Journal of Algebra 333 (2011) 427–457

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the isomorphism problem for the ring of monomial representations of a finite group

Michael Müller

Mathematisches Institut, Friedrich-Schiller-Universität, 07737 Jena, Germany

A R T I C L E I N F O

Article history:
Received 18 September 2010
Available online 23 February 2011
Communicated by Michel Broué

Keywords:
Representation rings
Isomorphism problem
Primitive idempotents
Torsion units

A B S T R A C T

In this paper we are concerned with the problem of finding properties of a finite group G in the ring $D(G)$ of monomial representations of G. We determine the conductors of the primitive idempotents of $\mathbb{Q}(\zeta) \otimes_{\mathbb{Z}} D(G)$, where $\zeta \in \mathbb{C}$ is a primitive $|G|$-th root of unity, and prove a structure theorem for the torsion units of $D(G)$. Using these results we show that an abelian group G is uniquely determined by the ring $D(G)$. We state an explicit formula for the primitive idempotents of $\mathbb{Z}[\zeta]_{\mathfrak{p}} \otimes_{\mathbb{Z}} D(G)$, where $\mathbb{Z}[\zeta]_{\mathfrak{p}}$ is a localization of $\mathbb{Z}[\zeta]$. We get further results for nilpotent and p-nilpotent groups and we obtain properties of Sylow subgroups of G from $D(G)$.

1. Introduction

The ring $D(G)$ of monomial representations of a finite group G has been investigated by Andreas Dress and Robert Boltje (the letter D is paying tribute to Dress who studied similar rings in [8]). A motivation to consider this ring arised from the *Brauer induction theorem* which says that there is a canonical way of writing complex characters as an integral linear combination of induced linear characters (cf. [1,17]). Detailed information about construction, species and idempotent formulae of $D(G)$ can be found in [3].

We are mainly interested in finding properties of G by analyzing the structure of $D(G)$. Since the Burnside ring $B(G)$ can be embedded in $D(G)$, there is a connection to the similar problem concerning the ring $B(G)$. This problem has been studied in [6,14,16], among others. Considering results for the isomorphism problem for Burnside rings it seems to be useful to work with primitive idempotents of $R \otimes_{\mathbb{Z}} D(G)$, where R is a subring of $\mathbb{C}$, with conductors of such idempotents and with torsion units of $D(G)$.

E-mail address: m_mueller76@gmx.net.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.02.013

In the second section we give a survey over the construction of $D(G)$, the species and the primitive idempotents of $\mathbb{Q}(\zeta)\otimes_{\mathbb{Z}} D(G)$ ($\zeta\in\mathbb{C}$ primitive $|G|$-th root of unity). The third section contains the determination of the conductors of the primitive idempotents of $\mathbb{Q}(\zeta)\otimes_{\mathbb{Z}} D(G)$ (i.e. the minimal natural number $n_e\in\mathbb{N}$ for a primitive idempotent $e\in\mathbb{Q}(\zeta)\otimes_{\mathbb{Z}} D(G)$ such that $n_e\cdot e\in\mathbb{Z}[\zeta]\otimes_{\mathbb{Z}} D(G)$) and a first application concerning the order of the center of the group G. Next we prove a structure theorem for the torsion units of $D(G)$. In Section 5 we show that an abelian group G is uniquely determined by the ring $D(G)$. In the sixth section we state an explicit formula for the primitive idempotents of $\mathbb{Z}[\zeta]_{\mathfrak{p}}\otimes_{\mathbb{Z}} D(G)$, where $\mathfrak{p}$ is a maximal ideal of $\mathbb{Z}[\zeta]$ and $\mathbb{Z}[\zeta]_{\mathfrak{p}}$ is the localization of $\mathbb{Z}[\zeta]$ at $\mathfrak{p}$. Using this result we obtain properties of the Sylow subgroups of G from $D(G)$. Among others we show that the case $D(G)\cong D(\tilde{G})$, where G has an abelian Sylow p-subgroup, implies the commutativity of the Sylow p-subgroups of $\tilde{G}$. In the last section we consider nilpotent and p-nilpotent groups. Among others we show that the ring $D(G)$ detects nilpotency of G.

Notation. For a group element $g\in G$ we write $\operatorname{ord}(g)$ for the order of g. Let G_p be the set of all p-elements and $G_{p'}$ be the set of all p-regular elements of G (p prime). For $g\in G$ let $g_p\in G_p$ and $g_{p'}\in G_{p'}$ be the uniquely determined elements with $g=g_pg_{p'}=g_{p'}g_p$. For a group G we denote by G' the commutator subgroup of G and by $Z(G)$ the center of G. For a subgroup H of G we use the notation $H\leqslant G$. We sometimes write $H<G$ in case H is a proper subgroup and $H\trianglelefteq G$ in case H is a normal subgroup of G. For $H\leqslant G$ let $C_G(H)$ be the centralizer and $N_G(H)$ be the normalizer of H in G. For $g\in G$ we set ${}^gH:=gHg^{-1}$ and $H^g:=g^{-1}Hg$. Moreover we set $\hat{G}:=\operatorname{Hom}(G,\mathbb{C}^\times)$.

2. The ring of monomial representations

Let G be a finite group. The *monomial category* of G is denoted by $\mathbf{mon}_{\mathbb{C}G}$. The objects of $\mathbf{mon}_{\mathbb{C}G}$ are pairs $(V,\mathcal{L})$ consisting of a finitely generated $\mathbb{C}G$-module V and a set $\mathcal{L}$ of one-dimensional subspaces of V with $\bigoplus_{L\in\mathcal{L}}L=V$ and $gL\in\mathcal{L}$ for $g\in G$ and $L\in\mathcal{L}$. A morphism $f:(V,\mathcal{L})\to(W,\mathcal{M})$ of $\mathbf{mon}_{\mathbb{C}G}$ is a homomorphism $f:V\to W$ of $\mathbb{C}G$-modules such that for all $L\in\mathcal{L}$ there exists $M\in\mathcal{M}$ with $f(L)\subseteq M$. In [4] a morphisms between monomial objects is defined in a different way, but this will not affect the results below. Two objects $(V,\mathcal{L})$ and $(W,\mathcal{M})$ are *isomorphic* if there exists a morphism $f:(V,\mathcal{L})\to(W,\mathcal{M})$ such that the according $\mathbb{C}G$-module homomorphism is an isomorphism. There is a direct sum and a tensor product on $\mathbf{mon}_{\mathbb{C}G}$ defined by

$$(V,\mathcal{L})\oplus(W,\mathcal{M}):=(V\oplus W,\mathcal{L}\cup\mathcal{M})$$

and

$$(V,\mathcal{L})\otimes(W,\mathcal{M}):=\bigl(V\otimes_{\mathbb{C}}W,\{L\otimes_{\mathbb{C}}M\colon L\in\mathcal{L},\ M\in\mathcal{M}\}\bigr)$$

for objects $(V,\mathcal{L}),(W,\mathcal{M})\in\mathbf{mon}_{\mathbb{C}G}$. An object $(V,\mathcal{L})$ of $\mathbf{mon}_{\mathbb{C}G}$ with $V\neq 0$ is *indecomposable* if $(V,\mathcal{L})=(V_1,\mathcal{L}_1)\oplus(V_2,\mathcal{L}_2)$ with objects $(V_1,\mathcal{L}_1),(V_2,\mathcal{L}_2)\in\mathbf{mon}_{\mathbb{C}G}$ implies $V_1=0$ or $V_2=0$.

We denote by $[V,\mathcal{L}]$ the isomorphism class of the object $(V,\mathcal{L})$ of $\mathbf{mon}_{\mathbb{C}G}$. The *ring of monomial representations* $D(G)$ is the $\mathbb{Z}$-module generated by the isomorphism classes of the objects of $\mathbf{mon}_{\mathbb{C}G}$ relative to the relations

$$[V,\mathcal{L}]+[W,\mathcal{M}]=\bigl[(V,\mathcal{L})\oplus(W,\mathcal{M})\bigr]$$

and

$$[V,\mathcal{L}]\cdot[W,\mathcal{M}]=\bigl[(V,\mathcal{L})\otimes(W,\mathcal{M})\bigr],$$

$(V,\mathcal{L}),(W,\mathcal{M})\in\mathbf{mon}_{\mathbb{C}G}$. Then $D(G)$ is a unitary ring with identity $[\mathbb{C},\{\mathbb{C}\}]$ (we consider $\mathbb{C}$ as the trivial $\mathbb{C}G$-module). Moreover $D(G)$ is a free $\mathbb{Z}$-module, and the isomorphism classes of the indecomposable objects of $\mathbf{mon}_{\mathbb{C}G}$ form a $\mathbb{Z}$-basis of $D(G)$ (cf. [4,9]).

Let $H \leqslant G$ and $\varphi \in \hat{H}$. The $\mathbb{C}G$-module $\mathbb{C}_\varphi$ is the $\mathbb{C}$-vectorspace $\mathbb{C}$ with the underlying G-action defined by $g * c := \varphi(g) \cdot c$, $g \in G$, $c \in \mathbb{C}$. Moreover for $g \in G$ we define a linear character ${}^g\varphi \in \widehat{{}^gH}$ by

$$ {}^g\varphi\left({}^gh\right) := \varphi(h), \quad h \in H. $$

We can describe the indecomposable objects of $\mathbf{mon}_{\mathbb{C}G}$ in the following way (cf. [4,9]):

Proposition 2.1.

(i) *Let $H \leqslant G$ and $\varphi \in \hat{H}$. Then $(\mathrm{ind}_H^G \mathbb{C}_\varphi, \{g \otimes \mathbb{C}_\varphi\colon\ g \in G\})$ is an indecomposable object in $\mathbf{mon}_{\mathbb{C}G}$.*

(ii) *Let $H, U \leqslant G$, $\varphi \in \hat{H}$ and $\psi \in \hat{U}$. The objects $(\mathrm{ind}_H^G \mathbb{C}_\varphi, \{g \otimes \mathbb{C}_\varphi\colon\ g \in G\})$ and $(\mathrm{ind}_U^G \mathbb{C}_\psi, \{g \otimes \mathbb{C}_\psi\colon\ g \in G\})$ are isomorphic if and only if there exists $g \in G$ with ${}^gH = U$ and ${}^g\varphi = \psi$.*

(iii) *Every indecomposable object in $\mathbf{mon}_{\mathbb{C}G}$ is isomorphic to an object $(\mathrm{ind}_H^G \mathbb{C}_\varphi, \{g \otimes \mathbb{C}_\varphi\colon\ g \in G\})$ with $H \leqslant G$ and $\varphi \in \hat{H}$.*

From now on we identify the object $(\mathrm{ind}_H^G \mathbb{C}_\varphi, \{g \otimes \mathbb{C}_\varphi\colon\ g \in G\})$ with the *monomial pair* (H, φ). We denote by

$$ \mathcal{M}(G) := \left\{(H, \varphi)\colon\ H \leqslant G,\ \varphi \in \hat{H}\right\} $$

the set of all monomial pairs of G and define by ${}^g(H, \varphi) := ({}^gH, {}^g\varphi)$ an action of G on $\mathcal{M}(G)$. We write $[H, \varphi]_G$ for the G-orbit of $(H, \varphi) \in \mathcal{M}(G)$ and we set

$$ \mathcal{M}(G)/G := \left\{[H, \varphi]_G\colon\ (H, \varphi) \in \mathcal{M}(G)\right\}. $$

Moreover for $(H, \varphi), (U, \psi) \in \mathcal{M}(G)$ we write $(H, \varphi) \leqslant (U, \psi)$ if $H \leqslant U$ and $\psi_{|H} = \varphi$. Therefore we get a partial order on $\mathcal{M}(G)$. By

$$ N_G(H, \varphi) := \left\{g \in G\colon\ {}^g(H, \varphi) = (H, \varphi)\right\} $$

we denote the stabilizer of $(H, \varphi) \in \mathcal{M}(G)$ in G. In particular we get the inclusion

$$ H \leqslant N_G(H, \varphi) \leqslant N_G(H). $$

By Proposition 2.1 we can identify the isomorphism classes of indecomposable objects with the elements of $\mathcal{M}(G)/G$. Thus the ring $D(G)$ is the free abelian group generated by the G-orbits $[H, \varphi]_G \in \mathcal{M}(G)/G$ together with the multiplication

$$ [H, \varphi]_G \cdot [U, \psi]_G = \sum_{HgU \in H\backslash G/U} \left[H \cap {}^gU, \varphi_{|H \cap {}^gU} \cdot {}^g\psi_{|H \cap {}^gU}\right]_G $$

for $[H, \varphi]_G, [U, \psi]_G \in \mathcal{M}(G)/G$. In particular $D(G)$ is finitely generated.

For a commutative unitary ring R and $H \leqslant G$ we set

$$ D_R(H) := R \otimes_{\mathbb{Z}} D(H). $$

Let $K \leqslant H \leqslant G$ and $g \in G$. The *conjugation map* $c_{g,H}$ is defined by

$$ \begin{aligned} c_{g,H} : D_R(H) &\to D_R\left({}^gH\right), \\ [U, \varphi]_H &\mapsto \left[{}^gU, {}^g\varphi\right]_{{}^gH}, \end{aligned} $$

the *restriction map* res_K^H is defined by

$$\begin{aligned}\mathrm{res}_K^H : D_R(H) &\to D_R(K),\\ [U,\varphi]_H &\mapsto \sum_{KhU \in K\backslash H/U} \big[K \cap {}^hU, {}^h\varphi_{|K\cap {}^hU}\big]_K\end{aligned}$$

and the *induction map* ind_K^H is defined by

$$\begin{aligned}\mathrm{ind}_K^H : D_R(K) &\to D_R(H),\\ [U,\varphi]_K &\mapsto [U,\varphi]_H.\end{aligned}$$

The conjugation and the restriction maps are R-algebra homomorphisms. The induction maps are morphisms of the additive groups. Together with these operations the functor D_R becomes an R-Green functor on G (cf. [4]).

A *species* of $D(G)$ is a ring homomorphism $s : D(G) \to \mathbb{C}$. In the following we give a short survey on the construction of the species of $D(G)$ according to [3].

Let $R(G)$ be the ordinary character ring of G. For $g \in G$ we define the ring homomorphism

$$\begin{aligned}t_g : R(G) &\to \mathbb{C},\\ \varphi &\mapsto \varphi(g).\end{aligned}$$

For $H \leqslant G$ we define the ring homomorphism

$$\begin{aligned}\pi_H : D(H) &\to R\big(H/H'\big),\\ [U,\psi]_H &\mapsto \begin{cases}\bar\psi & \text{if } U = H,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where $\bar\psi \in \widehat{H/H'}$ is defined by $\bar\psi(hH') := \psi(h)$. We set

$$\mathcal{D}(G) := \big\{\big(H, hH'\big):\ H \leqslant G,\ h \in H\big\}$$

and define an action of G on $\mathcal{D}(G)$ by ${}^g(H,hH') := ({}^gH, {}^gh{}^gH')$ for $g \in G$. We write $[H,hH']_G$ for the G-orbit of $(H,hH') \in \mathcal{D}(G)$ and we set

$$\mathcal{D}(G)/G := \big\{\big[H,hH'\big]_G:\ \big(H,hH'\big) \in \mathcal{D}(G)\big\}.$$

The stabilizer of $(H,hH') \in \mathcal{D}(G)$ in G is denoted by

$$N_G\big(H,hH'\big) := \big\{g \in G:\ {}^g\big(H,hH'\big) = \big(H,hH'\big)\big\}.$$

Moreover we obtain the inclusion

$$H \leqslant HC_G(H) \leqslant N_G\big(H,hH'\big) \leqslant N_G(H).$$

For every element $(H,hH') \in \mathcal{D}(G)$ we get a ring homomorphism

$$s_{(H,hH')}^{D(G)} := t_{hH'} \circ \pi_H \circ \mathrm{res}_H^G : D(G) \to D(H) \to R\big(H/H'\big) \to \mathbb{C}.$$

In particular the images of the elements $[U,\psi]_G \in \mathcal{M}(G)/G$ are given by

$$s^{D(G)}_{(H,hH')}\big([U,\psi]_G\big) = \sum_{\substack{gU\in G/U\\ H\leqslant {}^gU}} {}^g\psi(h).$$

We get the set of all species of $D(G)$ by this construction. Moreover $s^{D(G)}_{(H,hH')} = s^{D(G)}_{(U,uU')}$ if and only if $[H,hH']_G = [U,uU']_G$. Thus there is a 1-1-correspondence between the species of $D(G)$ and the elements of $\mathcal{D}(G)/G$. Moreover for $H \leqslant G$, $(U,uU') \in \mathcal{D}(H)$ and $g \in G$ it holds

$$s^{D({}^gH)}_{({}^gU,{}^gu{}^gU')} \circ c_{g,H} = s^{D(H)}_{(U,uU')} \quad \text{and} \quad s^{D(H)}_{(U,uU')} \circ \mathrm{res}^G_H = s^{D(G)}_{(U,uU')}.$$

Let $\zeta \in \mathbb{C}$ be a primitive $|G|$-th root of unity and $m := |\mathcal{D}(G)/G|$. The map

$$s^{D(G)} := \prod_{[H,hH']_G\in\mathcal{D}(G)/G} s^{D(G)}_{(H,hH')} : D(G) \to \mathbb{Z}[\zeta]^m$$

is a ring monomorphism. Thus we can identify the ring $D(G)$ with a subring of $\mathbb{Z}[\zeta]^m$. The image of $\mathcal{M}(G)/G$ under the map $s^{D(G)}$ is called *species table* of $D(G)$.

If we extend $D(G)$ with the coefficient ring $\mathbb{Q}(\zeta)$, we get a ring isomorphism $D_{\mathbb{Q}(\zeta)}(G) \cong \mathbb{Q}(\zeta)^m$. If we extend the species linearly to $D_{\mathbb{Q}(\zeta)}(G)$, the primitive idempotents of $D_{\mathbb{Q}(\zeta)}(G)$ are the elements $e^{D(G)}_{(H,hH')} \in D_{\mathbb{Q}(\zeta)}(G)$, $(H,hH') \in \mathcal{D}(G)$, determined by the property

$$s^{D(G)}_{(U,uU')}\big(e^{D(G)}_{(H,hH')}\big) = \begin{cases} 1 & \text{if } [U,uU']_G = [H,hH']_G,\\ 0 & \text{otherwise.}\end{cases}$$

An explicit formula for the primitive idempotents of $D_{\mathbb{Q}(\zeta)}(G)$ is given by

$$e^{D(G)}_{(H,hH')} = \frac{|H'|}{|N_G(H,hH')||H|} \sum_{L\leqslant H} |L|\mu(L,H) \sum_{\varphi\in\hat{H}} \varphi\big(h^{-1}\big)[L,\varphi_{|L}]_G, \quad \big(H,hH'\big) \in \mathcal{D}(G) \tag{1}$$

(cf. [3]). The map $\mu : \mathcal{V}(G) \times \mathcal{V}(G) \to \mathbb{Z}$ is called *Möbius function* which is recursively defined by $\sum_{H\leqslant K\leqslant U} \mu(H,K) = 0$ for $H < U$, $\mu(H,H) = 1$ and $\mu(H,U) = 0$ for $H \nleqslant U$ $(H,U \in \mathcal{V}(G))$ where $\mathcal{V}(G)$ is the subgroup lattice of G.

Considering isomorphism problems, the following fact will be very useful. Let $\tilde{G}$ be another finite group. For an isomorphism $\alpha : D(G) \to D(\tilde{G})$ and $(H,hH') \in \mathcal{D}(G)$ there exists $(\tilde{H},\tilde{h}\tilde{H}') \in \mathcal{D}(\tilde{H})$ with

$$s^{D(G)}_{(H,hH')} = s^{D(\tilde{G})}_{(\tilde{H},\tilde{h}\tilde{H}')} \circ \alpha.$$

Another important role plays the embedding of the Burnside ring into the ring of monomial representations. We will introduce the Burnside ring as a subring of $D(G)$ because for further results it is not necessary to work with the theory of G-sets (cf. [3]).

The free abelian subgroup generated by the elements $[H,1]_G \in \mathcal{M}(G)/G$, $H \leqslant G$, form a subring of $D(G)$, the *Burnside ring* $B(G)$ of G. The multiplication in $B(G)$ is given by

$$[H,1]_G \cdot [U,1]_G = \sum_{HgU\in H\backslash G/U} \big[H \cap {}^gU, 1\big]_G.$$

For a commutative unitary ring R and $H \leqslant G$ we set

$$B_R(H) := R \otimes_{\mathbb{Z}} B(H).$$

Since the conjugation maps, restriction maps and induction maps on $D_R(H)$ yield corresponding maps on $B_R(H)$, the functor B_R becomes a R-Green functor on G.

We get the species of $B(G)$ by restricting the species of $D(G)$. Therefore, the species of $B(G)$ are given by

$$s_H^{B(G)} : B(G) \to \mathbb{Z},$$
$$[U,1]_G \mapsto \sum_{\substack{gU \in G/U \\ H \leqslant {}^g U}} 1$$

for $H \leqslant G$. Moreover $s_H^{B(G)} = s_K^{B(G)}$ for $H, K \leqslant G$ if and only if $H = {}^g K$ for some $g \in G$. The primitive idempotents of $B_{\mathbb{Q}}(G)$ are exactly the elements $e_H^{B(G)} \in B_{\mathbb{Q}}(G)$ $(H \leqslant G)$ with

$$s_U^{B(G)}\big(e_H^{B(G)}\big) = \begin{cases} 1 & \text{if } U =_G H, \\ 0 & \text{else.} \end{cases}$$

An explicit formula for the primitive idempotents $e_H^{B(G)}$ is given by

$$e_H^{B(G)} = \frac{1}{|N_G(H)|} \sum_{U \leqslant H} |U| \mu(U, H) [U, 1]_G \tag{2}$$

(cf. [10]).

3. The conductors of the primitive idempotents

In the following let G always be a finite group and $\zeta \in \mathbb{C}$ be a $|G|$-th root of unity. In this section we determine the conductors of the primitive idempotents of $D_{\mathbb{Q}(\zeta)}(G)$. The *conductor* of a primitive idempotent $e \in D_{\mathbb{Q}(\zeta)}(G)$ is the minimal natural number $n_e \in \mathbb{N}$ with $n_e \cdot e \in D_{\mathbb{Z}[\zeta]}(G)$. First we state a result about restricted and induced primitive idempotents.

Lemma 3.1. *Let $H \leqslant G$ and $h \in H$.*

(i) $\mathrm{res}_H^G(e_{(H,hH')}^{D(G)}) = \sum_{\substack{[H,uH']_H \in \mathcal{D}(H)/H \\ [H,uH']_G = [H,hH']_G}} e_{(H,uH')}^{D(H)}$.

(ii) $\mathrm{ind}_H^G(e_{(H,hH')}^{D(H)}) = (N_G(H,hH') : H) e_{(H,hH')}^{D(G)}$.

(iii) $\mathrm{ind}_H^G(\mathrm{res}_H^G(e_{(H,hH')}^{D(G)})) = (N_G(H) : H) e_{(H,hH')}^{D(G)}$.

Proof. (i) It holds

$$s_{(K,kK')}^{D(H)}\big(\mathrm{res}_H^G\big(e_{(H,hH')}^{D(G)}\big)\big) = s_{(K,kK')}^{D(G)}\big(e_{(H,hH')}^{D(G)}\big) = 1$$

for $(K, kK') \in \mathcal{D}(H)$ if and only if (K, kK') and (H, hH') are conjugate in G.

(iii) Let $[K, \psi]_G \in \mathcal{M}(G)/G$. Then

$$\mathrm{ind}_H^G\big(\mathrm{res}_H^G([K,\psi]_G)\big) = \sum_{HgK \in H\backslash G/K} \big[H \cap {}^g K, {}^g \psi_{|H \cap {}^g K}\big]_G = [H,1]_G [K,\psi]_G.$$

Thus

$$\operatorname{ind}_H^G\bigl(\operatorname{res}_H^G\bigl(e_{(H,hH')}^{D(G)}\bigr)\bigr)=[H,1]_G e_{(H,hH')}^{D(G)}=s_{(H,hH')}^{D(G)}\bigl([H,1]_G\bigr)e_{(H,hH')}^{D(G)}=\frac{|N_G(H)|}{|H|}e_{(H,hH')}^{D(G)}.$$

(ii) Let $(H,vH')\in\mathcal{D}(G)$ and $g\in G$ with ${}^g(H,vH')=(H,hH')$. Since $s_{(H,hH')}^{D(H)}\circ c_{g,H}=s_{(H,vH')}^{D(H)}$, we get

$$c_{g,H}\bigl(e_{(H,vH')}^{D(H)}\bigr)=e_{(H,hH')}^{D(H)},$$

and since $\operatorname{ind}_H^G=c_{g,G}\circ\operatorname{ind}_H^G=\operatorname{ind}_H^G\circ c_{g,H}$, we obtain

$$\operatorname{ind}_H^G\bigl(e_{(H,vH')}^{D(H)}\bigr)=\operatorname{ind}_H^G\bigl(e_{(H,hH')}^{D(H)}\bigr).$$

Thus

$$\operatorname{ind}_H^G\bigl(\operatorname{res}_H^G\bigl(e_{(H,hH')}^{D(G)}\bigr)\bigr)=\operatorname{ind}_H^G\Biggl(\sum_{\substack{[H,uH']_H\in\mathcal{D}(H)/H\\ [H,uH']_G=[H,hH']_G}}e_{(H,uH')}^{D(H)}\Biggr)=\frac{|N_G(H)|}{|N_G(H,hH')|}\operatorname{ind}_H^G\bigl(e_{(H,hH')}^{D(H)}\bigr).$$

Together with part (iii) we get $\operatorname{ind}_H^G(e_{(H,hH')}^{D(H)})=(N_G(H,hH'):H)e_{(H,hH')}^{D(G)}$. $\square$

For using some important results of Boltje we have to introduce the ghost ring of the representation ring $D(G)$ (cf. [5]). Let

$$x=(x_H)_{H\leqslant G}\in\prod_{H\leqslant G}\mathbb{Z}\hat{H}$$

with $x_H=\sum_{\varphi\in\hat{H}}z_{H,\varphi}\varphi$ ($z_{H,\varphi}\in\mathbb{Z}$, $H\leqslant G$, $\varphi\in\hat{H}$). For $H\leqslant G$ and $\varphi\in\hat{H}$ we define

$$x(H,\varphi):=z_{H,\varphi}.$$

Note that this is well defined since the set of linear characters of H is a basis of $\mathbb{Z}\hat{H}$. The subring

$$\hat{D}(G):=\Biggl(\prod_{H\leqslant G}\mathbb{Z}\hat{H}\Biggr)^G:=\Biggl\{x\in\prod_{H\leqslant G}\mathbb{Z}\hat{H}\colon x(H,\varphi)=x\bigl({}^g(H,\varphi)\bigr)\ \forall(H,\varphi)\in\mathcal{M}(G)\ \forall g\in G\Biggr\}$$

of $\prod_{H\leqslant G}\mathbb{Z}\hat{H}$ is called the *ghost ring* of $D(G)$. Identifying $R(H/H')$ with $\mathbb{Z}\hat{H}$ for $H\leqslant G$, we get a ring monomorphism

$$\rho:=\bigl(\pi_H\circ\operatorname{res}_H^G\bigr)_{H\leqslant G}:D(G)\to\hat{D}(G).$$

Moreover we set

$$\rho_H:=\pi_H\circ\operatorname{res}_H:D(G)\to\mathbb{Z}\hat{H}$$

for $H \leqslant G$. Note that the image of a basis element $[U,\lambda]_G \in \mathcal{M}(G)/G$ under this map is given by

$$\rho_H\big([U,\lambda]_G\big) = \sum_{\substack{gU\in G/U\\ H\leqslant {}^gU}} {}^g\lambda_{|H} \in \mathbb{Z}\hat{H}.$$

By linear extension we get an isomorphism $\rho : \mathbb{Q}\otimes_{\mathbb{Z}} D(G) \to \mathbb{Q}\otimes_{\mathbb{Z}} \hat{D}(G)$ (cf. [2]). We will use the following integrality criteria for elements of the ghost ring:

Proposition 3.2. *Let $x \in \hat{D}(G)$. Then $x \in \rho(D(G))$ if and only if the congruence*

$$\sum_{(H,\varphi)\leqslant(I,\psi)\in\mathcal{M}(N_G(H,\varphi))} \mu(H,I)\cdot x(I,\psi) \equiv 0 \pmod{\big(N_G(H,\varphi):H\big)}$$

holds for all $(H,\varphi)\in\mathcal{M}(G)$.

Proof. See [5], Cor. 2.8. □

We also make use of the following two lemmata:

Lemma 3.3. *Let $H \leqslant G$ and $\hat{H}_0 := \{\varphi_{|H}\colon \varphi\in\hat{G}\}$. For $\psi\in\hat{H}_0$ we set $A_\psi := \{\varphi\in\hat{G}\colon \varphi_{|H}=\psi\}$.*

(i) *$\hat{H}_0$ is a subgroup of $\hat{H}$ with $\hat{H}_0 \cong HG'/G'$. Moreover $|A_\psi| = (G:HG')$.*
(ii) *Let $g\in G$. Then $\sum_{\varphi\in A_\psi}\varphi(g) = \begin{cases}(G:HG')\psi(g) & \text{if } gG'\in HG'/G',\\ 0 & \text{else.}\end{cases}$*

Part (i) is a well known consequence of the theory of irreducible characters of abelian groups (cf. [13]) and part (ii) can be easily proved by the second orthogonality relation.

Lemma 3.4. *Let $H\leqslant G$ and m be the squarefree part of $(G:G'H)$. Then $(N_G(H):H)$ divides $m\mu(H,G)$.*

Proof. See [11], Thm. 4.5. □

We can now state the main result of this section.

Theorem 3.5. *Let $(H,hH')\in\mathcal{D}(G)$. Then $(N_G(H,hH'):H')$ is the conductor of $e^{D(G)}_{(H,hH')}$.*

Proof. We first prove that $m := (G:G')$ is the conductor of $e^{D(G)}_{(G,gG')}$ for $g\in G$. By the explicit formula for the primitive idempotents (1) we obtain

$$\begin{aligned} e^{D(G)}_{(G,gG')} &= \frac{|G'|}{|G|^2}\sum_{L\leqslant G}|L|\mu(L,G)\sum_{\varphi\in\hat{G}}\varphi\big(g^{-1}\big)[L,\varphi_{|L}]_G \\ &= \frac{|G'|}{|G|}\sum_{\varphi\in\hat{G}}\varphi\big(g^{-1}\big)[G,\varphi]_G + \frac{|G'|}{|G|^2}\sum_{L<G}|L|\mu(L,G)\sum_{\varphi\in\hat{G}}\varphi\big(g^{-1}\big)[L,\varphi_{|L}]_G. \end{aligned}$$

We conclude that the coefficient of $[G,1]_G$ in $e^{D(G)}_{(G,gG')}$ is m^{-1}. Therefore m divides the conductor of $e^{D(G)}_{(G,gG')}$ for all $g\in G$.

Let $f \in B_{\mathbb{Q}}(G)$ be the primitive idempotent with $s_G^{B(G)}(f) = 1$ and $s_H^{B(G)}(f) = 0$ for $H < G$. Let $\mathcal{C}(G)$ be a system of representatives for the conjugacy classes of subgroups of G. Then $f = \sum_{U \in \mathcal{C}(G)} a_U [U, 1]_G$ with uniquely determined coefficients $a_U \in \mathbb{Q}$. Let $1 = \lambda_1, \ldots, \lambda_m$ be the linear characters of G. For $i = 1, \ldots, m$ we define

$$x_i := \sum_{U \in \mathcal{C}(G)} a_U [U, \lambda_{i|U}]_G \in D_{\mathbb{Q}}(G).$$

Note that $x_1 = f$. We now show that $\rho_H(x_i) = 0$ in the case $H < G$ and $\rho_G(x_i) = \lambda_i$ for $i = 1, \ldots, m$. It holds

$$0 = s_H^{B(G)}(x_1) = \left(t_{hH'} \circ \pi_H \circ \mathrm{res}_H^G\right)(x_1)$$

for $H < G$ and all $h \in H$. Therefore

$$\rho_H(x_1) = \left(\pi_H \circ \mathrm{res}_H^G\right)(x_1) = 0.$$

Moreover ${}^g\lambda_i = \lambda_i$ for $g \in G$ and $i = 1, \ldots, m$. Thus we get

$$\rho_H\left([U, \lambda_{i|U}]_G\right) = \sum_{\substack{gU \in G/U \\ H \leqslant {}^gU}} {}^g\lambda_{i|H} = \sum_{\substack{gU \in G/U \\ H \leqslant {}^gU}} \lambda_{i|H} = \lambda_{i|H}\, \rho_H\left([U, 1]_G\right)$$

for $H, U \leqslant G$ and $i = 1, \ldots, m$ and we obtain

$$\rho_H(x_i) = \sum_{U \in \mathcal{C}(G)} a_U\, \rho_H\left([U, \lambda_{i|U}]_G\right) = \lambda_{i|H} \sum_{U \in \mathcal{C}(G)} a_U\, \rho_H\left([U, 1]_G\right) = \lambda_{i|H}\, \rho_H(x_1) = 0$$

for $H < G$ and $i = 1, \ldots, m$. It holds

$$\rho_G(x_i) = \lambda_i \sum_{U \in \mathcal{C}(G)} a_U\, \rho_G\left([U, 1]_G\right) = \lambda_i a_G$$

for $i = 1, \ldots, m$, and by the explicit formula (2) for the primitive idempotents of $B_{\mathbb{Q}}(G)$ we get $a_G = 1$. Thus $\rho_G(x_i) = \lambda_i$ and

$$s_{(H,hH')}^{D(G)}(x_i) = \begin{cases} \lambda_i(h) & \text{if } H = G, \\ 0 & \text{else.} \end{cases}$$

Moreover $\rho(x_i) \in \hat{D}(G)$ for $i = 1, \ldots, m$. By the second orthogonality relation we obtain

$$s_{(H,hH')}^{D(G)}\left(\frac{1}{m} \sum_{i=1}^{m} \lambda_i\left(g^{-1}\right) x_i\right) = \begin{cases} 1 & \text{if } (H, hH') = (G, gG'), \\ 0 & \text{else} \end{cases}$$

and therefore

$$e_{(G,gG')}^{D(G)} = \frac{1}{m} \sum_{i=1}^{m} \lambda_i\left(g^{-1}\right) x_i$$

for $g \in G$.

We now show that the conductor of $e^{D(G)}_{(G,1G')}$ is equal to m. For $i=1,\dots,m$ we set $y_i := \rho(x_i) \in \hat{D}(G)$. Then

$$y_i(U,\lambda_{j|U}) = \begin{cases} 1 & \text{if } (U,\lambda_{j|U}) = (G,\lambda_i), \\ 0 & \text{else} \end{cases} \tag{3}$$

for $U \leqslant G$ and $i,j \in \{1,\dots,m\}$. By Proposition 3.2, $\sum_{i=1}^m y_i \in \rho(D(G))$ holds if and only if the congruence

$$\sum_{(H,\varphi)\leqslant(U,\psi)\in\mathcal{M}(N_G(H,\varphi))} \mu(H,U)\sum_{i=1}^m y_i(U,\psi) \equiv 0 \pmod{\bigl(N_G(H,\varphi):H\bigr)} \tag{4}$$

holds for all $(H,\varphi)\in\mathcal{M}(G)$. Since $\rho_U(x_i)=0$ for $U<G$ and $i=1,\dots,m$ we get

$$\sum_{i=1}^m y_i(U,\psi) = 0$$

for $U<G$. In the case $(H,\varphi)\in\mathcal{M}(G)$ with $(H,\varphi)\not\leqslant(G,\lambda_i)$ for $i=1,\dots,m$ and the case $H \ntrianglelefteq G$ congruence (4) is fulfilled. Let $(H,\varphi)\in\mathcal{M}(G)$ with $H \trianglelefteq G$ and $(H,\varphi)\leqslant(G,\lambda)$ for some $\lambda\in\hat{G}$. In this case we get exactly $k := (G:HG')$ extensions of φ on G by Lemma 3.3(i). Let $\lambda_{i_1},\dots,\lambda_{i_k}$ $(i_1,\dots,i_k \in \{1,\dots,m\})$ be these extensions. By equality (3) we obtain

$$\begin{aligned}\sum_{(H,\varphi)\leqslant(U,\psi)\in\mathcal{M}(N_G(H,\varphi))} \mu(H,U)\sum_{i=1}^m y_i(U,\psi) &= \mu(H,G)\sum_{j=1}^k y_{i_j}(G,\lambda_{i_j}) \\ &= \mu(H,G)\bigl(G:HG'\bigr).\end{aligned}$$

By Lemma 3.4 $(N_G(H,\varphi):H)$ divides $(G:HG')\mu(H,G)$. Thus congruence (4) holds for all $(H,\varphi)\in\mathcal{M}(G)$. Moreover

$$\rho\bigl((G:G')e^{D(G)}_{(G,1G')}\bigr) = \rho\left(\sum_{i=1}^m x_i\right) = \sum_{i=1}^m y_i \in \rho\bigl(D(G)\bigr).$$

Since ρ is injective we obtain $(G:G')e^{D(G)}_{(G,1G')} \in D(G)$. Therefore $(G:G')$ is the conductor of $e^{D(G)}_{(G,1G')}$.

For $U\leqslant G$ let $\tau_{U,1},\dots,\tau_{U,s_U}$ $(s_U=(UG':G'))$ be the distinct restrictions $\lambda_{1|U},\dots,\lambda_{m|U}$. For $j=1,\dots,s_U$ we set $M_{\tau_{U,j}} := \{\varphi\in\hat{G}\colon\ \varphi_{|U}=\tau_{U,j}\}$. By Lemma 3.3(ii) we get

$$\begin{aligned}\sum_{i=1}^m \lambda_i\bigl(g^{-1}\bigr)[U,\lambda_{i|U}]_G &= \sum_{j=1}^{s_U}[U,\tau_{U,j}]_G \sum_{\varphi\in M_{\tau_{U,j}}}\varphi\bigl(g^{-1}\bigr) \\ &= \begin{cases} (G:UG')\sum_{j=1}^{s_U}\tau_{U,j}(g^{-1})[U,\tau_{U,j}]_G & \text{if } gG'\in UG'/G', \\ 0 & \text{else}\end{cases}\end{aligned}$$

for $U\leqslant G$ and $g\in G$. Therefore

$$\sum_{i=1}^m \lambda_i\bigl(g^{-1}\bigr)x_i = \sum_{U\in\mathcal{C}(G)} a_U \sum_{i=1}^m \lambda_i\bigl(g^{-1}\bigr)[U,\lambda_{i|U}]_G$$

$$= \sum_{\substack{U \in \mathcal{C}(G) \\ gG' \in UG'/G'}} a_U\big(G : UG'\big) \sum_{j=1}^{s_U} \tau_{U,j}\big(g^{-1}\big)[U, \tau_{U,j}]_G$$

for $g \in G$. Note that this equation does not depend on the choice of $\mathcal{C}(G)$. Since

$$me^{D(G)}_{(G,1G')} = \sum_{i=1}^{m} x_i = \sum_{U \in \mathcal{C}(G)} a_U\big(G : UG'\big) \sum_{j=1}^{s_U} [U, \tau_{U,j}]_G \in D(G)$$

and $[U, \tau_{U,r}]_G \neq [U, \tau_{U,t}]_G$ for $r, t \in \{1, \dots, s_U\}$ with $r \neq t$ we get $a_U(G : UG') \in \mathbb{Z}$ for $U \in \mathcal{C}(G)$. Thus

$$me^{D(G)}_{(G,gG')} = \sum_{i=1}^{m} \lambda_i\big(g^{-1}\big)x_i \in D_{\mathbb{Z}[\zeta]}(G).$$

Therefore $m = (G : G')$ is the conductor of $e^{D(G)}_{(G,gG')}$, $g \in G$.

Let $(H, hH') \in \mathcal{D}(G)$. By Lemma 3.1(ii) we obtain

$$\big(N_G\big(H, hH'\big) : H'\big)e^{D(G)}_{(H,hH')} = \operatorname{ind}_H^G\big(\big(H : H'\big)e^{D(H)}_{(H,hH')}\big) \in D_{\mathbb{Z}[\zeta]}(G).$$

Moreover the coefficient of $[H, 1]_G$ in $e^{D(G)}_{(H,hH')}$ is equal to $|H'|/|N_G(H, hH')|$. Therefore $(N_G(H, hH') : H')$ is the conductor of $e^{D(G)}_{(H,hH')}$. □

We can now state the first consequences.

Theorem 3.6. *The group order $|G|$ is uniquely determined by $D(G)$.*

Proof. Let $W \subseteq \mathbb{C}$ be the set of all roots of unity and let $\mathcal{O}$ be the ring of the algebraic integers of $\mathbb{Q}(W)$. Every $e^{D(G)}_{(H,hH')}$ is a primitive idempotent of $D_{\mathbb{Q}(W)}(G)$ for $(H, hH') \in \mathcal{D}(G)$ and $(N_G(H, hH') : H')$ is the minimal natural number $n \in \mathbb{N}$ with $ne^{D(G)}_{(H,hH')} \in D_{\mathcal{O}}(G)$. Moreover $|G|$ is the conductor of $e^{D(G)}_{(1,1)}$ and therefore

$$|G| = \min\big\{n \in \mathbb{N}\colon ne^{D(G)}_{(H,hH')} \in D_{\mathcal{O}}(G) \text{ for all } \big(H, hH'\big) \in \mathcal{D}(G)\big\}.$$

Thus the theorem is proved. □

The following proposition is an immediate consequence of Theorem 3.5.

Proposition 3.7. *Let $(H, hH') \in \mathcal{D}(G)$. Then the conductor of $e^{D(G)}_{(H,hH')} \in D_{\mathbb{Q}(\zeta)}(G)$ is equal to $|G|$ if and only if H is a normal abelian subgroup and $h \in Z(G)$. Moreover G is abelian if and only if the conductors of the primitive idempotents of $D_{\mathbb{Q}(\zeta)}(G)$ are equal to $|G|$.*

Therefore the ring $D(G)$ detects commutativity of a finite group. We now state an interesting proposition concerning the orders of elements of the center of G.

Proposition 3.8. *Let G and $\tilde{G}$ be finite groups and $\alpha : D(G) \to D(\tilde{G})$ be an isomorphism. Let $h \in Z(G)$, $H := \langle h \rangle$, $n := |H|$ and $\alpha(e^{D(G)}_{(H,h)}) = e^{D(\tilde{G})}_{(\tilde{H},\tilde{h}\tilde{H}')}$ with $(\tilde{H}, \tilde{h}\tilde{H}') \in \mathcal{D}(\tilde{G})$. Then $\tilde{H}$ is a normal abelian subgroup of $\tilde{G}$, $\tilde{h} \in Z(\tilde{G})$ and $|\langle \tilde{h} \rangle| \in \{n, 2n, \frac{n}{2}\}$.*

Proof. The subgroup H is abelian and normal since $h \in Z(G)$. Moreover the conductor of $e^{D(G)}_{(H,h)}$ is equal to $|G|$. We set

$$M := \{x \in D_{\mathbb{Q}}(G)\colon\ s^{D(G)}_{(H,h)}(x) \in \mathbb{C} \text{ is root of unity}\}.$$

It holds $s^{D(G)}_{(H,h)}(x) \in \mathbb{Q}(\zeta)$ for all $x \in M$, and the set $\{\mathrm{ord}(\xi)\colon\ s^{D(G)}_{(H,h)}(x) = \xi,\ x \in M\}$ is bounded since $\pm\zeta^i$ ($i \in \mathbb{N}$) are the only roots of unity in $\mathbb{Q}(\zeta)$. We set

$$m := \max\{\mathrm{ord}(\xi)\colon\ s^{D(G)}_{(H,h)}(x) = \xi,\ x \in M\}.$$

Let $\lambda \in \hat{H}$ with $\lambda(h) = \omega$ where $\omega \in \mathbb{C}$ is a primitive n-th root of unity. Then

$$s^{D(G)}_{(H,h)}\big([H,\lambda]_G\big) = \sum_{gH \in G/H} {}^{g}\lambda(h) = (G:H)\omega.$$

Thus $y := (-1)^n (G:H)^{-1}[H,\lambda]_G \in M$. We obtain

$$\mathrm{ord}\big(s^{D(G)}_{(H,h)}(y)\big) = \begin{cases} 2\mathrm{n} & \text{if } n \text{ odd}, \\ \mathrm{n} & \text{if } n \text{ even}. \end{cases} \tag{5}$$

We now show the equality $m = \mathrm{ord}(s^{D(G)}_{(H,h)}(y))$. Let

$$x := \sum_{[U,\psi]_G \in \mathcal{M}(G)/G} a_{[U,\psi]}[U,\psi]_G \in M$$

with $a_{[U,\psi]} \in \mathbb{Q}$ for $[U,\psi]_G \in \mathcal{M}(G)/G$. In case $U \leqslant G$ with $H \nleqslant_G U$ we get $s^{D(G)}_{(H,h)}([U,\psi]_G) = 0$. In case $[U,\psi]_G \in \mathcal{M}(G)/G$ with $H \leqslant_G U$ we get $H \leqslant U$ and $\psi(h) \in \mathbb{Q}(\omega)$. Thus

$$s^{D(G)}_{(H,h)}(x) = \sum_{[U,\psi]_G \in \mathcal{M}(G)/G} a_{[U,\psi]} s^{D(G)}_{(H,h)}\big([U,\psi]_G\big) = \sum_{\substack{[U,\psi]_G \in \mathcal{M}(G)/G \\ H \leqslant U}} a_{[U,\psi]} \sum_{gU \in G/U} \psi(h) \in \mathbb{Q}(\omega).$$

Since ω^i ($i \in \mathbb{N}$) are the only roots of unity in $\mathbb{Q}(\omega)$ we get $m \leqslant 2n$ in case n is odd and $m \leqslant n$ in case n is even. Together with Eq. (5) we obtain

$$m = \begin{cases} 2n & \text{if } n \text{ odd}, \\ n & \text{if } n \text{ even}. \end{cases}$$

By Proposition 3.7, $\tilde{H}$ is abelian and normal and $\tilde{h} \in Z(\tilde{G})$ since the conductor of $e^{D(\tilde{G})}_{(\tilde{H},\tilde{h})}$ is equal to $|G| = |\tilde{G}|$. We set

$$\tilde{M} := \{\tilde{x} \in D_{\mathbb{Q}}(\tilde{G})\colon\ s^{D(\tilde{G})}_{(\tilde{H},\tilde{h})}(\tilde{x}) \in \mathbb{C} \text{ is root of unity}\}$$

and

$$\tilde{m} := \max\{\mathrm{ord}(\xi)\colon\ s^{D(\tilde{G})}_{(\tilde{H},\tilde{h})}(\tilde{x}) = \xi,\ \tilde{x} \in \tilde{M}\}.$$

Let $\tilde{n} := |\langle\tilde{h}\rangle|$ and $\tilde{\omega} \in \mathbb{C}$ a primitive $\tilde{n}$-th root of unity. Since $\tilde{H}$ is abelian there exists a linear character $\tilde{\lambda}$ of $\tilde{H}$ with $\tilde{\lambda}(\tilde{h}) = \tilde{\omega}$. Analogous to the above descriptions we set $\tilde{y} := (-1)^{\tilde{n}}(\tilde{G} : \tilde{H})^{-1}[\tilde{H}, \tilde{\lambda}]_{\tilde{G}} \in \tilde{M}$ and we obtain

$$\operatorname{ord}\left(s^{D(\tilde{G})}_{(\tilde{H},\tilde{h})}(\tilde{y})\right) = \begin{cases} 2\tilde{n} & \text{if } \tilde{n} \text{ odd}, \\ \tilde{n} & \text{if } \tilde{n} \text{ even}. \end{cases}$$

With the same argumentation as above we get

$$\tilde{m} = \begin{cases} 2\tilde{n} & \text{if } \tilde{n} \text{ odd}, \\ \tilde{n} & \text{if } \tilde{n} \text{ even}. \end{cases}$$

It holds $\alpha(M) = \tilde{M}$ since $s^{D(\tilde{G})}_{(\tilde{H},\tilde{h})} \circ \alpha = s^{D(G)}_{(H,h)}$. Thus $m = \tilde{m}$ and $n = \tilde{n}$, $n = 2\tilde{n}$ and $2n = \tilde{n}$ are the only cases that could arise. Therefore $\tilde{n} \in \{n, 2n, \frac{n}{2}\}$. □

A direct consequence of this proposition is the following theorem:

Theorem 3.9. *Let G and $\tilde{G}$ be finite groups with $D(G) \cong D(\tilde{G})$. If $2 \neq p$ is a prime number which divides $|Z(G)|$ then p divides $|Z(\tilde{G})|$. If there exists an element of order 4 in $Z(G)$ then 2 divides $|Z(\tilde{G})|$.*

4. The group of torsion units of $D(G)$

We develop some results on the group of torsion units of $D(G)$ following results for the Burnside ring in [15]. For a commutative unitary ring R let $U_T(R)$ be the group of torsion units of R.

Lemma 4.1. *Let R be a commutative unitary ring and let A and B be additive subgroups of R with the following properties:*

$$R = A \oplus B, \qquad A^2 \subseteq A, \qquad B^2 \subseteq B, \qquad AB \subseteq A, \qquad 1 \in B.$$

Therefore A is an ideal in R and B is a unitary subring of R. Moreover we require the existence of a natural number $n \in \mathbb{N}$ with $u^n = 1$ for all $u \in U_T(R)$. Then:

- (i) *Every torsion unit $u \in U_T(R)$ is of the form $u = b(1 + a)$ with uniquely determined elements $b \in U_T(B)$ and $a \in \tilde{A} := \{a \in A{:}\ \sum_{k=1}^{n} \binom{n}{k} a^k = 0\}$. Moreover every element $b(1 + a)$ with $b \in U_T(R)$ and $a \in \tilde{A}$ is a torsion unit of R.*
- (ii) *It is $|U_T(R)| = |U_T(B)||\tilde{A}|$ in case $U_T(R)$ is finite.*

Proof. $\tilde{A}$ is not empty since $0 \in \tilde{A}$. Let $b \in U_T(B)$ and $a \in \tilde{A}$. Then

$$\big(b(1 + a)\big)^n = (1 + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1.$$

Thus $b(1 + a) \in U_T(R)$.

Let $u \in U_T(R)$. Then there exist uniquely determined elements $a \in A$ and $b \in B$ with $u = a + b$. Therefore

$$1 = u^n = (a + b)^n = \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^k + b^n.$$

Note that $\sum_{k=0}^{n-1}\binom{n}{k}a^{n-k}b^k \in A$ and $b^n - 1 \in B$. We obtain $b^n - 1 = 0$ since $R = A \oplus B$. Thus $b \in U_T(B)$. Let $c := ab^{n-1} \in A$. Then $b(1+c) = b + a = u$, and since

$$\sum_{k=1}^{n}\binom{n}{k}c^k = \sum_{k=1}^{n}\binom{n}{k}\left(ab^{n-1}\right)^k = \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k} = (a+b)^n - b^n = 1 - b^n = 0$$

we get $c \in \tilde{A}$.

Let $b_1, b_2 \in U_T(B)$ and $c_1, c_2 \in \tilde{A}$ with $b_1(1+c_1) = b_2(1+c_2)$. Then $b_1 - b_2 + b_1c_1 - b_2c_2 = 0$, and since $b_1, b_2 \in B$, $b_1c_1, b_2c_2 \in A$ and $R = A \oplus B$ it follows that $b_1 = b_2$ and $c_1 = c_2$ and the proof of part (i) is complete. Part (ii) is a direct consequence of part (i). □

A partially ordered set $(I, \leqslant)$ is called *rigid* if

(i) I contains a greatest element e and a smallest element 0.
(ii) Every subset $M_{i,j} := \{k \in I\colon\ k \leqslant i, k \leqslant j\}$, $i, j \in I$, contains a greatest element $m(i,j)$. (Therefore every two elements $i, j \in I$ have an infimum in I.)

Proposition 4.2. *Let R be a commutative unitary ring and $(I, \leqslant)$ be a finite, partially ordered, rigid set. We assume the existence of a family $\{R(i)\colon\ i \in I\}$ of additive subgroups of R with the following properties*:

(1) $R = \bigoplus_{i\in I} R(i)$ *(direct sum of additive groups),*
(2) $R(e) = \mathbb{Z}H$ *with a finite subgroup* $H \leqslant U_T(R)$*,*
(3) $R(i)R(j) \subseteq R(m(i,j))$ *for all* $i, j \in I$*.*

Furthermore there exists $n \in \mathbb{N}$ with $u^n = 1$ for all $u \in U_T(R)$. For $i \in I\backslash\{e\}$ we set

$$R_i := \left\{a \in R(i)\colon \sum_{k=1}^{n}\binom{n}{k}a^k = 0\right\}.$$

Then:

(i) *Every torsion unit $u \in U_T(R)$ is of the form*

$$u = g\prod_{i\in I\backslash\{e\}}(1 + a_i)$$

with uniquely determined elements $a_i \in R_i$ and $g \in \pm H$. Moreover every element of this form is a torsion unit in R.

(ii) *It is $|U_T(R)| = 2|H|\prod_{i\in I\backslash\{e\}}|R_i|$ in case $U_T(R)$ is finite.*

Proof. We show the first part of (i) by induction on $|I|$. In case $|I| = 1$ we get $R = R(e) = \mathbb{Z}H$. Since H is an abelian group, $U_T(\mathbb{Z}H) = \pm H$ (cf. [12]).

Let $|I| = 2$. Then $R = R(0) \oplus R(e)$. Since $m(i,i) = i$ and $m(i,0) = 0$ for $i \in I$ we obtain

$$R(0)R(0) \subseteq R(0), \qquad R(e)R(e) \subseteq R(e), \quad \text{and} \quad R(0)R(e) \subseteq R(0).$$

Moreover $1 \in R(e)$. By Lemma 4.1 (with $A := R(0)$ and $B := R(e)$) every torsion unit $u \in U_T(R)$ is of the form $u = g(1+a)$ with uniquely determined elements $a \in R_0$ and $g \in U_T(R(e)) = U_T(\mathbb{Z}H) = \pm H$. Moreover every element $u = g(1+a)$ with $g \in \pm H$ and $a \in R_0$ is a torsion unit of R by Lemma 4.1.

Let $|I| \geqslant 3$ and k be a maximal element of $\{i \in I\colon\ i < e\}$. We set

$$J := I \backslash \{k\}, \qquad A := \bigoplus_{j \in J \backslash \{e\}} R(j) \quad \text{and} \quad B := R(e) \oplus R(k).$$

Then

$$R = A \oplus B, \qquad A^2 \subseteq A, \qquad B^2 \subseteq B, \qquad AB \subseteq A \quad \text{and} \quad 1 \in R(e) \subseteq B.$$

Let $u \in U_T(R)$. By Lemma 4.1 we can write $u = b(1 + a)$ with uniquely determined $b \in U_T(B) = U_T(R(e) \oplus R(k))$ and $a \in \tilde{A} := \{a \in A\colon\ \sum_{k=1}^{n} \binom{n}{k} a^k = 0\}$. Since

$$R(e)^2 \subseteq R(e), \qquad R(k)^2 \subseteq R(k), \qquad R(e)R(k) \subseteq R(k) \quad \text{and} \quad 1 \in R(e)$$

we can use Lemma 4.1 for the unitary subring $B = R(e) \oplus R(k)$. Thus b is of the form $b = g(1 + a_k)$ with uniquely determined elements $g \in U_T(R(e)) = \pm H$ and $a_k \in R_k$. Therefore $u = g(1 + a_k)(1 + a)$.

The ring $\bigoplus_{j \in J} R(j)$ is commutative and unitary and J is a finite, partial ordered, rigid set. Therefore the conditions of the propositions are fulfilled and we can use induction. Since

$$(1 + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1,$$

it holds $1 + a \in U_T(\bigoplus_{j \in J} R(j))$, and by induction follows that $1 + a = h \prod_{j \in J \backslash \{e\}} (1 + a_j)$ with uniquely determined $h \in \pm H$ and $a_j \in R_j$. Therefore $u = gh \prod_{i \in I \backslash \{e\}} (1 + a_j)$.

Let $u = g' \prod_{i \in I \backslash \{e\}} (1 + a_i')$ with $g' \in \pm H$ and $a_i' \in R_i$. Then

$$1 = gh\bigl(g'\bigr)^{-1} \prod_{i \in I \backslash \{e\}} (1 + a_i)\bigl(1 + a_i'\bigr)^{-1}.$$

Since $(1 + a_i') \in U_T(R)$ there exists $s_i \in \mathbb{N}$ with $(1 + a_i')^{s_i} = (1 + a_i')^{-1}$ for $i \in I \backslash \{e\}$. Since $R(i)^2 \subseteq R(i)$ there exists $c_i \in R(i)$ with $(1 + a_i)(1 + a_i')^{-1} = (1 + a_i)(1 + a_i')^{s_i} = 1 + c_i$ for $i \in I \backslash \{e\}$. Therefore

$$1 = gh\bigl(g'\bigr)^{-1} \prod_{i \in I \backslash \{e\}} (1 + c_i). \tag{6}$$

Since $R(e)R(i) \subseteq R(i)$ for $i \in I$ we get $1 = gh(g')^{-1} + r_1$ with $r_1 \notin R(e)$ by expanding Eq. (6). The decomposition $R = \bigoplus_{i \in I} R(i)$ implies $gh(g')^{-1} = 1$ and therefore $gh = g'$. Assume $c_i \neq 0$ for some $i \in I \backslash \{e\}$. We choose $i \in I \backslash \{e\}$ maximal with the property $c_i \neq 0$. In case $j \in I \backslash \{e, i\}$ with $c_j \neq 0$ we get $m(i, j) \neq i$ by the maximality of i. Thus $c_i c_j \notin R(i)$. By expanding Eq. (6) we get $1 = 1 + c_i + r_2$ with $r_2 \notin R(i)$ The decomposition $R = \bigoplus_{i \in I} R(i)$ implies $c_i = 0$ contradicting our assumption. Therefore $c_i = 0$ for all $i \in I \backslash \{e\}$. Thus $1 + a_i = 1 + a_i'$ for all $i \in I \backslash \{e\}$.

Conversely $g \prod_{i \in I \backslash \{e\}} (1 + a_i) \in U_T(R)$ since $g \in U_T(R)$ for $g \in \pm H$ and $1 + a_i \in U_T(R)$ for $a_i \in R_i$. Thus assertion (i) is proved.

Part (ii) follows immediately from part (i). □

Let G be a finite group and $\mathcal{N}(G)$ be the set of normal subgroups of G. We say that a subset $S \subseteq \mathcal{N}(G)$ has property $(*)$ in cases

1. $1, G \in S$,
2. $M, N \in S$ implies $MN \in S$ and $M \cap N \in S$.

Let $S \subseteq \mathcal{N}(G)$ with property $(*)$. For $N \in S$ let $S(N)$ be the set of all elements $[K,\psi]_G \in \mathcal{M}(G)/G$ with the following properties:

1. $N \leqslant K$,
2. $N \leqslant M \leqslant K$ with $M \in S$ implies $N = M$.

Remark 4.3. We should remark the following facts: For a nonempty subset $S \subseteq \mathcal{N}(G)$ we get $S(N) \neq \emptyset$ since $[N,1]_G \in S(N)$ for $N \in S$. The set $\{S(N)\colon N \in S\}$ is a partially ordered rigid set with $S(L) \leqslant S(M)$ in case $L \leqslant M$. Moreover $S(G)$ is the greatest and $S(1)$ is the smallest element of $\{S(N)\colon N \in S\}$. The infimum of two elements $S(L), S(N) \in \{S(N)\colon N \in S\}$ is given by $S(L \cap N)$. The group $(S(G),\cdot)$ is a subgroup of $U_T(D(G))$ with $S(G) \cong \hat{G}$. We should also remark that $[K,\psi]_G \in S(N)$ implies $N \leqslant {}^gK$ for all $g \in G$. Thus the above definition of $S(N)$ does not depend on the choice of the representative subgroup K.

Let $T \subseteq \mathcal{M}(G)/G$. The additive subgroup of $D(G)$ which is generated by the elements $[H,\varphi]_G \in T$ will be denoted by $D(G)_T$. We set $D(G)_T = \{0\}$ in case $T = \emptyset$.

Lemma 4.4. *Let $S \subseteq \mathcal{N}(G)$ with property $(*)$. Then*:

(i) $D(G) = \bigoplus_{N \in S} D(G)_{S(N)}$ (*direct sum of additive subgroups*),
(ii) $D(G)_{S(M)} D(G)_{S(N)} \subseteq D(G)_{S(M\cap N)}$ *for* $M, N \in S$,
(iii) $D(G)_{S(G)} = \mathbb{Z}S(G) \cong \mathbb{Z}\hat{G}$.

Proof. Let $[K,\psi]_G \in S(M) \cap S(N)$ with $M, N \in S$. Then $M \leqslant MN \leqslant K$ and $N \leqslant NM \leqslant K$. Since $MN \in S$ we get $M = MN = N$. Thus $S(M) \cap S(N) = \emptyset$ for $M, N \in S$ with $M \neq N$.

Let $[K,\psi]_G \in \mathcal{M}(G)/G$ and set $X_K := \{N \in S\colon N \leqslant K\}$. It is $X_K \neq \emptyset$ since $1 \in S$. Let $N_0 := \prod_{N \in X_K} N$. Since S has property $(*)$ we get $N_0 \in S$ and therefore $N_0 \in X_K$. Thus $[K,\psi]_G \in S(N_0)$ and we get

$$\mathcal{M}(G)/G = \biguplus_{N \in S} S(N).$$

Part (i) follows immediately.

Let $[H,\psi]_G \in S(M)$ and $[K,\psi]_G \in S(N)$ with $M, N \in S$. Since

$$[H,\varphi]_G[K,\psi]_G = \sum_{HgK \in H\backslash G/K} \bigl[H \cap {}^gK, \varphi \cdot {}^g\psi\bigr]_G$$

we have to show $[H \cap {}^gK, \varphi \cdot {}^g\psi]_G \in S(M \cap N)$ for all $g \in G$. It holds $M \leqslant H$ and $N \leqslant {}^gK$ for all $g \in G$. Therefore $M \cap N \leqslant H \cap {}^gK$ for all $g \in G$. Let $M \cap N \leqslant L \leqslant H \cap {}^gK$ for $L \in S$ and $g \in G$. Then

$$M \leqslant ML \leqslant M\bigl(H \cap {}^gK\bigr) \leqslant H$$

and

$$N \leqslant NL \leqslant N\bigl(H^g \cap K\bigr) \leqslant K.$$

Since $[H,\varphi]_G \in S(M)$ and $[K,\psi]_G \in S(N)$ we get $M = ML$ and $N = NL$. Thus $L \leqslant M \cap N$, and this implies $L = M \cap N$. Therefore $[H \cap {}^gK, \tau]_G \in S(M \cap N)$ for all $g \in G$ and all linear characters τ of $H \cap {}^gK$ and part (ii) is proved.

Part (iii) is a direct consequence of $S(G) \cong \hat{G}$ and the definition of $D(G)_{S(G)}$. □

Remark 4.5. Let $\zeta \in \mathbb{C}$ be a primitive $|G|$-th root of unity. Every torsion unit $u \in U_T(D(G))$ is of the form

$$u = \sum_{[H,hH']_G \in \mathcal{D}(G)/G} u_{[H,hH']} e^{D(G)}_{(H,hH')}$$

with $u_{[H,hH']} \in \{\pm\zeta^i\colon\ i \in \mathbb{N}\}$ for all $[H,hH]_G \in \mathcal{D}(G)/G$. Thus $U_T(D(G))$ is a finite group. Moreover the exponent $\exp(U_T(D(G)))$ of $U_T(D(G))$ divides $2|G|$.

We can now state the main theorem of this section which is a direct consequence of Proposition 4.2, Lemma 4.4 and Remark 4.5.

Theorem 4.6. *Let G be a finite group and S be a subset of $\mathcal{N}(G)$ with property $(*)$. Let $n \in \mathbb{N}$ be a multiple of $\exp(U_T(D(G)))$. For $H \in S$ we set*

$$H^* := \left\{a \in D(G)_{S(H)}\colon \sum_{k=1}^{n} \binom{n}{k} a^k = 0\right\}.$$

Then every torsion unit $u \in U_T(D(G))$ is of the form

$$u = \pm[G,\psi]_G \prod_{H \in S\setminus\{G\}} (1 + u_H)$$

with uniquely determined $u_H \in H^*$ and $\psi \in \hat{G}$. Moreover

$$\bigl|U_T\bigl(D(G)\bigr)\bigr| = 2|\hat{G}|\left(\prod_{H\in S\setminus\{G\}} \bigl|H^*\bigr|\right).$$

5. Abelian groups

In Proposition 3.7 we proved that the ring $D(G)$ detects commutativity of the group G. With the help of Theorem 4.6 we will show that $D(G) \cong D(\tilde{G})$ with an abelian group G implies $G \cong \tilde{G}$. In the following we will use the notation C_2 for the group with 2 elements.

Proposition 5.1. *Let G be an abelian group. Then*

$$U_T\bigl(D(G)\bigr) \cong G \times C_2^{m+1},$$

where m is the number of subgroups of G with index 2.

Proof. For $G = 1$ the assumption is clear. Let $G \neq 1$. We use the notations of Theorem 4.6 and set $S := \{H\colon\ H \leqslant G\}$ and $n := 2|G|$. Then S has property $(*)$, and for $H \in S$, $S(H) = \{[H,\psi]_G\colon\ \psi \in \hat{H}\}$ holds. Let $U < G$ be a proper subgroup and $a \in U^*$. Then $a+1$ is a torsion unit in $D(G)$. Let $\rho : D(G) \to \hat{D}(G)$ be the embedding of D(G) in the ghost ring $\hat{D}(G)$ and ρ_U the projection in $\mathbb{Z}\hat{U}$. Then $\rho_U(a+1) \in \mathbb{Z}\hat{U}$ is a torsion unit in $\mathbb{Z}\hat{U}$. Since $\hat{U}$ is abelian, the set of all torsion units of $\mathbb{Z}\hat{U}$ is $\pm\hat{U}$ (cf. [12]). Thus there exists $\tau \in \hat{U}$ with $\rho_U(a+1) = \pm\tau$. The element a is of the form $a = \sum_{\lambda\in\hat{U}} a_{[U,\lambda]}[U,\lambda]_G$ with $a_{[U,\lambda]} \in \mathbb{Z}$. Since G is abelian, we obtain

$$\pm\tau - 1 = \rho_U(a) = \sum_{\lambda\in\hat{U}} a_{[U,\lambda]} \sum_{gU \in G/U} {}^{g}\lambda = (G:U) \sum_{\lambda\in\hat{U}} a_{[U,\lambda]}\lambda.$$

Note that in the above equation we use $\rho_U([U,\lambda]_G) = \sum_{gU\in G/U} {}^g\lambda$. In case $2 < (G:U)$ we get $a_{[U,\lambda]} = 0$ for all $\lambda \in \hat{U}$ and therefore $a = 0$. Let $(G:U) = 2$. We obtain $\rho_U(a) \in \{0, -2\}$, and in case $\rho_U(a) = 0$ we get $a_{[U,\lambda]} = 0$ for all $\lambda \in \hat{U}$ and therefore $a = 0$. Let $\rho_U(a) = -2$. Then $a_{[U,1]} = -1$ and $a_{[U,\lambda]} = 0$ for all $\lambda \in \hat{U}\backslash\{1\}$. Moreover

$$\bigl(1 - [U,1]_G\bigr)^2 = 1 - 2[U,1]_G + [U,1]_G^2 = 1 - 2[U,1] + \sum_{gU\in G/U} [U,1]_G = 1. \tag{7}$$

Then $(1 - [U,1]_G)^{2|G|} = 1$ and therefore $-[U,1]_G \in U^*$. Thus $U^* = \{0, -[U,1]_G\}$. All in all we get

$$\bigl|U^*\bigr| = \begin{cases} 2 & \text{if } (G:U) = 2, \\ 1 & \text{else.} \end{cases} \tag{8}$$

Since every torsion unit $u \in U_T(D(G))$ is of the form

$$u = \pm[G,\psi]_G \prod_{H\in S\backslash\{G\}} (1 + u_H)$$

with uniquely determined $u_H \in H^*$ and $\psi \in \hat{G}$ we get the desired isomorphism by Eq. (7) and (8). □

Theorem 5.2. *Let G be a finite abelian group and let $\tilde{G}$ be a finite group with $D(G) \cong D(\tilde{G})$. Then $G \cong \tilde{G}$.*

Proof. By Proposition 3.7 the group $\tilde{G}$ is abelian. Moreover $U_T(D(G)) \cong U_T(D(\tilde{G}))$. By Proposition 5.1 we get $G \times C_2^{m+1} \cong \tilde{G} \times C_2^{\tilde{m}+1}$ where m and $\tilde{m}$ are the numbers of subgroups of G and $\tilde{G}$ with index 2. Then $|G \times C_2^{m+1}| = |\tilde{G} \times C_2^{\tilde{m}+1}|$, and since $|G| = |\tilde{G}|$ we obtain $m = \tilde{m}$ and therefore $G \cong \tilde{G}$. □

6. The primitive idempotents of $\mathbb{Z}[\zeta]_{\mathfrak{p}} \otimes_{\mathbb{Z}} D(G)$

Let $\mathfrak{p}$ be a maximal ideal in $\mathbb{Z}[\zeta]$, $p := \mathrm{char}(\mathbb{Z}[\zeta]/\mathfrak{p})$ and $R := \mathbb{Z}[\zeta]_{\mathfrak{p}}$ the localization of $\mathbb{Z}[\zeta]$ at $\mathfrak{p}$. In this section we will state a formula for the primitive idempotents of $D_R(G)$.

We write

$$\bigl(H, hH'\bigr) \equiv_p \bigl(U, uU'\bigr)$$

for $(H, hH'), (U, uU') \in \mathcal{D}(G)$ in case

$$s^{D(G)}_{(H,hH')}(x) \equiv s^{D(G)}_{(U,uU')}(x) \pmod{\mathfrak{p}}$$

for all $x \in D(G)$. Then $\equiv_p$ is an equivalence relation on $\mathcal{D}(G)$. The equivalence classes of this relation are called $\mathfrak{p}$-equivalence classes of $\mathcal{D}(G)$. We define

$$\mathcal{D}_p(G) := \bigl\{\bigl(K, kK'\bigr) \in \mathcal{D}(G)\colon\ \bigl|\langle k\rangle\bigr| \not\equiv 0 \not\equiv \bigl(N_G\bigl(K, kK'\bigr) : K\bigr) \pmod{p}\bigr\}.$$

The following proposition summarizes some results of [9].

Proposition 6.1.

(i) *It holds $(H, hH') \equiv_p (H, h_{p'}H')$ for all $(H, hH') \in \mathcal{D}(G)$.*
(ii) *Let $(H, hH') \in \mathcal{D}(G)$ and K/H be a p-subgroup of $N_G(H, hH')/H$. Then $(H, hH') \equiv_p (K, hK')$.*
(iii) *Let $(H, hH'), (K, kK') \in \mathcal{D}_p(G)$. Then $(H, hH') \equiv_p (K, kK')$ if and only if (H, hH') and (K, kK') are conjugate in G.*

Proof. See [9], Lem. 1, Lem. 2, Prop. 3. □

Let $(H, hH') \in \mathcal{D}(G)$. By Proposition 6.1(i) we get $(H, hH') \equiv_p (H, h_{p'}H')$, and for a Sylow p-subgroup H_1/H of $N_G(H, hH')/H$ we conclude $(H, h_{p'}H') \equiv_p (H_1, h_{p'}H'_1)$ by Proposition 6.1(ii). With the same argument we get $(H_2, h_{p'}H'_2) \equiv_p (H_1, h_{p'}H'_1)$ for a Sylow p-subgroup H_2/H_1 of $N_G(H_1, h_{p'}H'_1)/H_1$. If we go on like this we obtain $(H_n, h_{p'}H'_n) \in \mathcal{D}_p(G)$ for some $n \in \mathbb{N}$. We call $(H_n, h_{p'}H'_n)$ a *p-regularization* of (H, hH'). Moreover $(H_n, h_{p'}H'_n)$ is uniquely determined up to conjugation in G (cf.[9]). By Proposition 6.1 we conclude that every $\mathfrak{p}$-equivalence class of $\mathcal{D}(G)$ is represented by exactly one orbit $[H, hH']_G \in \mathcal{D}(G)/G$ with $(H, hH') \in \mathcal{D}_p(G)$.

We use the notation $O^p(G)$ for the smallest normal subgroup of G such that $G/O^p(G)$ is a p-group. The group G is called *p-perfect* in case $O^p(G) = G$. The subgroup $O^p(G)$ is p-perfect and characteristic in G. For a p-regularization $(H_n, h_{p'}H'_n)$ of $(H, hH') \in \mathcal{D}(G)$ it holds $O^p(H_n) = O^p(H) \leqslant H$.

We also use the following well-known lemmata.

Lemma 6.2. *Let G be a finite group, A a normal abelian Hall-subgroup of G and $[A, G]$ the commutator of A with G. Then $A = C_A(G) \oplus [A, G]$.*

Proof. See [13], Kapitel III, Satz 13.4. □

Lemma 6.3. *Let G be a finite group and H be an abelian Hall-subgroup of G. Then $H \cap G' \cap Z(G) = 1$.*

Proof. See [13], Kapitel IV, Satz 2.2. □

Let H be a p-perfect subgroup of G and $h \in G$. We define

$$S^p(H, hH') := \{U \leqslant G\colon\ O^p(U) = H,\ U \leqslant N_G(H, hH')\}.$$

For $U \in S^p(H, hH')$ and $u \in U$ we get $u_{p'} \in H$. Since p does not divide $(H : H')$, the group H/H' is a normal abelian Hall-subgroup of U/H'. It follows that

$$H/H' = C_{H/H'}(U/H') \oplus [H/H', U/H']$$

by Lemma 6.2. In the following we write $u_{p',c}H'$ for the $C_{H/H'}(U/H')$-part of $u_{p'}H'$ in H/H'. We can now state the main theorem of this section.

Theorem 6.4. *There is a* 1-1*-correspondence between the primitive idempotents of $D_R(G)$ and the elements of the set*

$$I := \{[H, hH']_G \in \mathcal{D}(G)/G\colon\ H = O^p(H)\}.$$

An explicit formula for the primitive idempotents is given by

$$e^{D(G),p}_{(H,hH')} = \sum_{\substack{[U,uU']_G \in \mathcal{D}(G)/G \\ U \in S^p(H,hH') \\ u_{p',c}H' = hH'}} e^{D(G)}_{(U,uU')}, \quad [H, hH']_G \in I.$$

Proof. There is a 1-1-correspondence between the primitive idempotents of $D_R(G)$ and the $\mathfrak{p}$-equivalence classes of $\mathcal{D}(G)$ (cf. [7], Satz 1.12). We will show that every $\mathfrak{p}$-equivalence class of $\mathcal{D}(G)$ contains exactly one G-orbit $[H, hH']_G$ with a p-perfect subgroup H.

Let $(U, uU') \in \mathcal{D}(G)$. We set $H := O^p(U)$, $\bar{H} := H/H'$ and $\bar{U} := U/H'$. Then H is p-perfect and $\bar{H}$ is a normal abelian Hall-subgroup of $\bar{U}$. By Lemma 6.2 we get

$$\bar{H} = C_{\bar{H}}(\bar{U}) \oplus [\bar{H}, \bar{U}],$$

where $[\bar{H}, \bar{U}]$ is the commutator of $\bar{H}$ and $\bar{U}$. It holds $u_{p'}H' \in \bar{H}$ since $(\bar{U} : \bar{H})$ is a p-power. Thus there exist $hH' \in C_{\bar{H}}(\bar{U})$ and $vH' \in [\bar{H}, \bar{U}]$ with $u_{p'}H' = hvH'$. Therefore $u_{p'}U' = hvU' \in U/U'$ holds. Moreover $v \in U'$ since $vH' \in [\bar{H}, \bar{U}] \leqslant \bar{U}' = U'/H'$. Thus

$$\left(U, u_{p'}U'\right) = \left(U, hU'\right).$$

It is $H \trianglelefteq U$ and since $hH' \in C_{\bar{H}}(\bar{U})$ we get $whw^{-1}H' = hH'$ for all $w \in U$. Thus $U \leqslant N_G(H, hH')$ and U/H is a p-subgroup of $N_G(H, hH')/H$. By Proposition 6.1(i) and (ii) it holds

$$\left(H, hH'\right) \equiv_p \left(U, hU'\right) = \left(U, u_{p'}U'\right) \equiv_p \left(U, uU'\right).$$

All in all we can say at this point that for $(U, uU') \in \mathcal{D}(G)$ it holds

$$\left(U, uU'\right) \equiv_p \left(O^p(U), u_{p',c}O^p(U)'\right). \tag{9}$$

Let K be a p-perfect subgroup of G and $k \in K$ with $(H, hH') \equiv_p (K, kK')$. We will show $[H, hH']_G = [K, kK']_G$. Since $O^p(K) = K$, the group K/K' is a p'-group. Thus $k_p \in K'$, and it follows that $kK' = k_{p'}K'$. Therefore we can assume $k = k_{p'}$. With the same argumentation we assume $h = h_{p'}$. Let $(\tilde{H}, h\tilde{H}')$ and $(\tilde{K}, k\tilde{K}')$ be p-regularizations of (H, hH') and (K, kK'). Then

$$\left(\tilde{H}, h\tilde{H}'\right) \equiv_p \left(H, hH'\right) \equiv_p \left(K, kK'\right) \equiv_p \left(\tilde{K}, k\tilde{K}'\right).$$

By Lemma 6.1(iii) $(\tilde{H}, h\tilde{H}')$ and $(\tilde{K}, k\tilde{K}')$ are conjugate in G. Thus

$$H = O^p(\tilde{H}) =_G O^p(\tilde{K}) = K.$$

In the following we assume $H = K$. We will show that hH' and kH' are conjugate in $N_G(H)$. Let V/H be a Sylow p-subgroup of $N_G(H, hH')/H$ and set $\bar{V} := V/H'$. It holds $\bar{H} = C_{\bar{H}}(\bar{V}) \oplus [\bar{H}, \bar{V}]$ by Lemma 6.2. Obviously it holds $[\bar{H}, \bar{V}] \subseteq \bar{V}' \cap \bar{H}$. Conversely we assume $x \in \bar{V}' \cap \bar{H}$. Then $x = cd$ with $c \in C_{\bar{H}}(\bar{V})$ and $d \in [\bar{H}, \bar{V}]$. We get

$$c = xd^{-1} \in C_{\bar{H}}(\bar{V}) \cap \bar{V}' = Z(\bar{V}) \cap \bar{H} \cap \bar{V}' = 1$$

by Lemma 6.3. Thus $x \in [\bar{H}, \bar{V}]$ and

$$\bar{V}' \cap \bar{H} = [\bar{H}, \bar{V}]. \tag{10}$$

The group H is normal in $N_G(V)$ since $H = O^p(V)$ is characteristic in V. Since H' is characteristic in H we get $H' \trianglelefteq N_G(V)$. It holds $C_{N_G(V)/H'}(\bar{V}) \trianglelefteq N_G(V)/H'$, and since $\bar{H} \trianglelefteq N_G(V)/H'$ we get

$$C_{\bar{H}}(\bar{V}) = C_{N_G(V)/H'}(\bar{V}) \cap \bar{H} \trianglelefteq N_G(V)/H'. \tag{11}$$

We now show that (V, hV') is a p-regularization of (H, hH'). Let $t \in N_G(V, hV') \leqslant N_G(H)$. It is $hH' \in C_{\bar{H}}(\bar{V})$ since $V \leqslant N_G(H, hH')$. Moreover it is $tht^{-1}H' \in C_{\bar{H}}(\bar{V})$ since $C_{\bar{H}}(\bar{V}) \trianglelefteq N_G(V)/H'$ (by Eq. (11)).

Thus $h^{-1}tht^{-1}H' \in C_{\bar{H}}(\bar{V})$. It holds $h^{-1}tht^{-1} \in V'$, therefore we get $h^{-1}tht^{-1}H' \in V'/H' = \bar{V}'$. By Eq. (10) we obtain

$$C_{\bar{H}}(\bar{V}) \cap \bar{V}' = C_{\bar{H}}(\bar{V}) \cap \bar{V}' \cap \bar{H} = C_{\bar{H}}(\bar{V}) \cap [\bar{H}, \bar{V}] = 1.$$

It follows that $tht^{-1}H' = hH'$. Thus $t \in N_G(H, hH')$ and we get $N_G(V, hV') \leqslant N_G(H, hH')$. Then

$$\bigl(N_G\bigl(V, hV'\bigr) : V\bigr) = \bigl(N_G\bigl(V, hV'\bigr)/H : V/H\bigr) \not\equiv 0 \pmod{p}.$$

Therefore (V, hV') is a p-regularization of (H, hH'). We can now assume

$$\bigl(\tilde{H}, h\tilde{H}'\bigr) = \bigl(V, hV'\bigr).$$

In particular $hH' \in C_{\bar{H}}(\tilde{H}/H')$, and with the same argumentation we get $kH' \in C_{\bar{H}}(\tilde{K}/H')$. Since

$$\bigl(V, hV'\bigr) = \bigl(\tilde{H}, h\tilde{H}'\bigr) \equiv_p \bigl(\tilde{K}, k\tilde{K}'\bigr)$$

we obtain by Proposition 6.1(iii) the existence of $g \in G$ with ${}^{g}(\tilde{K}, k\tilde{K}') = (V, hV')$. Since $O^p(\tilde{K}) = H = O^p(V)$ it holds $g \in N_G(H)$. Thus $gkg^{-1}H' \in C_{\bar{H}}({}^{g}(\tilde{K}/H')) = C_{\bar{H}}(\bar{V})$. Since $hH' \in C_{\bar{H}}(\bar{V})$ it follows that $h^{-1}gkg^{-1}H' \in C_{\bar{H}}(\bar{V})$. Since $h^{-1}gkg^{-1} \in V'$ we get $h^{-1}gkg^{-1}H' \in \bar{V}'$ and therefore

$$h^{-1}gkg^{-1}H' \in C_{\bar{H}}(\bar{V}) \cap \bar{V}' = Z(\bar{V}) \cap \bar{H} \cap \bar{V}' = 1$$

by Lemma 6.3. Thus $hH' = gkg^{-1}H'$ with $g \in N_G(H)$ and therefore every $\mathfrak{p}$-equivalence class is represented by exactly one orbit $[H, hH']_G$ with a p-perfect subgroup H.

Let H be any p-perfect subgroup of G, $h \in H$ and let X be the equivalence class represented by $[H, hH']_G$. We set

$$T := \bigl\{\bigl[U, uU'\bigr]_G \in \mathcal{D}(G)/G\colon \bigl(U, uU'\bigr) \in X\bigr\}$$

and

$$Y := \bigl\{\bigl[U, uU'\bigr]_G \in \mathcal{D}(G)/G\colon U \in S^p\bigl(H, hH'\bigr),\ u_{p',c}H' = hH'\bigr\}.$$

Let $[U, uU']_G \in T$ with $O^p(U) = H$. We get $[H, u_{p',c}H']_G = [H, hH']_G$ by the above argumentations. Thus there exists $g \in N_G(H)$ with $g^{-1}hgH' = u_{p',c}H'$. Since $U \leqslant N_G(H, u_{p',c}H')$ it follows that ${}^{g}U \leqslant N_G(H, hH')$. Moreover $u_{p'}H' = u_{p',c}vH'$ with $vH' \in [\bar{H}, U/H']$. Thus

$$\bigl({}^{g}u\bigr)_{p'}H' = {}^{g}(u_{p'})H' = {}^{g}(u_{p',c}){}^{g}vH'$$

with ${}^{g}(u_{p',c})H' \in C_{\bar{H}}({}^{g}U/H')$ and ${}^{g}vH' \in [\bar{H}, {}^{g}U/H']$. It holds

$$\bigl({}^{g}u\bigr)_{p'}H' = \bigl({}^{g}u\bigr)_{p',c}wH'$$

with $({}^{g}u)_{p',c} \in C_{\bar{H}}({}^{g}U/H')$ and $wH' \in [\bar{H}, {}^{g}U/H']$. Since $\bar{H} = C_{\bar{H}}({}^{g}U/H') \oplus [\bar{H}, {}^{g}U/H']$ we get

$${}^{g}(u_{p',c})H' = \bigl({}^{g}u\bigr)_{p',c}H'.$$

Thus $({}^{g}u)_{p',c}H' = hH'$ and we get $[U, uU']_G = [{}^{g}U, {}^{g}u{}^{g}U']_G \in Y$.

Let conversely be $[U, uU']_G \in Y$. We can assume $O^p(U) = H$ and $u_{p',c}H' = hH'$. We get

$$\big(U, uU'\big) \equiv_p \big(H, u_{p',c}H'\big) = \big(H, hH'\big)$$

by Eq. (9). Thus we get $[U, uU']_G \in T$ and so $Y = T$.

Every primitive idempotent of $D_R(G)$ corresponding to X is of the form

$$\sum_{[U,uU']_G \in T} e^{D(G)}_{(U,uU')}.$$

Since $Y = T$ we obtain the idempotent formula stated in the theorem. □

7. Sylow subgroups

In this section we present some results about Sylow subgroups of two finite groups G and $\tilde{G}$ with $D(G) \cong D(\tilde{G})$.

Proposition 7.1. *Let G and $\tilde{G}$ be finite groups, $\alpha : D(\tilde{G}) \to D(G)$ an isomorphism, p a prime divisor of $|G|$ and P a Sylow p-subgroup of G. Let $\alpha(e^{D(\tilde{G})}_{(1,1)}) = e^{D(G)}_{(U,uU')}$. Then the group $H := O^p(U)$ is a normal abelian p'-subgroup of G and $h := u_{p'} \in Z(G)$. We set*

$$I := \big\{\big[K, kK'\big]_G \in \mathcal{D}(G)/G\colon\ K = HV,\ V \leqslant P,\ k = hv,\ v \in V\big\}.$$

Then

$$\alpha\big(e^{D(\tilde{G}),p}_{(1,1)}\big) = \sum_{[K,kK']_G \in I} e^{D(G)}_{(K,kK')}.$$

Proof. We get $|G| = |\tilde{G}|$ by Theorem 3.6. Moreover by Proposition 3.8 U is a normal abelian subgroup of G and $u \in Z(G)$ with $|\langle u\rangle| \in \{1, 2\}$. Thus H is a normal abelian p'-subgroup of G and $h \in \{1, u\} \subseteq Z(G)$. It holds

$$U \in S^p(H, h) := \big\{K \leqslant G\colon\ O^p(K) = H,\ K \leqslant N_G(H, h)\big\} = \big\{K \leqslant G\colon\ O^p(K) = H\big\},$$

and since $u_{p'} \in Z(G)$ we get $u_{p',c} = u_{p'}$. Thus the idempotent $e^{D(G)}_{(U,u)}$ is included in the sum

$$e^{D(G),p}_{(H,h)} = \sum_{\substack{[K,kK']_G \in \mathcal{D}(G)/G \\ K \in S^p(H,h) \\ k_{p',c} = h}} e^{D(G)}_{(K,kK')}.$$

Therefore $\alpha(e^{D(\tilde{G}),p}_{(1,1)}) = e^{D(G),p}_{(H,h)}$. Let

$$J := \big\{\big[K, kK'\big]_G \in \mathcal{D}(G)/G\colon\ O^p(K) = H,\ k_{p',c} = h\big\}.$$

We show $I = J$. Let $[K, kK']_G \in I$. Then $O^p(K) = H$. Moreover we can assume $k = hv$ with $v \in V$ for some subgroup $V \leqslant P$. Since $h \in Z(G)$ it holds $h = k_{p'} = k_{p',c}$. Thus $[K, kK']_G \in J$.

Let conversely be $[K, kK']_G \in J$. We can assume $k_{p',c} = h$. It holds $H = O^p(K)$ and by Lemma 6.2 we get $H = C_H(K) \oplus [H, K]$. Since $k_{p'} \in H$ it holds $k_{p'} = k_{p',c}y = hy$ with some $y \in [H, K] \leqslant K'$. Thus

$$\big[K, kK'\big]_G = \big[K, k_p k_{p'} K'\big]_G = \big[K, k_p h y K'\big]_G = \big[K, h k_p K'\big]_G.$$

By the Schur–Zassenhaus theorem there exists a p-subgroup $V \leqslant G$ with $K = HV$. Moreover there exists $g \in G$ with ${}^gV \leqslant P$. Then ${}^gK = H({}^gV)$. Since ${}^gV$ is a Sylow p-subgroup of ${}^gK$ there exists $w \in {}^gK$ with ${}^{wg}k_p \in {}^gV$. Thus

$$\left[K, kK'\right]_G = \left[{}^gK, h\left({}^gk_p\right){}^gK'\right]_G = \left[{}^gK, h\left({}^{wg}k_p\right){}^gK'\right]_G \in I$$

and the proposition is proved. □

We can now state the first result.

Theorem 7.2. *Let G and $\tilde{G}$ be finite groups with $D(G) \cong D(\tilde{G})$ and let p be a prime divisor of $|G|$. If $\tilde{G}$ has a non-trivial normal p-subgroup then G has a non-trivial normal p-subgroup.*

Proof. Let $\tilde{P}$ be a Sylow p-subgroup of $\tilde{G}$. By Theorem 6.4 we get

$$e_{(1,1)}^{D(\tilde{G}),p} = \sum_{\substack{[\tilde{K},\tilde{k}\tilde{K}']_G \in \mathcal{D}(\tilde{G})/\tilde{G} \\ \tilde{K} \leqslant \tilde{P}}} e_{(\tilde{K},\tilde{k}\tilde{K}')}^{D(\tilde{G})}. \tag{12}$$

By the assumption there exists a normal p-subgroup $1 \neq \tilde{U}$ of $\tilde{G}$ with $\tilde{U} \leqslant \tilde{P}$. Then $\tilde{K} := Z(\tilde{U}) \neq 1$ is an abelian p-subgroup of $\tilde{G}$ which is characteristic in $\tilde{U}$. Thus $\tilde{K}$ is normal in $\tilde{G}$ and therefore $e_{(\tilde{K},1)}^{D(\tilde{G})}$ has conductor $|\tilde{G}|$. Thus the sum in Eq. (12) includes at least two primitive idempotents with conductor $|\tilde{G}|$ (consider $e_{(1,1)}^{D(\tilde{G})}$ and $e_{(\tilde{K},1)}^{D(\tilde{G})}$). Let $\alpha : D(\tilde{G}) \to D(G)$ be an isomorphism, P a Sylow p-subgroup of G and let $\alpha(e_{(1,1)}^{D(\tilde{G})}) = e_{(U,uU')}^{D(G)}$ with a normal abelian subgroup $U \leqslant G$ and $u \in Z(G)$. By Proposition 7.1,

$$\alpha\left(e_{(1,1)}^{D(\tilde{G}),p}\right) = \sum_{[K,kK']_G \in I} e_{(K,kK')}^{D(G)}$$

holds with

$$I = \left\{\left[K, kK'\right]_G \in \mathcal{D}(G)/G\colon\ K = O^p(U)V,\ k = u_{p'}v,\ v \in V,\ V \leqslant P\right\}.$$

There exists at least one element $[K, kK']_G \in I$ with $[K, kK']_G \neq [O^p(U), u_{p'}]_G$ such that $e_{(K,kK')}^{D(G)}$ has conductor $|G| = |\tilde{G}|$. Thus K is an abelian normal subgroup of G. Since $K/O^p(U)$ is a non-trivial p-group, the Sylow p-subgroup of K is non-trivial and normal in G. □

Theorem 7.3. *Let G and $\tilde{G}$ be finite groups with $D(G) \cong D(\tilde{G})$. Let p be a prime divisor of $|G|$ and let P and $\tilde{P}$ be Sylow p-subgroups of G and $\tilde{G}$. If $\tilde{P}$ is abelian then P is abelian.*

Proof. Let $\alpha : D(\tilde{G}) \to D(G)$ be an isomorphism, $\alpha(e_{(1,1)}^{D(\tilde{G})}) = e_{(U,uU')}^{D(G)}$ with a normal abelian subgroup $U \leqslant G$ and $u \in Z(G)$. Let $H := O^p(U)$ and $h := u_{p'}$. By Proposition 7.1 we obtain

$$\alpha\left(e_{(1,1)}^{D(\tilde{G}),p}\right) = \sum_{[K,kK']_G \in I} e_{(K,kK')}^{D(G)}$$

with

$$I := \left\{\left[K, kK'\right]_G \in \mathcal{D}(G)/G\colon\ K = HV,\ V \leqslant P,\ k = hv,\ v \in V\right\}.$$

Let $\tilde{P}$ be abelian. Then the conductors of all primitive idempotents $e^{D(\tilde{G})}_{(\tilde{K},\tilde{k}\tilde{K}')}$, $\tilde{K} \leqslant \tilde{P}$, $\tilde{k} \in \tilde{K}$, are divisible by $|\tilde{P}|$. Since

$$e^{D(\tilde{G}),p}_{(1,1)} = \sum_{\substack{[\tilde{K},\tilde{k}\tilde{K}']_{\tilde{G}} \in \mathcal{D}(\tilde{G})/\tilde{G} \\ \tilde{K} \leqslant \tilde{P}}} e^{D(\tilde{G})}_{(\tilde{K},\tilde{k}\tilde{K}')}$$

$|P| = |\tilde{P}|$ divides the conductor of $e^{D(G)}_{(K,kK')}$ for all $[K,kK']_G \in I$. We set $K := HP$. Then $[K,hK']_G \in I$ and p does not divide $(N_G(K,hK') : K)$. Thus $|P|$ divides $(K : K')$ and therefore $P \cap K' = 1$. It follows that $P' \leqslant K' \cap P = 1$ and therefore P is abelian. □

The next theorem is concerned with Sylow 2-subgroups of groups G and $\tilde{G}$ with $D(G) \cong D(\tilde{G})$. We first need the following lemma.

Lemma 7.4. *Let G be a finite group and $(H,hH') \in \mathcal{D}(G)$. We assume the existence of $x \in D_{\mathbb{Q}}(G)$ and $n \in \mathbb{N}$ such that $s^{D(G)}_{(H,hH')}(x)$ is a primitive n-th root of unity.*

(i) *If $2 \nmid n$ or $4 \mid n$ then n divides $|\langle h\rangle|$.*
(ii) *If $n = 2m$ with $m \in \mathbb{N}$ and $2 \nmid m$ then m divides $|\langle h\rangle|$.*

Proof. Let $\omega \in \mathbb{C}$ be a primitive $|\langle h\rangle|$-th root of unity. For every subgroup $U \leqslant G$ with $H \leqslant U$ and every linear character $\psi \in \hat{U}$ it holds $\psi(h) = \omega^i$ for some $i \in \mathbb{N}$. For $[U,\psi]_G \in \mathcal{M}(G)/G$ we get

$$s^{D(G)}_{(H,hH')}\big([U,\psi]_G\big) = \sum_{\substack{gU \in G/U \\ H \leqslant {}^gU}} {}^g\psi(h) \in \mathbb{Q}(\omega).$$

Therefore $s^{D(G)}_{(H,hH')}(x) \in \mathbb{Q}(\omega)$. Since $\pm\omega^i$ $(i \in \mathbb{N})$ are the only roots of unity in $\mathbb{Q}(\omega)$ we get $s^{D(G)}_{(H,hH')}(x) \in \{\pm\omega^i\colon\ i \in \mathbb{N}\}$. Therefore

$$n \mid \max\big\{\mathrm{ord}\big(\pm\omega^i\big)\colon\ i \in \mathbb{N}\big\} \in \big\{\mathrm{ord}(\omega), \mathrm{ord}(-\omega)\big\}.$$

In case $\mathrm{ord}(\omega) \geqslant \mathrm{ord}(-\omega)$ we obtain that n divides $|\langle h\rangle|$ and (i) and (ii) is proved. Let $2 \cdot \mathrm{ord}(\omega) = \mathrm{ord}(-\omega)$. Then $2 \nmid \mathrm{ord}(\omega)$ and since $n \mid \mathrm{ord}(-\omega)$ we get $4 \nmid n$. If $2 \nmid n$ we get $n \mid \mathrm{ord}(\omega)$ and therefore (i). Let $2 \mid n$. Since $n \mid \mathrm{ord}(-\omega) = 2 \cdot \mathrm{ord}(\omega)$ we obtain that $\frac{n}{2}$ divides $\mathrm{ord}(\omega)$ and we proved (ii). □

Theorem 7.5. *Let G and $\tilde{G}$ be finite groups with $D(G) \cong D(\tilde{G})$ and let P and $\tilde{P}$ be Sylow 2-subgroups of G and $\tilde{G}$. If P is cyclic then $\tilde{P}$ is cyclic.*

Proof. Let $P = \langle h\rangle$ and $|P| = 2^n$ with $n \in \mathbb{N}$. We assume $n \geqslant 2$. Note that $(N_G(P) : C_G(P))$ divides $|\mathrm{Aut}(P)| = 2^{n-1}$. Since $2 \nmid (N_G(P) : C_G(P))$ we get $N_G(P) = C_G(P)$. Let $\lambda \in \hat{P}$ such that $\lambda(h)$ is a primitive 2^n-th root of unity. Then

$$s^{D(G)}_{(P,h)}\left(\frac{1}{(N_G(P):P)}[P,\lambda]_G\right) = \frac{1}{(N_G(P):P)} \sum_{gP \in N_G(P)/P} {}^g\lambda(h) = \lambda(h).$$

Let $\alpha : D(G) \to D(\tilde{G})$ be an isomorphism. Then $s^{D(G)}_{(P,h)} = s^{D(\tilde{G})}_{(\tilde{H},\tilde{h}\tilde{H}')} \circ \alpha$ with $(\tilde{H},\tilde{h}\tilde{H}') \in \mathcal{D}(\tilde{G})$. We set

$$\tilde{x} := \alpha\left(\frac{1}{(N_G(P):P)}[P,\lambda]_G\right) \in D_{\mathbb{Q}}(\tilde{G}).$$

Then $s^{D(\tilde{G})}_{(\tilde{H},\tilde{h}\tilde{H}')}(\tilde{x}) = \lambda(h)$ is a primitive 2^n-th root of unity. Moreover 2^n divides $|\langle\tilde{h}\rangle|$ by Lemma 7.4. Thus $\tilde{G}$ contains an element of order 2^n. Therefore $\tilde{P}$ is cyclic. □

8. Nilpotent and p-nilpotent groups

In the first theorem of this section we prove that the ring of monomial representations of a finite group detects nilpotency.

Theorem 8.1. *Let G be a finite nilpotent group and $\tilde{G}$ a finite group with $D(G) \cong D(\tilde{G})$. Then $\tilde{G}$ is nilpotent.*

Proof. Let $\alpha : D(\tilde{G}) \to D(G)$ be an isomorphism and let

$$\alpha\big(e^{D(\tilde{G})}_{(1,1)}\big) = e^{D(G)}_{(U,uU')}.$$

By Proposition 3.7 U is a normal abelian subgroup of G and $u \in Z(G)$. Let p be a prime divisor of G, P the Sylow p-subgroup of G and $H := O^p(U)$. Then H is a normal abelian subgroup of G with $p \nmid |H|$. Since $u \in Z(G)$ we get $h := u_{p'} \in Z(G) \cap H$. Since G is nilpotent we obtain

$$\alpha\big(e^{D(\tilde{G}),p}_{(1,1)}\big) = \sum_{[K,kK']_G \in I} e^{D(G)}_{(K,kK')} \tag{13}$$

with

$$I = \big\{\big[K,kK'\big]_G \in \mathcal{D}(G)/G\colon\ K = H \times V,\ V \leqslant P,\ k = hv,\ v \in V\big\}$$

by Proposition 7.1. Let $K := H \times V$ with $V \leqslant P$ and $k := hv$ with $v \in V$. Since G is nilpotent it holds $G_{p'} \leqslant C_G(V)$, and since H is normal in G we get $G_{p'} \leqslant N_G(K)$. Since $h \in Z(G)$ we get

$$gkK'g^{-1} = ghvg^{-1}K' = hvK' = kK'$$

for all $g \in G_{p'}$. Thus $G_{p'} \leqslant N_G(K,kK')$. Moreover $K' = (H \times V)' = V'$ is a p-subgroup of G. Thus $|G_{p'}|$ divides $(N_G(K,kK'):K')$. Therefore $|G_{p'}|$ divides the conductor of the primitive idempotents $e^{D(G)}_{(K,kK')}$ with $[K,kK']_G \in I$. Let $\tilde{P}$ be a Sylow p-subgroup of $\tilde{G}$. By

$$e^{D(\tilde{G}),p}_{(1,1)} = \sum_{\substack{[\tilde{U},\tilde{u}\tilde{U}'] \in \mathcal{D}(\tilde{G})/\tilde{G} \\ \tilde{U} \leqslant \tilde{P}}} e^{D(\tilde{G})}_{(\tilde{U},\tilde{u}\tilde{U}')}$$

and Eq. (13) we obtain that $|G_{p'}| = |\tilde{G}|_{p'}$ divides the conductor of the primitive idempotents $e^{D(\tilde{G})}_{(\tilde{U},\tilde{u}\tilde{U}')}$ with $\tilde{U} \leqslant \tilde{P}$. In particular $|\tilde{G}|_{p'}$ divides the conductor of $e^{D(\tilde{G})}_{(\tilde{P},1\tilde{P}')}$. Therefore $|\tilde{G}|_{p'}$ divides $(N_{\tilde{G}}(\tilde{P}) : \tilde{P}')_{p'} = (N_{\tilde{G}}(\tilde{P}) : \tilde{P})$ and therefore $\tilde{P}$ is normal in $\tilde{G}$ and the theorem is proved. □

Next we will show that the isomorphy $D(G) \cong D(\tilde{G})$ with nilpotent groups G and $\tilde{G}$ implies the isomorphy $D(P) \cong D(\tilde{P})$ where P and $\tilde{P}$ are Sylow p-subgroups of G and $\tilde{G}$. We need the following two propositions.

Proposition 8.2. *Let G and H be finite groups with* $\gcd(|G|, |H|) = 1$. *Then*

$$D(G \times H) \cong D(G) \otimes_{\mathbb{Z}} D(H).$$

Proof. Since $\gcd(|G|, |H|) = 1$, every subgroup of $G \times H$ is of the form $U \times V$ with subgroups $U \leqslant G$ and $V \leqslant H$. Moreover every linear character of a subgroup $U \times V \leqslant G \times H$ is of the form $\varphi \times \psi$ with $(\varphi, \psi) \in \hat{U} \times \hat{V}$. Therefore the map

$$D(G \times H) \to D(G) \otimes_{\mathbb{Z}} D(H),$$
$$[U \times V, \varphi \times \psi]_{G \times H} \mapsto [U, \varphi]_G \otimes [V, \psi]_H$$

is well defined and an isomorphism. □

Proposition 8.3. *Let A_1, A_2, B_1, B_2 be commutative rings with unit element which are finitely generated and free as a $\mathbb{Z}$-module. Moreover, assume that the rings A_1 and A_2 have $\mathbb{Z}$-bases which contain the respective unit element. Further, assume that there exists a unitary subring $R \subseteq \mathbb{C}$ such that the only idempotents in $R \otimes_{\mathbb{Z}} A_i$ $(i = 1, 2)$ are 0 and 1 and such that the R-algebra $R \otimes_{\mathbb{Z}} B_i$ $(i = 1, 2)$ is isomorphic to a direct product of copies of R. If $A_1 \otimes_{\mathbb{Z}} B_1 \cong A_2 \otimes_{\mathbb{Z}} B_2$ then $B_1 \cong B_2$.*

Proof. Let $\{a_1, \dots, a_s\} \subseteq A_1$, $\{\tilde{a}_1, \dots, \tilde{a}_t\} \subseteq A_2$, $\{b_1, \dots, b_n\} \subseteq B_1$ and $\{\tilde{b}_1, \dots, \tilde{b}_m\} \subseteq B_2$ the respective $\mathbb{Z}$-bases with the unit elements $a_1 = 1_{A_1}$ and $\tilde{a}_1 = 1_{A_2}$. Then $\{a_i \otimes b_j\colon\ i = 1, \dots, s,\ \ j = 1, \dots, n\}$ is a $\mathbb{Z}$-basis of $A_1 \otimes_{\mathbb{Z}} B_1$ and $\{\tilde{a}_i \otimes \tilde{b}_j\colon\ i = 1, \dots, t,\ \ j = 1, \dots, m\}$ is a $\mathbb{Z}$-basis of $A_2 \otimes_{\mathbb{Z}} B_2$. Consider the canonical embeddings

$$\varphi : B_1 \to R \otimes_{\mathbb{Z}} B_1,$$
$$b_i \mapsto 1_R \otimes b_i,$$

$$\delta : B_1 \to A_1 \otimes_{\mathbb{Z}} B_1,$$
$$b_i \mapsto 1_{A_1} \otimes b_i$$

$$\psi := 1 \otimes \delta : R \otimes_{\mathbb{Z}} B_1 \to R \otimes_{\mathbb{Z}} A_1 \otimes_{\mathbb{Z}} B_1,$$
$$1_R \otimes b_i \mapsto 1_R \otimes 1_{A_1} \otimes b_i,$$

$$\mu : A_1 \otimes_{\mathbb{Z}} B_1 \to R \otimes_{\mathbb{Z}} A_1 \otimes_{\mathbb{Z}} B_1,$$
$$a_j \otimes b_i \mapsto 1_R \otimes a_j \otimes b_i$$

$(i = 1, \dots, n,\ j = 1, \dots, s)$. Then $\psi \circ \varphi = \mu \circ \delta$. We define the canonical embeddings $\tilde{\varphi} : B_2 \to R \otimes_{\mathbb{Z}} B_2$, $\tilde{\delta} : B_2 \to A_2 \otimes_{\mathbb{Z}} B_2$, $\tilde{\psi} : R \otimes_{\mathbb{Z}} B_2 \to R \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_2$ and $\tilde{\mu} : A_2 \otimes_{\mathbb{Z}} B_2 \to R \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_2$ in an analogous way. Let

$$\alpha : A_1 \otimes_{\mathbb{Z}} B_1 \to A_2 \otimes_{\mathbb{Z}} B_2$$

be an isomorphism. We extend α linearly to the isomorphism

$$\hat{\alpha} : R \otimes_{\mathbb{Z}} A_1 \otimes_{\mathbb{Z}} B_1 \to R \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_2.$$

Then $\hat{\alpha} \circ \mu = \tilde{\mu} \circ \alpha$. Let $e_1, \dots, e_n$ be the primitive idempotents of $R \otimes_{\mathbb{Z}} B_1$ and $\tilde{e}_1, \dots, \tilde{e}_m$ be the primitive idempotents of $R \otimes_{\mathbb{Z}} B_2$. Then

$$R \otimes_{\mathbb{Z}} B_1 = \bigoplus_{i=1}^{n} Re_i \quad \text{and} \quad R \otimes_{\mathbb{Z}} B_2 = \bigoplus_{i=1}^{m} R\tilde{e}_i.$$

Moreover 0 and 1 are the only idempotents in $R \otimes_{\mathbb{Z}} A_1$ and $R \otimes_{\mathbb{Z}} A_2$. Then $1_{R\otimes_{\mathbb{Z}} A_1} \otimes e_i$, $i = 1, \dots, n$, are the primitive idempotents of $(R \otimes_{\mathbb{Z}} A_1) \otimes_R (R \otimes_{\mathbb{Z}} B_1)$. Since

$$R \otimes_{\mathbb{Z}} A_1 \otimes_{\mathbb{Z}} B_1 \cong (R \otimes_{\mathbb{Z}} A_1) \otimes_R (R \otimes_{\mathbb{Z}} B_1)$$

the elements $\psi(e_i)$, $i = 1, \dots, n$, are the primitive idempotents of $R \otimes_{\mathbb{Z}} A_1 \otimes_{\mathbb{Z}} B_1$. Similarly $\tilde{\psi}(\tilde{e}_i)$, $i = 1, \dots, m$, are the primitive idempotents of $R \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_2$. Thus

$$\hat{\alpha}\big(\big\{\psi(e_1), \dots, \psi(e_n)\big\}\big) = \big\{\tilde{\psi}(\tilde{e}_1), \dots, \tilde{\psi}(\tilde{e}_m)\big\}.$$

In particular $n = m$. We assume $\hat{\alpha}(\psi(e_i)) = \tilde{\psi}(\tilde{e}_i)$ for $i = 1, \dots, n$. Let $c \in B_1$. Then there exist $r_1, \dots, r_n \in R$ with $\varphi(c) = \sum_{i=1}^{n} r_i e_i$ and we get

$$(\hat{\alpha} \circ \psi \circ \varphi)(c) = (\hat{\alpha} \circ \psi)\left(\sum_{i=1}^{n} r_i e_i\right) = \sum_{i=1}^{n} r_i \tilde{\psi}(\tilde{e}_i).$$

Thus there exist $t_1, \dots, t_n \in R$ with $(\hat{\alpha} \circ \psi \circ \varphi)(c) = \sum_{i=1}^{n} t_i(1_R \otimes 1_{A_2} \otimes \tilde{b}_i)$. It holds

$$(\hat{\alpha} \circ \psi \circ \varphi)(c) = (\hat{\alpha} \circ \mu \circ \delta)(c) = (\tilde{\mu} \circ \alpha \circ \delta)(c),$$

and there exist $z_{i,j} \in \mathbb{Z}$ $(i = 1, \dots, t,\ j = 1, \dots, n)$ with

$$(\alpha \circ \delta)(c) = \sum_{i=1}^{t} \sum_{j=1}^{n} z_{i,j}(\tilde{a}_i \otimes \tilde{b}_j).$$

Therefore

$$\sum_{i=1}^{n} t_i(1_R \otimes 1_{A_2} \otimes \tilde{b}_i) = (\tilde{\mu} \circ \alpha \circ \delta)(c) = \sum_{i=1}^{t} \sum_{j=1}^{n} z_{i,j}(1_R \otimes \tilde{a}_i \otimes \tilde{b}_j).$$

Since $\tilde{a}_1 = 1_{A_2}$ the set $\{1_R \otimes 1_{A_2} \otimes \tilde{b}_j\colon\ j = 1, \dots, n\}$ is a subset of the canonical basis $\{1_R \otimes \tilde{a}_i \otimes \tilde{b}_j\colon\ i = 1, \dots, t,\ j = 1, \dots, n\}$ of $R \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_2$. Thus $t_j = z_{1,j} \in \mathbb{Z}$ for all $j = 1, \dots, n$ and $z_{i,j} = 0$ for $i \neq 1$, $j = 1, \dots, n$. It follows that $(\hat{\alpha} \circ \psi \circ \varphi)(c) \in (\tilde{\psi} \circ \tilde{\varphi})(B_2)$ and therefore

$$\beta := \tilde{\varphi}^{-1} \circ \tilde{\psi}^{-1} \circ \hat{\alpha} \circ \psi \circ \varphi : B_1 \to B_2$$

is a ring monomorphism. Considering $\psi \circ \varphi = \mu \circ \delta$ and $\tilde{\delta}^{-1} \circ \tilde{\mu}^{-1} = \tilde{\varphi}^{-1} \circ \tilde{\psi}^{-1}$ we get

$$\beta = \tilde{\delta}^{-1} \circ \alpha \circ \delta. \tag{14}$$

With the same argumentation we get a ring monomorphism

$$\tilde{\beta} = \delta^{-1} \circ \alpha^{-1} \circ \tilde{\delta} : B_2 \to B_1.$$

Moreover $\beta \circ \tilde{\beta} = \mathrm{id}_{B_2}$ and $\tilde{\beta} \circ \beta = \mathrm{id}_{B_1}$. Therefore β is an isomorphism. □

Theorem 8.4. *Let p be a prime number. Let $G = P \times H$ and $\tilde{G} = \tilde{P} \times \tilde{H}$ be finite groups with p-groups P, $\tilde{P}$ and p'-groups H, $\tilde{H}$. If $D(G) \cong D(\tilde{G})$ then $D(H) \cong D(\tilde{H})$.*

Proof. Let $\xi \in \mathbb{C}$ be a primitive $|H|$-th root of unity, $\mathfrak{p}$ be a prime ideal in $\mathbb{Z}[\xi]$ with $\mathrm{char}(Z[\xi]/\mathfrak{p}) = p$ and $R := \mathbb{Z}[\xi]_{\mathfrak{p}}$ be the localization at $\mathfrak{p}$. By Theorem 6.4 the rings $D_R(H)$ and $D_{\mathbb{Q}(\xi)}(H)$ have the same primitive idempotents. Similarly the primitive idempotents of $D_R(\tilde{H})$ and $D_{\mathbb{Q}(\xi)}(\tilde{H})$ are corresponding. Therefore $D_R(H)$ and $D_R(\tilde{H})$ are completely reducible. Moreover by Theorem 6.4 we obtain that 0 and 1 are the only idempotents in $D_R(P)$ and $D_R(\tilde{P})$. By Proposition 8.2 we get the isomorphy

$$D(P) \otimes_{\mathbb{Z}} D(H) \cong D(G) \cong D(\tilde{G}) \cong D(\tilde{P}) \otimes_{\mathbb{Z}} D(\tilde{H}).$$

We set $A_1 := D(P)$, $A_2 := D(\tilde{P})$, $B_1 := D(H)$ and $B_2 := D(\tilde{H})$. Then all conditions in Theorem 8.3 are valid and we get the isomorphy $D(H) \cong D(\tilde{H})$. □

Corollary 8.5. *Let G and $\tilde{G}$ be finite nilpotent groups with $D(G) \cong D(\tilde{G})$. Let $p_1, \ldots, p_n$ be the different prime divisors of $|G|$, and for $i = 1, \ldots, n$ let G_i and $\tilde{G}_i$ be the Sylow p_i-subgroups of G and $\tilde{G}$. Let*

$$\alpha : D(G_1) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} D(G_n) \to D(\tilde{G}_1) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} D(\tilde{G}_n)$$

be an isomorphism. Then there exist isomorphisms $\alpha_i : D(G_i) \to D(\tilde{G}_i)$ for $i = 1, \ldots, n$ with $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$.

Proof. For $i = 2, \ldots, n$ let

$$H_i := G_i \times \cdots \times G_n, \qquad \tilde{H}_i := \tilde{G}_i \times \cdots \times \tilde{G}_n$$

and

$$\delta_i : D(H_i) \to D(H_{i-1}), \qquad \tilde{\delta}_i : D(\tilde{H}_i) \to D(\tilde{H}_{i-1})$$

be the canonical embeddings. Applying Theorem 8.4 under consideration of Eq. (14) we get the isomorphism $\beta_2 := \tilde{\delta}_2^{-1} \circ \alpha \circ \delta_2 : D(H_2) \to D(\tilde{H}_2)$. Applying Theorem 8.4 again we get the isomorphism $\beta_3 := \tilde{\delta}_3^{-1} \circ \beta_2 \circ \delta_3 : D(H_3) \to D(\tilde{H}_3)$. If we go on like this we obtain the isomorphism

$$\beta_n := \tilde{\delta}_n^{-1} \circ \cdots \circ \tilde{\delta}_2^{-1} \circ \alpha \circ \delta_2 \circ \cdots \circ \delta_n : D(G_n) \to D(\tilde{G}_n)$$

where $\delta_2 \circ \cdots \circ \delta_n : D(G_n) \to D(G)$ and $\tilde{\delta}_2 \circ \cdots \circ \tilde{\delta}_n : D(\tilde{G}_n) \to D(\tilde{G})$ are the canonical embeddings. In this way we get isomorphisms $D(G_i) \to D(\tilde{G}_i)$ for all $i = 1, \ldots, n$. If we let $\tau_i : D(G_i) \to D(G)$ and $\tilde{\tau}_i : D(\tilde{G}_i) \to D(\tilde{G})$ be the canonical embeddings, the maps

$$\alpha_i := \tilde{\tau}_i^{-1} \circ \alpha \circ \tau_i : D(G_i) \to D(\tilde{G}_i), \quad i = 1, \ldots, n,$$

are these isomorphisms. Let $x = x_1 \otimes \cdots \otimes x_n \in D(G_1) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} D(G_n)$. Then

$$\alpha(x) = \alpha\big((x_1 \otimes 1_{D(G_2)} \otimes \cdots \otimes 1_{D(G_n)}) \cdot \ldots \cdot (1_{D(G_1)} \otimes \cdots \otimes 1_{D(G_{n-1})} \otimes x_n)\big)$$
$$= (\alpha \circ \tau_1)(x_1) \cdot \ldots \cdot (\alpha \circ \tau_n)(x_n) = (\tilde{\tau}_1 \circ \alpha_1)(x_1) \cdot \ldots \cdot (\tilde{\tau}_n \circ \alpha_n)(x_n)$$
$$= \alpha_1(x_1) \otimes \cdots \otimes \alpha_n(x_n).$$

Therefore $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$. $\square$

The next result is concerned with the group of torsion units of $D(G)$ where G is a nilpotent group of odd order.

Theorem 8.6. *Let G be a nilpotent group of odd order. Then $U_T(D(G)) \cong \hat{G} \times C_2$.*

Proof. We assume $U_T(D(G)) \ncong \hat{G} \times C_2$. By Theorem 4.6 with $S = 1, G$, there exists

$$0 \neq u = \sum_{[H,\varphi]_G \in \mathcal{M}(G)/G} z_{[H,\varphi]}[H,\varphi]_G \in D(G), \quad z_{[H,\varphi]} \in \mathbb{Z},$$

with $\sum_{k=1}^{2|G|} \binom{2|G|}{k} u^k = 0$ and $z_{[G,\varphi]} = 0$ for all $\varphi \in \hat{G}$. Thus $1 + u \in U_T(D(G))$. Choose $U \leqslant G$ such that $|U|$ is maximal with the property $z_{[U,\psi]} \neq 0$ for some $\psi \in \hat{U}$. Then $U < G$. Since $\pm\hat{U}$ is the set of all torsion units in $\mathbb{Z}\hat{U}$ there exists $\tau \in \hat{U}$ with

$$\rho_U(u) = \sum_{[U,\varphi]_G \in \mathcal{M}(G)/G} z_{[U,\varphi]} \sum_{gU \in N_G(U)/U} {}^g\varphi = \pm\tau - 1.$$

In case $\tau \neq 1$ we get $z_{[U,1]} = -1$ and $(N_G(U) : U) = 1$. Since G is nilpotent, $(N_G(U) : U) \neq 1$ holds in contradiction to the above case. Therefore $\tau = 1$. Since $(N_G(U) : U) \not\equiv 0 \pmod 2$ the case $\rho_U(u) = -2$ is not possible. Therefore $\rho_U(u) = 0$. This implies $z_{[U,\varphi]} = 0$ for all $\varphi \in \hat{U}$ contradicting the assumption $z_{[U,\psi]} \neq 0$. Therefore $U_T(D(G)) \cong \hat{G} \times C_2$. $\square$

Corollary 8.7. *Let G and $\tilde{G}$ be finite nilpotent groups with $D(G) \cong D(\tilde{G})$. Then the $2'$-Hall subgroups of G/G' and $\tilde{G}/\tilde{G}'$ are isomorphic.*

Proof. Let H and $\tilde{H}$ be the $2'$-Hallgroups of G and $\tilde{G}$. By Theorem 8.4 we obtain the isomorphy $D(H) \cong D(\tilde{H})$. Moreover we get $H/H' \times C_2 \cong \tilde{H}/\tilde{H}' \times C_2$ by Theorem 8.6. Therefore we get $H/H' \cong \tilde{H}/\tilde{H}'$. $\square$

For p-nilpotent groups we get the following result.

Theorem 8.8. *Let G and $\tilde{G}$ be finite groups with $D(G) \cong D(\tilde{G})$. Assume that for a prime divisor p of $|G|$ the Sylow p-subgroups of G and $\tilde{G}$ are cyclic. If G is p-nilpotent then $\tilde{G}$ is p-nilpotent.*

Proof. Let P be a Sylow p-subgroup of G and let $\alpha : D(\tilde{G}) \to D(G)$ be an isomorphism. By Proposition 3.7, $\alpha(e_{(1,1)}^{D(\tilde{G})}) = e_{(U,u)}^{D(G)}$ holds with a normal abelian subgroup U of G and $u \in Z(G)$. Let $H := O^p(U)$ and $h := u_{p'} \in Z(G)$. By Proposition 7.1 we obtain

$$\alpha\big(e_{(1,1)}^{D(\tilde{G}),p}\big) = \sum_{[K,kK']_G \in I} e_{(K,kK')}^{D(G)} \tag{15}$$

with

$$I = \{[K, kK']_G \in \mathcal{D}(G)/G\colon\ K = HV,\ V \leqslant P,\ k = hv,\ v \in V\}.$$

Let $V \leqslant P$, $v, w \in V$ and $K := HV$. Assume $[K, hvK']_G = [K, hwK']_G$. We first prove $v = w$.

There exists $gK' \in N_G(K)/K'$ with ${}^g(hv)K' = hwK'$. Since $h \in Z(G)$ we get ${}^g vK' = wK'$. Since P is cyclic $K/H \cong V$ is cyclic. Thus $K' \leqslant H$ and $VK'/K' \cong V$. In particular VK'/K' is a cyclic p-subgroup of $N_G(K)/K'$. It holds $vK', wK' \in VK'/K'$, and since $|\langle wK'\rangle| = |\langle {}^g vK'\rangle|$ we get $\langle wK'\rangle = \langle vK'\rangle =: T$. Thus $gK' \in N_{N_G(K)/K'}(T)$. Since G is p-nilpotent the subgroup $N_G(K)$ is p-nilpotent and therefore $N_G(K)/K'$ is p-nilpotent. By the p-nilpotency-criteria of Frobenius follows that $N_{N_G(K)/K'}(T)/C_{N_G(K)/K'}(T)$ is a p-group. Since P is abelian every Sylow p-subgroup of $N_G(K)/K'$ is abelian. Therefore a Sylow p-subgroup of $N_G(K)/K'$ is included in $C_{N_G(K)/K'}(T)$. Thus $|N_G(K)/K'|_p$ divides $|C_{N_G(K)/K'}(T)|$. It follows that $N_{N_G(K)/K'}(T) = C_{N_G(K)/K'}(T)$ and therefore $wK' = vK'$. We get $v^{-1}w \in K' \cap V = 1$. Thus $v = w$.

Let $|P| = p^n$ with $n \in \mathbb{N}$. For every divisor p^m, $m \in \mathbb{N}$, of $|P|$ there exists exactly one subgroup $V \leqslant P$ with $|V| = p^m$. By the above part of the proof there exist exactly p^m different orbits $[HV, hv(HV)']_G$, $v \in V$, for every subgroup $V \leqslant P$ with $|V| = p^m$. Therefore

$$|I| = \sum_{i=0}^{n} p^i = \frac{p^{n+1} - 1}{p - 1}.$$

Let $\tilde{P}$ be a Sylow p-subgroup of $\tilde{G}$. Since $\tilde{P}$ is cyclic we get

$$e_{(1,1)}^{D(\tilde{G}),p} = \sum_{[\tilde{K}, \tilde{k}\tilde{K}']_{\tilde{G}} \in J} e_{(\tilde{K}, \tilde{k}\tilde{K}')}^{D(\tilde{G})}$$

with

$$J = \{[\tilde{V}, \tilde{v}]_{\tilde{G}} \in \mathcal{D}(\tilde{G})/\tilde{G}\colon\ \tilde{V} \leqslant \tilde{P},\ \tilde{v} \in \tilde{V}\}$$

by Theorem 6.4. Since $|\tilde{P}| = p^n$ it holds $|J| \leqslant \sum_{i=0}^{n} p^i$, and by Eq. (15) we get $|I| = |J|$. Hence $[\tilde{V}, \tilde{v}]_{\tilde{G}} = [\tilde{W}, \tilde{w}]_{\tilde{G}}$ with $\tilde{V}, \tilde{W} \leqslant \tilde{P}$, $\tilde{v} \in \tilde{V}$, $\tilde{w} \in \tilde{W}$ if and only if $\tilde{V} = \tilde{W}$ and $\tilde{v} = \tilde{w}$. Therefore $N_{\tilde{G}}(\tilde{V}) = C_{\tilde{G}}(\tilde{V})$ for all $\tilde{V} \leqslant \tilde{P}$. In particular $N_{\tilde{G}}(\tilde{P}) = C_{\tilde{G}}(\tilde{P})$. Thus $\tilde{G}$ is p-nilpotent by the p-nilpotency criterion of Burnside. □

Acknowledgments

This paper contains a part of the author's doctoral thesis which was supervised by Professor Burkhard Külshammer.

References

[1] R. Boltje, A canonical Brauer induction formula, Astérisque 181–182 (1990) 31–59.
[2] R. Boltje, A general theory of canonical induction formulae, J. Algebra 206 (1) (1998) 293–343.
[3] R. Boltje, Representation rings of finite groups, their species and idempotent formulae, J. Algebra, in press.
[4] R. Boltje, Monomial resolutions, J. Algebra 246 (2) (2001) 811–848.
[5] R. Boltje, Integrality conditions for elements in ghost rings of generalized Burnside rings, J. Algebra, in press.
[6] R. Brandl, T. Huckle, On the isomorphism problem for Burnside rings, Proc. Amer. Math. Soc. 123 (12) (1995) 3623–3626.
[7] M. Deiml, Zur Darstellungstheorie von Darstellungsringen, Dissertation, Jena 1997.
[8] A. Dress, The ring of monomial representations. I. Structure theory, J. Algebra 18 (1971) 137–157.
[9] B. Fotsing, B. Külshammer, Modular species and prime ideals for the ring of monomial representations of a finite group, Comm. Algebra 33 (10) (2005) 3667–3677.

[10] D. Gluck, Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex, Illinois J. Math. 25 (1) (1981) 63–67.
[11] T. Hawkes, I.M. Isaacs, M. Özaydin, On the Möbius function of a finite group, Rocky Mountain J. Math. 19 (4) (1989) 1003–1034.
[12] G. Higman, The units of group rings, Proc. London Math. Soc. 46 (1940) 231–248.
[13] B. Huppert, Endliche Gruppen I, Springer-Verlag, 1967.
[14] W. Kimmerle, F. Luca, A.G. Raggi-Cárdenas, Irreducible components and isomorphisms of the Burnside ring, preprint, 2006.
[15] T. Matsuda, On the unit groups of Burnside rings, Jpn. J. Math. 8 (1) (1982) 71–93.
[16] A.G. Raggi-Cárdenas, Groups with isomorphic Burnside rings, Arch. Math. 84 (3) (2005) 193–197.
[17] V.P. Snaith, Explicit Brauer Induction. With Applications to Algebra and Number Theory, Cambridge University Press, 1994.