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Perron–Frobenius operators and representations of the Cuntz–Krieger algebras for infinite matrices

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A B S T R A C T

In this paper we extend the work of Kawamura, see [K. Kawamura, The Perron–Frobenius operators, invariant measures and representations of the Cuntz–Krieger algebras, J. Math. Phys. 46 (2005)], for Cuntz–Krieger algebras O_A for infinite matrices A. We generalize the definition of branching systems, prove their existence for any given matrix A and show how they induce some very concrete representations of O_A. We use these representations to describe the Perron–Frobenius operator, associated to a nonsingular transformation, as an infinite sum and under some hypothesis we find a matrix representation for the operator. We finish the paper with a few examples.

1. Introduction

The interactions between the theory of dynamical systems and operator algebras are one of the main venues in modern mathematics. Exploring this interplay, Kawamura, see [3], recently showed that the theory of representations of the Cuntz–Krieger algebras is closely related to the theory involving the Perron–Frobenius operator. The work of Kawamura is done for the Cuntz–Krieger algebras O_A, for finite matrices A. In this paper we generalize many of the results in [3] for the Cuntz–Krieger algebras for infinite matrices (a concept introduced by Exel and Laca in [1]). For example, under some mild assumptions, we are able to give an explicit characterization of the Perron–Frobenius operator, associated to a nonsingular transformation, as an infinite sum, using a representation of an infinite Cuntz–Krieger algebra. In our efforts to generalize the notions of [3] we found two problems with the work done in there that we believe are worth mentioning. First is the necessity of an extra hypothesis in the definition of a branching function system given in [3]. The other problem is in the statement of Theorem 1.2 of [3], where BA should read $A^T B$. We will deal with both these cases when introducing our generalized versions of the theory of [3].

We organize the paper in the following way: In the remaining of the introduction we quickly recall the reader the main definitions of [3] and show the need for an extra hypothesis in the definition of a branching function system. In Section 2, we define branching systems for infinite matrices A, which we denote by A_∞. We deal with the existence of A_∞-branching systems for any given matrix A (infinite or not) and show how they induce representations of O_A in Section 3. Next, in Section 4, we use the representations introduced in Section 3 to describe the Perron–Frobenius operator as an infinite sum; we also present the generalized and corrected version of Theorem 1.2 of [3] in this section. We finish the paper in Section 5 with a few examples.

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Given a measure space (X,μ), let $L_p(X,\mu)$ be the set of all complex valued measurable functions f such that $\|f\|_p<\infty$. For a nonsingular transformation $F:X\to X$ (that is, $\mu(F^{-1}(E))=0$ if $\mu(E)=0$) let $P_F:L_1(X,\mu)\to L_1(X,\mu)$ be the Perron–Frobenius operator, that is, P_F is such that

$$\int_E P_F\psi(x)\,d\mu=\int_{F^{-1}(E)}\psi(x)\,d\mu$$

for each measurable subset E of X, for all $\psi\in L_1(X,\mu)$. Notice that, for $\psi\in L_1(X,\mu)$, $P_F(\psi)$ is the Radon–Nikodym derivative of the measure μ_{P_F}, given by $\mu_{P_F}(E)=\int_{F^{-1}(E)}\psi(x)\,d\mu$, with respect to μ (see [4] for more details about the Perron–Frobenius operator).

In order to describe the Perron–Frobenius operators and representations of the Cuntz–Krieger algebras, Kawamura, in [3], introduces the notion of A-branching function system on a measure space (X,μ): a family $(\{f_i\}_{i=1}^N,\{D_i\}_{i=1}^N)$ of measurable maps and measurable subsets of X, respectively, together with a nonsingular transformation $F:X\to X$ such that $f_i:D_i\to f_i(D_i)=R_i$, $\mu(X\setminus\bigcup_{i=1}^N R_i)=0$, $\mu(R_i\cap R_j)=0$ for all $i\neq j$, there exists the Radon–Nikodym derivative Φ_{f_i} of $\mu\circ f_i$ with respect to μ (where $\mu\circ f_i$ denotes the measure defined by $\mu\circ f_i(E)=\mu(f_i(E))$, for all measurable set E in D_i) and $\Phi_{f_i}>0$ almost everywhere in D_i for $i=1,\dots,N$, $F\circ f_i=id_{D_i}$ for each $i\in\mathbb{N}$ and $\mu(D_i\setminus\bigcup_{j:\ a_{ij}=1}R_i)=0$, where a_{ij} are the entries of the matrix A defining O_A.

Next, a family $\{S(f_i)\}_{i=1}^N$ of partial isometries in $L_2(X,\mu)$ is defined by $S(f_i)(\phi)=\chi_{R_i}\cdot(\Phi_F)^{\frac12}\cdot\phi\circ F$, where χ_{R_i} denotes the characteristic function of R_i, and a representation of O_A in $L_2(X,\mu)$ is obtained by defining $\pi_f(s_i)=S(f_i)$ $(i=1,\dots,N)$ (where s_i is one of the generating partial isometry in O_A), and using the universal property of O_A. But it happens that the definition given above for an A-branching function system is not enough to guarantee that we get a representation of O_A, in fact, it is not enough to prove most of the theorems in [3]. For example, let $X=[0,2]$, μ be the Lebesgue measure, $R_1=[0,1]=D_1$, $R_2=[1,2]=D_2$, $F:X\to X$ defined by $F(x)=x$ for each $x\in[0,2]$ (so, $f_i(x)=x$ for each $x\in D_i$) and $A=\begin{pmatrix}1&1\\1&0\end{pmatrix}$. Following [3], $(\{f_i\}_{i=1}2,\{D_i\}_{i=1}2)$ is an A-branching function system, but $S(f_1)^*S(f_1)(\phi)=\chi_{[0,1]}\cdot\phi$ and $(S(f_1)S(f_1)^*+S(f_2)S(f_2)^*)(\phi)=\chi_{[0,2]}\cdot\phi$, for each $\phi\in L_2(X,\mu)$, so that $S(f_1)^*S(f_1)\neq\sum_{i=1}^2 S(f_i)S(f_i)^*$. Therefore, the existence of a representation of O_A in $L_2([0,2],\mu)$ is not guaranteed.

As we have seen, we need to add some extra hypothesis to the definition of an A-branching function system. Namely, we also have to ask that $\mu(\bigcup_{j:\ a_{ij}=1}R_j\setminus D_i)=0$, for each $i=1,\dots,N$. We should mention that this extra condition is satisfied in all the examples given in [3]. With this new definition of an A-branching function system in mind, we are now able to generalize it to the countable infinite case.

2. A_∞-branching systems

For a measure space (X,μ) and for measurable subsets Y,Z of X, we write $Y\overset{\mu\text{-a.e.}}{=}Z$ if $\mu(Y\setminus Z)=0=\mu(Z\setminus Y)$ or equivalently, if there exist $Y',Z'\subset X$ such that $Y\cup Y'=Z\cup Z'$ with $\mu(Y')=0=\mu(Z')$.

Let A be an infinite matrix, with entries $A(i,j)\in\{0,1\}$, for $(i,j)\in\mathbb{N}\times\mathbb{N}$, and let (X,μ) be a measurable space. For each pair of finite subsets U,V of $\mathbb{N}$ and $j\in\mathbb{N}$ define

$$A(U,V,j)=\prod_{u\in U}A_{uj}\prod_{v\in V}(1-A_{vj}).$$

Definition 2.1. An A_∞-branching system on a σ-finite measure space (X,μ) is a family $(\{f_i\}_{i=1}^\infty,\{D_i\}_{i=1}^\infty)$ together with a nonsingular transformation $F:X\to X$ such that:

1. $f_i:D_i\to R_i$ is a measurable map, D_i,R_i are measurable subsets of X and $f_i(D_i)\overset{\mu\text{-a.e.}}{=}R_i$ for each $i\in\mathbb{N}$;
2. F satisfies $F\circ f_i=id_{D_i}$ μ-a.e. in D_i for each $i\in\mathbb{N}$;
3. $\mu(R_i\cap R_j)=0$ for all $i\neq j$;
4. $\mu(R_j\cap D_i)=0$ if $A(i,j)=0$ and $\mu(R_j\setminus D_i)=0$ if $A(i,j)=1$;
5. For each pair U,V of finite subsets of $\mathbb{N}$ such that $A(U,V,j)=1$ only for a finite number of j's,
$$\bigcap_{u\in U}D_u\cap\bigcap_{v\in V}(X\setminus D_v)\overset{\mu\text{-a.e.}}{=}\bigcup_{j\in\mathbb{N}:\ A(U,V,j)=1}R_j;$$
6. There exist the Radon–Nikodym derivatives Φ_{f_i} of $\mu\circ f_i$ with respect to μ in D_i and $\Phi_{f_i^{-1}}$ of $\mu\circ f_i^{-1}$ with respect to μ in R_i.

The existence of the Radon–Nikodym derivative Φ_{f_i} of $\mu\circ f_i$ with respect to μ in D_i together with the fact that $F\circ f_i=id_{D_i}$ μ-a.e. imply that $f_i\circ F_{|R_i}=id_{R_i}$ μ-a.e. So, the function f_i is μ-a.e. invertible, with inverse $f_i^{-1}:=F_{|R_i}$. These are the functions that appear in condition 6 above. If follows from the same condition that Φ_{f_i} and $\Phi_{f_i^{-1}}$ are measurable functions in D_i and R_i, respectively. We will also consider these functions as measurable functions in X, defining it as being zero out of D_i and R_i, respectively.

The functions Φ_{f_i} and $\Phi_{f_i^{-1}}$ are nonnegative μ-a.e., because μ is a (positive) measure. It is possible to show that $\Phi_{f_i} > 0$ and $\Phi_{f_i^{-1}} > 0$ μ-a.e. in D_i and R_i, respectively, and $\Phi_{f_i}(x)\Phi_{f_i^{-1}}(f_i(x)) = 1$ μ-almost everywhere in D_i. This equality will be used in the next section.

3. Representations of Cuntz–Krieger algebras for infinite matrices

Representations of the Cuntz–Krieger algebras are of great importance, having applications both to operator algebras and to dynamical systems. In this section we show that for each A_∞-branching system, there exists a representation of the unital Cuntz–Krieger C^*-algebra O_A on $\mathcal{B}(L_2(X,\mu))$, the bounded operators on $L_2(X,\mu)$.

Following [1], recall that the unital Cuntz–Krieger algebra of an infinite matrix A, with $A(i,j) \in \{0,1\}$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$ is the unital universal C^*-algebra generated by a family $\{S_i\}_{i\in\mathbb{N}}$ of partial isometries that satisfy:

1. $S_i S_i^* S_j S_j^* = 0$ if $i \neq j$;
2. $S_i^* S_i$ and $S_j^* S_j$ commute, for all i, j;
3. $S_i^* S_i S_j S_j^* = A(i,j) S_j S_j^*$, for all i, j;
4. $\prod_{u\in U} S_u S_u^* \prod_{v\in V}(1 - S_v S_v^*) = \sum_{j=1}^{\infty} A(U,V,j) S_j S_j^*$, for each pair of finite subsets $U, V \subseteq \mathbb{N}$ such that $A(U,V,j) := \prod_{u\in U} A(u,j) \prod_{v\in V}(1 - A(v,j))$ vanishes for all but a finite number of j's.

Theorem 3.1. *For a given A_∞-branching system (see 2.1), there exists a $*$-homomorphism $\pi : O_A \to \mathcal{B}(L_2(X,\mu))$ such that $\pi(S_i)\phi = \chi_{R_i} \cdot (\Phi_{f_i^{-1}})^{\frac{1}{2}} \cdot \phi \circ F$ for each $\phi \in L_2(X,\mu)$.*

Proof. First notice that for a given $\phi \in L_2(X,\mu)$ we have that

$$\int_X \left|\chi_{R_i}(x)\Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi\big(F(x)\big)\right|^2 d\mu = \int_{R_i} \Phi_{f_i^{-1}}(x)\left|\phi\big(f_i^{-1}(x)\big)\right|^2 d\mu = \int_{R_i} \left|\phi\big(f_i^{-1}(x)\big)\right|^2 d\big(\mu \circ f_i^{-1}\big)$$
$$= \int_{D_i} \left|\phi(x)\right|^2 d\mu \leqslant \int_X \left|\phi(x)\right|^2 d\mu.$$

To obtain the second equality we have considered the Radon–Nikodym derivative of $\mu \circ f_i^{-1}$ with respect to μ in R_i and the last equality is an application of the change of variable theorem.

So, we define the operator $\pi(S_i) : \mathcal{L}(L_2(X,\mu)) \to \mathcal{L}(L_2(X,\mu))$ by

$$\pi(S_i)\phi = \chi_{R_i} \cdot (\Phi_{f_i^{-1}})^{\frac{1}{2}} \cdot (\phi \circ F),$$

for each $\phi \in L_2(X,\mu)$. By using the above computation, we see that $\pi(S_i) \in \mathcal{B}(L_2(X,\mu))$.

Our aim is to show that $\{\phi(S_i)\}_{i\in\mathbb{N}}$ satisfies the relations 1–4 which define the Cuntz–Krieger algebra O_A. With this in mind, let us first determine the operator $\psi(S_i)^*$.

For each $\phi, \psi \in L_2(X,\mu)$,

$$\big\langle \pi(S_i)\phi, \psi\big\rangle = \int_X \chi_{R_i}(x)\Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi\big(F(x)\big)\overline{\psi(x)}\,d\mu = \int_{R_i} \Phi_{f_i^{-1}}(x)^{\frac{1}{2}}\phi\big(f_i^{-1}(x)\big)\overline{\psi(x)}\,d\mu = \cdots$$

by using the change of variable theorem

$$\cdots = \int_{D_i} \Phi_{f_i^{-1}}\big(f_i(x)\big)^{\frac{1}{2}}\phi(x)\overline{\psi\big(f_i(x)\big)}\,d\big(\mu \circ f_i\big) = \cdots$$

considering the Radon derivative Φ_{f_i} of $\mu \circ f_i$

$$\cdots = \int_{D_i} \Phi_{f_i}(x)\Phi_{f_i^{-1}}\big(f_i(x)\big)^{\frac{1}{2}}\phi(x)\overline{\psi\big(f_i(x)\big)}\,d\mu = \int_{D_i} \Phi_{f_i}(x)^{\frac{1}{2}}\phi(x)\overline{\psi\big(f_i(x)\big)}\,d\mu$$
$$= \int_X \phi(x)\chi_{D_i}\Phi_{f_i}(x)^{\frac{1}{2}}\,\overline{\psi\big(f_i(x)\big)}\,d\mu = \big\langle \phi, \chi_{D_i} \cdot \Phi_{f_i}^{\frac{1}{2}} \cdot (\psi \circ f_i)\big\rangle.$$

Then

$$\pi(S_i)^*\psi = \chi_{D_i} \cdot \Phi_{f_i}^{\frac{1}{2}} \cdot (\psi \circ f_i).$$

It is easy to show that

$$\pi(S_i)^*\pi(S_i)\psi = \chi_{D_i} \cdot \psi = M_{\chi_{D_i}}(\psi),$$

for each $\psi \in L_2(X,\mu)$ (that is, $\pi(S_i)^*\pi(S_i)$ is the multiplication operator by χ_{D_i}). In the same way $\pi(S_i)\pi(S_i)^* = M_{\chi_{R_i}}$.

Now we verify if $\{\pi(S_i)\}_{i\in\mathbb{N}}$ satisfies the relations 1–4, which define the C^*-algebra O_A. The first relation follows from the fact that $\mu(R_i \cap R_j) = 0$ for $i \neq j$. The second one is trivial.

To see that the third relation is also satisfied, recall that if $A(i,j) = 0$ then $\mu(R_j \cap D_i) = 0$ and hence

$$\pi(S_i)^*\pi(S_i)\pi(S_j)\pi(S_j)^* = M_{\chi_{D_i}} M_{\chi_{R_j}} = M_{\chi_{D_i \cap R_j}} = 0,$$

and if $A(i,j) = 1$ then $\mu(R_j \setminus D_i) = 0$ and hence

$$\pi(S_i)^*\pi(S_i)\pi(S_j)\pi(S_j)^* = M_{\chi_{D_i}} M_{\chi_{R_j}} = M_{\chi_{D_i \cap R_j}} = M_{\chi_{R_j}} = \pi(S_j)\pi(S_j)^*.$$

So, for each $i, j \in \mathbb{N}$,

$$\pi(S_i)^*\pi(S_i)\pi(S_j)\pi(S_j)^* = A(i,j)\pi(S_j)\pi(S_j)^*.$$

To verify the last relation, let U, V be finite subsets of $\mathbb{N}$ such that $A(U,V,j) = 1$ only for finitely many j's.

Then, by Definition 2.1.5,

$$M_{\chi_{(\bigcap_{u\in U} D_u \bigcap_{v\in V}(X\setminus D_v))}} = M_{\chi_{(\bigcup_{A(U,V,j)=1} R_j)}}.$$

Note that

$$M_{\chi_{(\bigcap_{u\in U} D_u \bigcap_{v\in V}(X\setminus D_v))}} = \prod_{u\in U} M_{\chi_{D_u}} \prod_{v\in V}(Id - M_{\chi_{D_v}}) = \prod_{u\in U} \pi(S_u)^*\pi(S_u) \prod_{v\in V}\bigl(Id - \pi(S_v)^*\pi(S_v)\bigr).$$

On the other hand,

$$M_{\chi_{(\bigcup_{j\in\mathbb{N}:\ A(U,V,j)=1} R_j)}} = \sum_{j\in\mathbb{N}:\ A(U,V,j)=1} M_{\chi_{R_j}} = \sum_{j\in\mathbb{N}:\ A(U,V,j)=1} \pi(S_j)\pi(S_j)^*.$$

This shows that the last relation defining O_A is also verified.

So, there exists a $*$-homomorphism $\pi : O_A \to \mathcal{B}(L_2(X,\mu))$ satisfying $\pi(S_i)\phi = \chi_{R_i} \cdot (\Phi_{f_i^{-1}})^{\frac{1}{2}} \cdot \phi \circ F$. $\square$

The previous theorem applies only if an A_∞-branching system is given. Our next step is to guarantee the existence of A_∞-branching systems for any matrix A. First we prove a lemma, which will be helpful in some situations.

Lemma 3.2. *Let A be an infinite matrix with entries in $\mathbb{N} \times \mathbb{N}$ having no identically zero rows, (X,μ) be a measure space, and let $\{R_j\}_{j=1}^\infty$ and $\{D_j\}_{j=1}^\infty$ be families of measurable subsets of X such that*

(a) $\mu(R_i \cap R_j) = 0$ *for all* $i \neq j$;
(b) $X \overset{\mu\text{-a.e.}}{=} \bigcup_{j=1}^\infty R_j$;
(c) $D_i \overset{\mu\text{-a.e.}}{=} \bigcup_{j\in\mathbb{N}:\ A(i,j)=1} R_j$.

Then conditions 4 *and* 5 *of* 2.1 *are satisfied.*

Proof. Condition 4 follows from (a) and (b). To show 5 first we note that $X \setminus D_v \overset{\mu\text{-a.e.}}{=} \bigcup_{j\in\mathbb{N}:\ A(v,j)=0} R_j$. Then, given U, V finite subsets of N, we have that

$$\bigcap_{u\in U} D_u \cap \bigcap_{v\in V}(X\setminus D_v) \overset{\mu\text{-a.e.}}{=} \Biggl(\bigcup_{j\in\mathbb{N}:\ A(u,j)=1\ \forall u\in U} R_j\Biggr) \cap \Biggl(\bigcup_{j\in\mathbb{N}:\ A(v,j)=0\ \forall v\in V} R_j\Biggr)$$
$$\overset{\mu\text{-a.e.}}{=} \Biggl(\bigcup_{j\in\mathbb{N}:\ \prod_{u\in U} A(u,j)=1} R_j\Biggr) \cap \Biggl(\bigcup_{j\in\mathbb{N}:\ \prod_{v\in V}(1-A(v,j))=1} R_j\Biggr) \overset{\mu\text{-a.e.}}{=} \bigcup_{j\in\mathbb{N}:\ A(U,V,j)=1} R_j. \quad \square$$

Theorem 3.3. *For each infinite matrix A, without identically zero rows, there exists an A_∞-branching system in the measure space $([0,\infty),\mu)$, where μ is the Lebesgue measure.*

Proof. Consider $[0,\infty)$ with the Lebesgue measure μ. Define $R_i=[i,i+1]$ and $D_i=\bigcup_{j:\ A(i,j)=1} R_j$. Note that $\mu(R_i\cap R_j)=0$ for $i\neq j$. Then, by the previous lemma, conditions 4 and 5 of Definition 2.1 are satisfied. So, it remains to define maps $f_i: D_i\to R_i$ and $F:[0,+\infty)\to[0,+\infty)$ satisfying the conditions of Definition 2.1. For a fixed $i_0\in\mathbb{N}$ we define f_{i_0} as follows. First divide the interval $\mathring{R}_{i_0}$ (where $\mathring{R}_{i_0}$ denotes the interior of R_{i_0}) in $\#\{j:\ A(i_0,j)=1\}$ intervals I_j. Then, define $\tilde{f_{i_0}}:\bigcup_{j:\ A(i_0,j)=1}\mathring{R}_j\to\bigcup_{j:\ A(i_0,j)=1}\mathring{I}_j$ such that $\tilde{f_{i_0}}:\mathring{R}_j\to\mathring{I}_j$ is a $C1$-diffeomorphism. We now define $f_{i_0}:D_{i_0}\to R_{i_0}$ by

$$f_{i_0}(x)=\begin{cases}\tilde{f_{i_0}}(x) & \text{if } x\in\bigcup_{j:\ A(i_0,j)=1}\mathring{R}_j,\\ i_0 & \text{if } x\in D_{i_0}\setminus\bigcup_{j:\ A(i_0,j)=1}\mathring{R}_j,\end{cases}$$

and $F:[0,\infty)\to[0,\infty)$ by

$$F(x)=\begin{cases}\tilde{f_{i_0}}^{-1}(x) & \text{if } x\in\bigcup_{j:\ A(i_0,j)=1}\mathring{I}_j,\\ 0 & \text{if } x\in R_{i_0}\setminus\bigcup_{j:\ A(i_0,j)=1}\mathring{I}_j.\end{cases}$$

Note that f_i and F are measurable maps. Moreover, $\mu\circ f_i$ and $\mu\circ f_i^{-1}$ are σ-finite measures in D_i and R_i. Next we show that there exist the Radon–Nikodym derivatives Φ_{f_i} of $\mu\circ f_i$ with respect do μ in D_i. Let $E\subseteq D_i$ be such that $\mu(E)=0$. To show that $\mu\circ f_i(E)=0$ it is enough to show that $\mu\circ f_i(E\cap(\bigcup_{j:\ A(i,j)=1}\mathring{R}_j))=0$, and this equality is true by [5]. Then, by [2], there exists the desired nonnegative Radon–Nikodym derivative Φ_{f_i}. In the same way there exists the (nonnegative) Radon–Nikodym derivative $\Phi_{f_i^{-1}}$ of $\mu\circ f_i^{-1}$ with respect to μ in R_i. We still need to show that F is nonsingular. For this, let $E\subseteq[0,\infty)$ be such that $\mu(E)=0$. Notice that it is enough to prove that $\mu(F^{-1}(E)\cap R_j)=0$ for each j. Now $\mu(F^{-1}(E)\cap R_j)=\mu(f_j(E\cap D_j))=0$ (where the last equality follows from the fact that $\mu\circ f_j\ll\mu$ in D_j), and hence $\mu(F^{-1}(E))=0$ as desired. □

Corollary 3.4. *Given an infinite matrix A, there exists a representation of O_A in $L_2([0,\infty),\mu)$ where μ is the Lebesgue measure. If A is $N\times N$ then there exists a representation of O_A in $L_2([0,N),\mu)$ where μ is the Lebesgue measure.*

4. The Perron–Frobenius operator

We now describe the Perron–Frobenius operator using the representations introduced in the previous section.

Theorem 4.1. *Let (X,μ) be a measure space with a branching system as in Definition* 2.1 *and let $\varphi\in L_1(X,\mu)$ be such that $\varphi(x)\geqslant 0$ μ-a.e.*

1. *If* $\operatorname{supp}(\varphi)\subseteq\bigcup_{i=1}^N R_j$, *then*

$$P_F(\varphi)=\sum_{i=1}^N\big(\pi(S_i^*)\sqrt{\varphi}\big)2.$$

2. *If* $\operatorname{supp}(\varphi)\subseteq\bigcup_{i=1}^\infty R_j$, *then*

$$P_F(\varphi)=\lim_{N\to\infty}\sum_{i=1}^N\big(\pi(S_i^*)\sqrt{\varphi}\big)2,$$

where the convergence occurs in the norm of $L_1(X,\mu)$.

Proof. The first assertion follow from the fact that for each measurable set $E\subseteq X$,

$$\int_E P_F(\varphi)(x)\,d\mu=\int_E\sum_{i=1}^N\big(\pi(S_i^*)\sqrt{\varphi}(x)\big)^2\,d\mu.$$

To prove this equality, we will use the Radon–Nikodym derivative of $\mu\circ f_i$, the change of variable theorem and the fact that $F^{-1}(E)\cap R_i=f_i(E\cap D_i)$. Given $E\subseteq X$ a measurable set we have that

$$\sum_{i=1}^{N}\int_E \big(\pi\big(S_i^*\big)\sqrt{\varphi}(x)\big)^2\,d\mu=\sum_{i=1}^{N}\int_E \chi_{D_i}(x)\Phi_{f_i}(x)\varphi\big(f_i(x)\big)\,d\mu=\sum_{i=1}^{N}\int_{E\cap D_i}\Phi_{f_i}(x)\varphi\big(f_i(x)\big)\,d\mu=\sum_{i=1}^{N}\int_{E\cap D_i}\varphi\big(f_i(x)\big)\,d(\mu\circ f_i)$$
$$=\sum_{i=1}^{N}\int_{f_i(E\cap D_i)}\varphi(x)\,d\mu=\sum_{i=1}^{N}\int_{F^{-1}(E)\cap R_i}\varphi(x)\,d\mu=\sum_{i=1}^{N}\int_{F^{-1}(E)}\chi_{R_i}\varphi(x)\,d\mu=\int_{F^{-1}(E)}\sum_{i=1}^{N}\chi_{R_i}\varphi(x)\,d\mu$$
$$=\int_{F^{-1}(E)}\varphi(x)\,d\mu=\int_E P_F(\varphi)(x)\,d\mu.$$

We now prove the second assertion. For each $N\in\mathbb{N}$, define $\varphi_N:=\sum_{i=1}^{N}\chi_{R_i}\cdot\varphi$. Note that $(\varphi_N)_{N\in\mathbb{N}}$ is an increasing sequence, bounded above by φ. Then

$$\lim_{N\to\infty}\int_X P_F(\varphi_N)(x)\,d\mu=\lim_{N\to\infty}\int_X \varphi_N(x)\,d\mu=\cdots$$

by the Lebesgue's Dominated Convergence Theorem

$$\cdots=\int_X \varphi(X)\,d\mu=\int_X P_F(\varphi)(x)\,d\mu.$$

Moreover, the sequence $(P_F(\varphi_N))_{N\in\mathbb{N}}$ is μ-a.e. increasing and bounded above by $P_F(\varphi)$.
Then,

$$\lim_{N\to\infty}\big\|P_F(\varphi)-P_F(\varphi_N)\big\|_1=\lim_{N\to\infty}\int_X\big|P_F(\varphi)(x)-P_F(\varphi_N)(x)\big|\,d\mu=\lim_{N\to\infty}\int_X P_F(\varphi)(x)-P_F(\varphi_N)(x)\,d\mu=0.$$

Therefore, $\lim_{N\to\infty}P_F(\varphi_N)=P_F(\varphi)$. By the first assertion, $P_F(\varphi_N)=\sum_{i=1}^{N}(\pi(S_i^*)\sqrt{\varphi_N})2$, and a simple calculation shows that

$$\sum_{i=1}^{N}\big(\pi\big(S_i^*\big)\sqrt{\varphi_N}\big)2=\sum_{i=1}^{N}\big(\pi\big(S_i^*\big)\sqrt{\varphi}\big)2.$$

So, we conclude that

$$\lim_{N\to\infty}\sum_{i=1}^{N}\big(\pi\big(S_i^*\big)\sqrt{\varphi}\big)2=P_F(\varphi).\qquad\square$$

Theorem 4.2. *Let A be a matrix such that each row has a finite number of* 1*s and let (X,μ) be an A_∞-branching system. Suppose $\mu(R_i)<\infty$ for each i (so that $\chi_{R_i}\in L_1(X,\mu)$). Moreover, suppose Φ_{f_i} is a constant positive function for each i, say $\Phi_{f_i}=b_i$ (for example, if f_i is linear). Let $W\subseteq L_1(X,\mu)$ be the vector subspace*

$$W=\operatorname{span}\{\chi_{R_i}:\ i\in\mathbb{N}\},$$

that is, W is the subspace of all finite linear combinations of χ_{R_i}. Then the Perron–Frobenius operator restricted to W, $P_{F|_W}:W\to W$, has a matrix representation given by A^TB, where B is the diagonal infinite matrix with nonzero entries $B_{i,i}=b_i$.

Although A and B are infinite matrices, we are considering the matrix multiplication A^TB as the usual multiplication for finite matrices, since B is column-finite.

Proof. Since each row z of A has a finite number of 1s, then, by Definition 2.1.5, taking $Z=\{z\}$ and $Y=\emptyset$, we obtain $D_z\overset{\mu\text{-a.e.}}{=}\bigcup_{j:\ A(z,j)=1}R_j$ so that $\chi_{D_z}=\sum_{j:\ A(z,j)=1}\chi_{R_j}$. Note that

$$P_F(\chi_{R_z})=b_z\chi_{D_z}=\sum_{j:\ A(z,j)=1}b_zR_j,$$

and so the element (j,z) of the matrix representation of $P_{F|_W}$ is $b_zA(z,j)$. $\square$

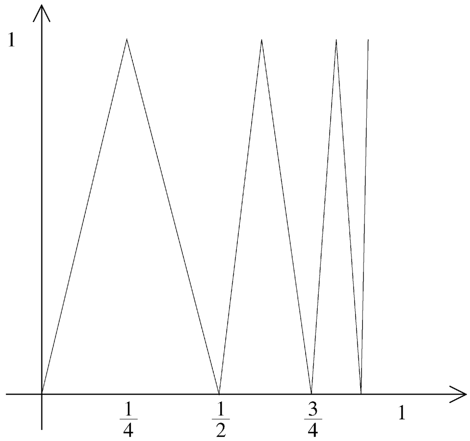

Fig. 1.

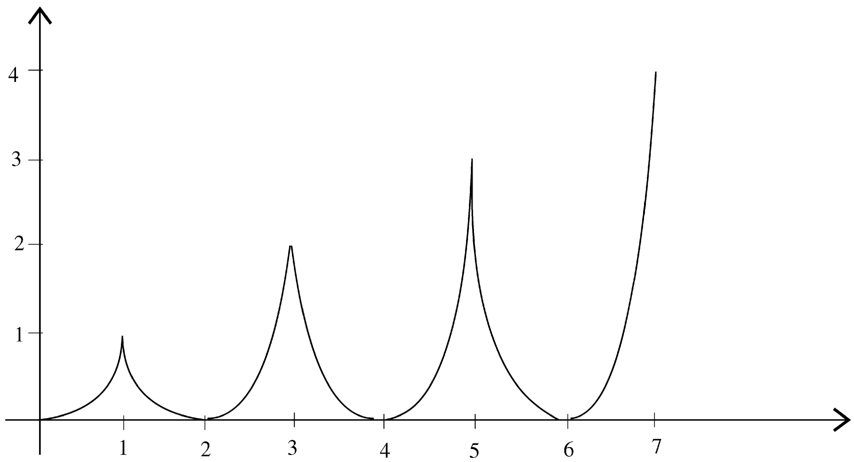

Fig. 2.

5. Examples

Example 5.1 *(O_∞ (O_A where all entries of the matrix A are 1)).* Consider $X = [0, 1]$ with Lebesgue measure and define $D_i = [0, 1]$, for $i = 1, 2, \ldots$. To define the R_i's we first need to define recursively the following sequences in X: Let $a_1 = 0$, $a_i = a_{i-1} + \frac{1}{2^i}$, $i = 2, 3, \ldots$, and let $b_i = \frac{a_i + a_{i+1}}{2}$, $i = 1, 2, \ldots$. Now define $R_i = [a_{\frac{i+1}{2}}, b_{\frac{i+1}{2}}]$ for i odd and $R_i = [b_{\frac{i}{2}}, a_{\frac{i}{2}+1}]$ for i even and define a map F on X by

$$F(x) = \frac{x}{b_{\frac{i+1}{2}} - a_{\frac{i+1}{2}}} + \frac{a_{\frac{i+1}{2}}}{a_{\frac{i+1}{2}} - b_{\frac{i+1}{2}}}, \quad \text{for } x \in R_i,\ i \text{ odd},$$

and

$$F(x) = \frac{x}{a_{\frac{i}{2}+1} - b_{\frac{i}{2}}} + \frac{b_{\frac{i}{2}}}{b_{\frac{i}{2}} - a_{\frac{i}{2}+1}}, \quad \text{for } x \in R_i,\ i \text{ even}.$$

Notice that F is nothing more than an affine transformation that takes the interval R_i onto $D_i = [0, 1]$, as shown in Fig. 1.

Finally, let $f_i = (F_{|R_i})^{-1}$. Then $(\{f_i\}_{i=1}^\infty, \{D_i\}_{i=1}^\infty)$ is an A_∞-branching system and hence induces a representation of the Cuntz–Krieger algebra O_∞.

Example 5.2. Let X be the measure space $[0, \infty)$, with the Lebesgue measure. Consider the map $F : [0, \infty) \to [0, \infty)$ defined by $F(x) = \frac{i}{2}(x - i)2$ for $x \in [i-1, i]$ and i odd and $F(x) = [\frac{i}{2}](x - (i-1))2$ for $x \in [i-1, i]$ and i even ($[\frac{i}{2}]$ is the least integer greater than or equal to $\frac{i}{2}$). In Fig. 2 we see the graph of F.

Define $R_i = [i-1, i]$ for $i = 1, 2, 3, \ldots,$ set $D_i = [0, [\frac{i}{2}]]$ and let $f_i : D_i \to R_i$ be defined by $f_i = (F_{|R_i})^{-1}$. Then $(\{f_i\}_{i=1}^{\infty}, \{D_i\}_{i=1}^{\infty})$ is an A_∞-branching system. This branching system induces a representation of the C^*-algebra O_A, for

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}.$$

Acknowledgments

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