle index polynomial of the exponentiation group was given by Palmer and Robinson [11]. They defined the following operators I_k for positive integers k.

Let $\mathfrak{R}=\mathbf{Q}[p_1,p_2,\ldots]$ be the ring of polynomials with the operation $\boxtimes$ as defined in Theorem 3.5. Palmer and Robinson defined for positive integers k the $\mathbf{Q}$-linear operators I_k on $\mathfrak{R}$ as follows:

Let $\lambda=(\lambda_1,\lambda_2,\ldots)$ be a partition of n. The action of I_k on the monomial p_λ is given by

$$I_k(p_\lambda)=p_\gamma, \tag{3.6}$$

where $\gamma=(\gamma_1,\gamma_2,\ldots)$ is the partition of n^k with

$$c_j(\gamma)=\frac{1}{j}\sum_{l\mid j}\mu\Bigl(\frac{j}{l}\Bigr)\Bigl(\sum_{i\mid l/\gcd(k,l)}ic_i(\lambda)\Bigr)^{\gcd(k,l)}.$$

Furthermore, $\{I_k\}$ generates a $\mathbf{Q}$-algebra Ω of $\mathbf{Q}$-linear operators on $\mathfrak{R}$. For any elements $I,J\in\Omega$, any $r\in\mathfrak{R}$ and $a\in\mathbf{Q}$, we set

$$(aI)(r)=a\bigl(I(r)\bigr),$$
$$(I+J)(r)=I(r)+J(r),$$
$$(IJ)(r)=I(r)\boxtimes J(r). \tag{3.7}$$

As discussed in Palmer and Robinson's paper [11], if $I_m(p_\mu)=p_\nu$, then ν is the cycle type of an element (α,τ) of the exponentiation group B^A acting on $[n]^m$, where α is a permutation in A with a single m-cycle, and $\tau\in B^m$ is such that μ is the cycle type of the permutation $\tau(m)\tau(m-1)\cdots\tau(2)\tau(1)$.

Definition 3.13. Let f_1 and f_2 be elements of the ring $\mathfrak{R} = \mathbf{Q}[p_1, p_2, \ldots]$. We define the *exponential composition* of f_1 and f_2, denoted $f_1 * f_2$, to be the image of f_2 under the operator obtained by substituting the operator I_r for the variables p_r in f_1.

Note that the operation $*$ is linear in the left parameters, but not on the right parameters. We call this the *partial linearity* of the operation $*$.

Let A be a subgroup of $\mathfrak{S}_m$, and let B be a subgroup of $\mathfrak{S}_n$. Palmer and Robinson [11, pp. 128–131] proved that the cycle index polynomial of B^A is the exponential composition of $Z(A)$ with $Z(B)$. That is,

$$Z\big(B^A\big) = Z(A) * Z(B).$$

As a consequence of Theorem 3.8, we get the cycle index of the species $(X^m/A)\langle X^n/B\rangle$:

$$Z_{(X^m/A)\langle X^n/B\rangle} = Z(A) * Z(B).$$

Next we generalize Palmer and Robinson's result to get the formula for the cycle index of the exponential composition of an arbitrary species. First, we introduce a lemma that is a generalization of the Cauchy–Frobenius Theorem, alias Burnside's Lemma. For the proof of a more general result, with applications and further references, see Robinson [13]. Another application is given in [4].

Lemma 3.14 *(Cauchy–Frobenius). Suppose that a finite group $M \times N$ acts on a set S. The groups M and N, considered as subgroups of $M \times N$, also act on S. The group N acts on the set of M-orbits. Then for any $g \in N$, the number of M-orbits fixed by g is given by*

$$\frac{1}{|M|} \sum_{f \in M} \operatorname{fix}(f, g),$$

where $\operatorname{fix}(f, g)$ denotes the number of elements in S that are fixed by $(f, g) \in M \times N$.

Theorem 3.15 *(Cycle index of the exponential composition). Let A be a subgroup of $\mathfrak{S}_k$, and let F be a species of structures concentrated on the cardinality n. Then the cycle index of the species $(X^k/A)\langle F\rangle$ is given by*

$$Z_{(X^k/A)\langle F\rangle} = Z(A) * Z_F, \tag{3.8}$$

where the expression $Z(A) * Z_F$ denotes the image of Z_F under the operator obtained by substituting the operator I_r for the variables p_r in $Z(A)$.

Remark 3.16 *(Notation and set-up).* We denote by Par_n the set of partitions of n, and by Par_n^k the set of k-sequences of partitions of n.

For fixed integers n, k, and $N = n^k$, we denote by $\mathscr{N}_N$ the species of *k-dimensional cubes, or k-cubes, on* $[N]$, defined by

$$\mathscr{N}_N = \mathscr{E}_n^{\boxdot k}[N].$$

We also call the elements of the set $(X^n)^{\boxdot k}[N]$ *k-dimensional ordered cubes on* $[N]$.

Let σ be a permutation on $[k]$ with cycle type

$$\operatorname{ct}(\sigma) = (r_1, r_2, \ldots, r_d).$$

Then σ acts on the $F^{\boxdot k}$-structures by permuting the subscripts. Let ν be a partition of N. Let δ be a permutation of $[N]$ with cycle type ν. Then δ acts on the $F^{\boxdot k}$-structures by transport of structures. We also introduce the notation

$$I\big(\mathrm{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}\big) = I_{r_1}(p_{\lambda^{(1)}}) \boxtimes I_{r_2}(p_{\lambda^{(2)}}) \boxtimes \cdots \boxtimes I_{r_d}(p_{\lambda^{(d)}}).$$

We denote by $\mathrm{Rec}_F(\sigma, \nu)$ a function on the pair (σ, ν) defined by

$$\mathrm{Rec}_F(\sigma, \nu) := \sum \frac{\prod_{i=1}^{d} \mathrm{fix}\, F[\lambda^{(i)}]}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(d)}}}, \tag{3.9}$$

where the summation is over all sequences $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ in Par_n^d with

$$I\big(\mathrm{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}\big) = p_\nu.$$

We denote by $\mathrm{fix}_F(\sigma, \delta)$ the number of $F^{\boxdot k}$-structures on the set $[N]$ fixed by the joint action of the pair (σ, δ).

Proof of Theorem 3.15. Let ν be a partition of N. It suffices to prove that the coefficients of p_ν on both sides of Eq. (3.8) are equal.

The right-hand side of Eq. (3.8) is

$$Z(A) * Z_F = \left(\frac{1}{|A|}\sum_{\sigma\in A} p_{\mathrm{ct}(\sigma)}\right) * \left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right) = \frac{1}{|A|}\sum_{\sigma\in A} I_{\mathrm{ct}(\sigma)}\left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right).$$

For $\sigma \in A$ with $\mathrm{ct}(\sigma) = (r_1, r_2, \dots, r_d)$, we have

$$I_{\mathrm{ct}(\sigma)} = I_{r_1} \cdots I_{r_d},$$

and

$$\begin{aligned} &I_{\mathrm{ct}(\sigma)}\left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right) \\ &\quad = I_{r_1}\left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right) \boxtimes I_{r_2}\left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right) \boxtimes \cdots \boxtimes I_{r_d}\left(\sum_{\lambda\vdash n} \mathrm{fix}\, F[\lambda]\frac{p_\lambda}{z_\lambda}\right). \end{aligned}$$

Therefore, the coefficient of p_ν in the expression $Z(A) * Z_F$ is

$$\frac{1}{|A|}\sum \frac{\prod_{i=1}^{d} \mathrm{fix}\, F[\lambda^{(i)}]}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(d)}}} = \frac{1}{|A|}\sum_{\sigma\in A} \mathrm{Rec}_F(\sigma, \nu), \tag{3.10}$$

where the summation on the left-hand side is taken over all sequences $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)})$ in Par_n^d for some $d \geqslant 1$ and all $\sigma \in A$ with $\mathrm{ct}(\sigma) = (r_1, r_2, \dots, r_d)$ such that

$$I\big(\mathrm{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}\big) = p_\nu,$$

and $\mathrm{Rec}_F(\sigma, \nu)$ on the right-hand side is as defined by (3.9) in Remark 3.16.

The left-hand side of Eq. (3.8) is

$$Z_{F^{\boxdot k}/A} = \sum_{\nu\vdash N} \mathrm{fix}\, \frac{F^{\boxdot k}}{A}[\nu]\frac{p_\nu}{z_\nu}.$$

Therefore, the coefficient of p_ν in the expression $Z_{F^{\boxdot k}/A}$ is

$$\frac{1}{z_\nu}\operatorname{fix}\frac{F^{\boxdot k}}{A}[\nu].$$

We then apply Theorem 3.14 to get that the number of A-orbits of $F^{\boxdot k}$-structures on $[N]$ fixed by a permutation $\delta \in \mathfrak{S}_N$ of cycle type ν is

$$\operatorname{fix}\frac{F^{\boxdot k}}{A}[\nu] = \operatorname{fix}\frac{F^{\boxdot k}}{A}[\delta] = \frac{1}{|A|}\sum_{\sigma\in A}\operatorname{fix}_F(\sigma,\delta), \tag{3.11}$$

where $\operatorname{fix}_F(\sigma,\delta)$ is as defined in Remark 3.16.

Therefore, combining (3.11) and (3.10), the proof of Eq. (3.8) is reduced to showing that

$$\operatorname{fix}_F(\sigma,\delta) = z_\nu \operatorname{Rec}_F(\sigma,\nu), \tag{3.12}$$

for any δ, ν and σ.

To prove (3.12), we start with observing that in order for an $F^{\boxdot k}$-structure on $[N]$ of the form

$$\big((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k)\big)$$

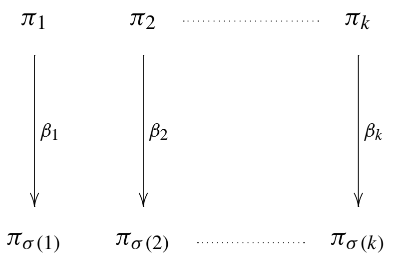
to be fixed by the pair (σ,δ), it is necessary that (σ,δ) fixes the k-cube of the form $(\pi_1,\pi_2,\ldots,\pi_k) \in \mathscr{N}_N$. This is equivalent to saying that

$$\mathscr{N}_N[\delta](\pi_1,\pi_2,\ldots,\pi_k) = (\pi_{\sigma(1)},\pi_{\sigma(2)},\ldots,\pi_{\sigma(k)}). \tag{3.13}$$

Suppose (3.13) holds for some k-cube $(\pi_1,\pi_2,\ldots,\pi_k) \in \mathscr{N}_N$. We let $\beta_i \in \mathfrak{S}_n$ be the induced action of δ on the blocks of π_i, for $i = 1, 2, \ldots, k$. That is,

$$\mathscr{N}_N[\delta](\pi_i) = \beta_i(\pi_{\sigma(i)})$$

for all $i \in [k]$.



Now we consider the simpler case when σ is a k-cycle, say, $\sigma = (1, 2, \ldots, k)$. Then the action of δ sends $(\pi_1,\pi_2,\ldots,\pi_k)$ to $(\pi_2,\pi_3,\ldots,\pi_1)$. Let $\beta = \beta_1\beta_2\cdots\beta_k$. The above discussion is saying that

$$I_k(p_{\mathrm{ct}(\beta)}) = p_\nu.$$

On the other hand, given a partition λ of n satisfying $I_k(p_\lambda) = p_\nu$, there are $n!/z_\lambda$ permutations in $\mathfrak{S}_n$ with cycle type λ. Let β be one of such. Then the number of sequences $(\beta_1,\beta_2,\ldots,\beta_k)$ whose product equals β is $(n!)^{k-1}$, since we can choose β_1 up to β_{k-1} freely, and β_k is therefore determined. All such sequences $(\beta_1,\beta_2,\ldots,\beta_k)$ will satisfy $I_k(p_{\mathrm{ct}(\beta_1\cdots\beta_k)}) = p_\nu$, thus their action on an arbitrary k-dimensional ordered cube, combined with the action of σ on the subscripts, would result in a permutation on $[N]$ with cycle type ν. But there are $N!/z_\nu$ permutations with cycle type ν, and only one of them is the δ that we started with. Considering that the k-cubes are just $\mathfrak{S}_n^k$-orbits of the k-dimensional ordered cubes, we count the number of k-cubes that are fixed by the pair (σ, δ) with the further condition that the product of the induced permutations on the π_i by δ has cycle type λ:

$$\frac{\#\left\{\substack{(\beta_1,\beta_2,\ldots,\beta_k)\in\mathfrak{S}_n^k \\ \text{with } \mathrm{ct}(\beta_1\cdots\beta_k)=\lambda}\right\}\cdot\#\{k\text{-dimensional ordered cubes}\}}{\#\left\{\substack{\text{permutations on } [N] \\ \text{with cycle type } \nu}\right\}\cdot\#\left\{\substack{k\text{-dimensional ordered cubes} \\ \text{in each equivalence class}}\right\}} = \frac{[(n!)^{k-1}\cdot n!/z_\lambda]\cdot N!}{N!/z_\nu\cdot(n!)^k} = \frac{z_\nu}{z_\lambda}.$$

Now we try to compute how many $F^{\boxdot k}$-structures of the form

$$\big((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k)\big),$$

based on a given rectangle $(\pi_1, \pi_2, \ldots, \pi_k)$ that is fixed by the β_i with

$$\mathrm{ct}\left(\prod_i \beta_i\right) = \lambda,$$

are fixed by the pair (σ, δ). We observe that the action of (σ, δ) determines that $f_k = F[\beta_1]f_1$ and $f_i = F[\beta_{i-1}]f_{i-1}$ for $i = 2, 3, \ldots, k$, and hence

$$f_k = F[\beta_1]F[\beta_2]\cdots F[\beta_k]f_k = F[\beta]f_k = F[\lambda]f_k.$$

In other words,

$$f_k \in \mathrm{Fix}\, F[\lambda].$$

Hence as long as we choose an f_k from $\mathrm{Fix}\, F[\lambda]$, then all the other f_i for $i < k$ are determined by our choice of f_k. There are $\mathrm{fix}\, F[\lambda]$ such choices for f_k.

Therefore, in the case when σ is a k-cycle, we get that the number of $F^{\boxdot k}$-structures on the set $[N]$ fixed by the pair (σ, δ) is

$$\mathrm{fix}_F(\sigma, \delta) = \sum_{\substack{\lambda\vdash n \\ I_k(p_\lambda)=p_\nu}} \mathrm{fix}\, F[\lambda]\frac{z_\nu}{z_\lambda} = z_\nu\, \mathrm{Rec}_F(\sigma, \nu).$$

Now let us consider the general case when σ contains d cycles of lengths $r_1, r_2, \ldots, r_d$. Let $(\pi_1, \pi_2, \ldots, \pi_k)$ be a k-cube fixed by the pair (σ, δ). Again we have (3.13), and we get an induced β_i on the blocks of $\pi_{\sigma^{-1}i}$ for each i.

We observe that the action of σ on the subscripts of the k-cube partitions the list $\pi_1, \pi_2, \ldots, \pi_k$ into d parts, of lengths $r_1, r_2, \ldots, r_d$, within each of which we get a r_i-cycle. We group the β_i on each of the d parts and get d permutations in the group $\mathfrak{S}_n$, whose cycle types are denoted by $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}$. This construction gives that such a sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ will be those that satisfy

$$I\big(\mathrm{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}\big) = p_\nu.$$

Therefore, the number of k-cubes fixed by (σ, δ) corresponding to such a sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ is

$$\begin{aligned}\frac{(n!)^{r_1-1}\cdot n!/z_{\lambda^{(1)}}\cdot\cdots\cdot(n!)^{r_d-1}\cdot n!/z_{\lambda^{(d)}}}{N!/z_\nu}\cdot\frac{N!}{(n!)^k} &= \frac{(n!)^{r_1+\cdots+r_d}}{N!}\frac{z_\nu}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(d)}}} \\ &= \frac{z_\nu}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(d)}}}.\end{aligned}$$

The number of F-structures that are assigned to this k-cube $(\pi_1, \pi_2, \ldots, \pi_k)$ that will be fixed under the action of the pair (σ, δ) corresponding to the sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ is hence

$$\operatorname{fix} F\big[\lambda^{(1)}\big] \cdots \operatorname{fix} F\big[\lambda^{(d)}\big],$$

since, similarly to our previous discussion, within each of the d parts, we only need to pick an F-structure that is fixed by a permutation of cycle type $\lambda^{(i)}$, and all other F-structures are left determined.

Therefore, we get that for any pair (σ, δ),

$$\operatorname{fix}_F(\sigma, \delta) = z_\nu \operatorname{Rec}_F(\sigma, \nu),$$

which concludes our proof. □

Remark 3.17. We can use the molecular decomposition to define *the exponential composition of a species F with a species H*. That is, if the molecular decomposition of H is given by

$$H = \sum_{\substack{M \subseteq H \\ M \text{ molecular}}} M,$$

then we define $H\langle F\rangle$ by

$$H\langle F\rangle = \sum_{\substack{M \subseteq H \\ M \text{ molecular}}} M\langle F\rangle.$$

The left-linearity of the operation $*$ gives that the cycle index of $H\langle F\rangle$ is

$$Z_{H\langle F\rangle} = Z_H * Z_F = \Bigg(\sum_{\substack{M \subseteq H \\ M \text{ molecular}}} Z_M\Bigg) * Z_F = \sum_{\substack{M \subseteq H \\ M \text{ molecular}}} Z_M * Z_F.$$

3.4. Cycle index of the species of prime graphs

Now we are ready to come back to the species of prime graphs.

Lemma 3.18. *Let P be any prime graph, and let k be any nonnegative integer. Then the species associated to the kth power of P is the exponential composition of $\mathscr{O}_P$ of order k. That is,*

$$\mathscr{O}_{P^k} = \mathscr{E}_k\langle \mathscr{O}_P\rangle.$$

Proof. We apply Theorem 3.8 and get

$$\mathscr{E}_k\langle \mathscr{O}_P\rangle = \mathscr{O}_P^{\boxdot k}/\mathfrak{S}_k = \left(\frac{X^n}{\operatorname{aut}(P)}\right)^{\boxdot k}/\mathfrak{S}_k = \frac{X^{n^k}}{\operatorname{aut}(P)^{\mathfrak{S}_k}}.$$

It follows from Proposition 2.3 that

$$\mathscr{E}_k\langle \mathscr{O}_P\rangle = \frac{X^{n^k}}{\operatorname{aut}(P^k)} = \mathscr{O}_{P^k}. \qquad \square$$

We can verify Lemma 3.18 in an intuitive way. Note that the set of $\mathscr{E}_k\langle \mathscr{O}_P\rangle$-structures on a finite set U is the set of $\mathfrak{S}_k$-orbits of $\mathscr{O}_P^{\boxdot k}$-structures on U, and an element of $\mathscr{E}_k\langle \mathscr{O}_P\rangle[U]$

of the form $\{(\pi_1, f_1), \dots, (\pi_k, f_k)\}$ is such that $(\pi_1, \pi_2, \dots, \pi_k)$ is a k-rectangle on U, and each f_i is a graph isomorphic to P whose vertex set equal to the blocks of π_i. Such a set $\{(\pi_1, f_1), \dots, (\pi_k, f_k)\}$ corresponds to a graph G isomorphic to P^k with vertex set U. More precisely, G is the Cartesian product of the f_i in which each vertex $u \in U$ is of the form $u = B_1 \cap B_2 \cap \cdots \cap B_k$, where each B_i is one of the blocks of π_i. In this way, we get a one-to-one correspondence between the $\mathscr{E}_k\langle \mathscr{O}_P\rangle$-structures on U and the set of graphs isomorphic to P^k with vertex set U.

Theorem 3.19. *The species $\mathscr{G}^c$ of connected graphs and $\mathscr{P}$ of prime graphs satisfy*

$$\mathscr{G}^c = \mathscr{E}\langle \mathscr{P}\rangle.$$

Proof. In this proof, all graphs considered are unlabeled.

The molecular decomposition of the species of prime graphs is

$$\mathscr{P} = \sum_{P \text{ prime}} \mathscr{O}_P,$$

where each $\mathscr{O}_P$ is a molecular species which is isomorphic to $X^{l(P)}/\operatorname{aut}(P)$.

Let $\{P_1, P_2, \dots\}$ be the set of unlabeled prime graphs. We have

$$\begin{aligned}
\mathscr{E}\langle \mathscr{P}\rangle &= \mathscr{E}\langle \mathscr{O}_{P_1} + \mathscr{O}_{P_2} + \cdots\rangle \\
&= \mathscr{E}\langle \mathscr{O}_{P_1}\rangle \boxdot \mathscr{E}\langle \mathscr{O}_{P_2}\rangle \boxdot \cdots \\
&= (X + \mathscr{O}_{P_1} + \mathscr{O}_{P_1^2} + \cdots) \boxdot (X + \mathscr{O}_{P_2} + \mathscr{O}_{P_2^2} + \cdots) \boxdot \cdots \\
&= \sum_{i_1, i_2, \dots \geqslant 0} \mathscr{O}_{P_1^{i_1}} \boxdot \mathscr{O}_{P_2^{i_2}} \boxdot \cdots \\
&= \sum_{i_1, i_2, \dots \geqslant 0} \mathscr{O}_{P_1^{i_1} \boxdot P_2^{i_2} \boxdot \cdots} \\
&= \sum_{C \text{ connected}} \mathscr{O}_C \\
&= \mathscr{G}^c. \qquad \square
\end{aligned}$$

Note that Theorem 2.8 follows as a corollaries of Theorems 3.11 and 3.19.

Remark 3.20. Recall that the exponential composition of a species F is the sum of $\mathscr{E}_k\langle F\rangle$ on all nonnegative integers k:

$$\mathscr{E}\langle F\rangle = \mathscr{E}_0\langle F\rangle + \mathscr{E}_1\langle F\rangle + \mathscr{E}_2\langle F\rangle + \cdots = X + F + \mathscr{E}_2\langle F\rangle + \cdots.$$

Theorem 3.19 gives that

$$\begin{aligned}
\mathscr{G}^c &= X + \mathscr{P} + \text{higher terms}, \\
\mathscr{P} &= \mathscr{G}^c - X - \text{higher terms}, \\
Z_{\mathscr{P}} &= Z_{\mathscr{G}^c} - p_1 - \text{higher terms}.
\end{aligned}$$

Now we can compute the cycle index of the species of prime graphs $Z_{\mathscr{P}}$ from the cycle index of the species of connected graphs $Z_{\mathscr{G}^c}$, given by formula (1.2), recursively using maple:

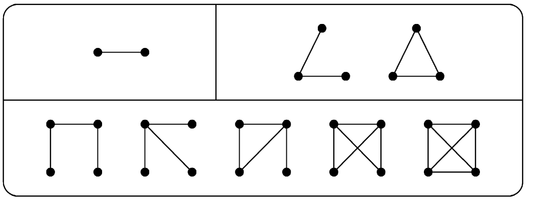


Fig. 5. Unlabeled prime graphs with n vertices, $n \leqslant 4$.

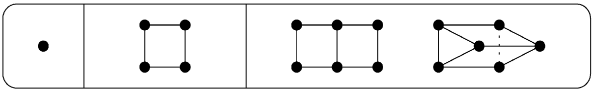


Fig. 6. Unlabeled non-prime graphs on n vertices, $n \leqslant 6$.

$$\begin{aligned}
Z_{\mathscr{P}} = {}& \left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2\right) + \left(\frac{2}{3}p_1^3 + p_1p_2 + \frac{1}{3}p_3\right) \\
& + \left(\frac{35}{24}p_1^4 + \frac{7}{4}p_1^2p_2 + \frac{2}{3}p_1p_3 + \frac{7}{8}p_2^2 + \frac{1}{4}p_4\right) \\
& + \left(\frac{91}{15}p_1^5 + \frac{19}{3}p_1^3p_2 + \frac{4}{3}p_1^3p_3 + 5p_1p_2^2 + p_1p_4 + \frac{2}{3}p_2p_3 + \frac{3}{5}p_5\right) \\
& + \left(\frac{1654}{45}p_1^6 + \frac{91}{3}p_1^4p_2 + \frac{38}{9}p_1^3p_3 + 21p_1^2p_2^2 + 2p_1^2p_4 + \frac{8}{3}p_1p_2p_3 \right. \\
& \left. + \frac{4}{5}p_1p_5 + \frac{47}{6}p_2^3 + \frac{5}{2}p_2p_4 + \frac{11}{9}p_3^2 + \frac{2}{3}p_6\right) + \cdots.
\end{aligned}$$

Fig. 5 shows the unlabeled prime graphs on no more than 4 vertices.

Hence we write down the beginning terms of the molecular decomposition of the species $\mathscr{P}$:

$$\mathscr{P} = \mathscr{E}_2 + (X \cdot \mathscr{E}_2 + \mathscr{E}_3) + \left(\mathscr{E}_2 \circ X^2 + X \cdot \mathscr{E}_3 + X^2 \cdot \mathscr{E}_2 + \mathscr{E}_2 \cdot \mathscr{E}_2 + \mathscr{E}_4\right) + \cdots.$$

Comparing Fig. 5 with unlabeled connected graphs with no more than 4 vertices, we see that there is only one unlabeled connected graph with 4 vertices that is not prime. In fact, if we compare the first several terms of $Z_{\mathscr{G}^c}$, given by (1.2), and $Z_{\mathscr{P}}$ of order no more than 6, we get that

$$\begin{aligned}
Z_{\mathscr{G}^c} - Z_{\mathscr{P}} = {}& p_1 + \frac{1}{8}\left(p_1^4 + 2p_1^2p_2 + 3p_2^2 + 2p_4\right) + \frac{1}{4}\left(p_1^6 + p_1^2p_2^2 + 2p_2^3\right) \\
& + \frac{1}{12}\left(p_1^6 + 3p_1^2p_2^2 + 4p_2^3 + 2p_3^2 + 2p_6\right) \cdots,
\end{aligned}$$

which is the cycle index of connected non-prime graphs on no more than 6 vertices, as shown in Fig. 6, which consist of a single vertex, a graph with 4 vertices, and two graphs with 6 vertices.

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