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# Linear perturbations of differential or difference operators with polynomials as eigenfunctions

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**Abstract**

This paper deals with one-parameter linear perturbations of a family of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg[P_n(x)] = n$ of the form

$$P_n^{\mu}(x) = P_n(x) + \mu Q_n(x),$$

where $\mu$ is a real parameter and $\{Q_n(x)\}_{n=0}^{\infty}$ are polynomials with $\deg[Q_n(x)] \leqslant n$. Let the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ be eigenfunctions of a linear differential or difference operator $L$ with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. The purpose of this paper is to derive necessary and sufficient conditions for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ such that the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ are eigenfunctions of a linear difference or differential operator (possibly of infinite order) of the form

$$L + \mu A$$

with eigenvalues

$$\{\lambda_n + \mu\alpha_n\}_{n=0}^{\infty}.$$

*Keywords:* Differential operators; Difference operators; Orthogonal polynomials

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## 1. Introduction

In a number of recent papers polynomials are considered orthogonal with respect to an inner product consisting of the standard inner product of one of the classical orthogonal polynomials to which one or two linear perturbation terms are added. In all cases an explicit representation of these

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orthogonal polynomials is given and a linear differential or difference operator and eigenvalues are constructed with these orthogonal polynomials as eigenfunctions.

Polynomials orthogonal with respect to the inner product

$$(f,g) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)g(x)x^\alpha e^{-x}\,dx + \mu f(0)g(0),$$

with $\mu \geqslant 0$, $\alpha > -1$ were considered in [15] and a representation was given of the form

$$P_n^\mu(x) = P_n(x) + \mu Q_n(x), \quad n = 0, 1, \dots, \tag{1}$$

where the polynomials $\{P_n(x)\}_{n=0}^\infty$ are the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ and the polynomials $\{Q_n(x)\}_{n=0}^\infty$ can be expressed in terms of Laguerre polynomials. In [11] (see also [2]) it is shown that if $\mu > 0$ the polynomials are eigenfunctions of a uniquely determined linear differential operator of order $2\alpha+4$ if $\alpha$ is a nonnegative integer and of infinite order otherwise, which is a linear perturbation of the differential equation for Laguerre polynomials. In [10] a one-parameter perturbation of the inner product for ultraspherical polynomials is discussed and a linear differential operator of order $2\alpha+4$ if $\alpha$ is a nonnegative integer and of infinite order otherwise is derived having these polynomials as eigenfunctions.

In [4] the inner product

$$\langle f,g\rangle = \sum_{x=0}^\infty f(x)g(x)\frac{e^{-a}a^x}{x!} + \mu f(0)g(0) + \nu\Delta f(0)\Delta g(0),$$

$$a > 0, \quad \mu \geqslant 0, \quad \nu \geqslant 0, \quad \Delta f(x) = f(x+1) - f(x), \tag{2}$$

has been dealt with and the corresponding orthogonal polynomials which are generalizations of the Charlier polynomials have been constructed. In [6] the case $\mu > 0$ and $\nu = 0$ was studied and it is shown that in that case the polynomials are eigenfunctions of a linear difference equation of infinite order and in [4] it was proved that also in the case $\mu=0$ and $\nu > 0$ the polynomials are eigenfunctions of a difference operator of infinite order but the operator is no longer uniquely determined.

Later the inner product

$$\langle f,g\rangle = (1-c)^\beta \sum_{x=0}^\infty \frac{(\beta)_x c^x}{x!} f(x)g(x) + \mu f(0)g(0) + \nu\Delta f(0)\Delta g(0), \tag{3}$$

$\beta > 0$, $0 < c < 1$, $\mu \geqslant 0$, $\nu \geqslant 0$ was treated. Here $(\beta)_x = \beta(\beta+1)(\beta+2)\cdots(\beta+x-1)$. The case $\mu > 0$ and $\nu = 0$ has been considered in [5]. The corresponding orthogonal polynomials which are generalizations of the Meixner polynomials have been constructed and it is shown that in that case the polynomials are eigenfunctions of a linear difference equation of infinite order. In [7], by using a special normalization of the polynomials, the situations for Meixner and for Laguerre polynomials are compared and the case $\mu = 0$ and $\nu > 0$ is treated. In the Meixner case again a difference operator of infinite order was found, not uniquely determined, which in the limit tends to a differential operator for the Laguerre case.

In all these cases the orthogonal polynomials $\{P_n^\mu(x)\}_{n=0}^\infty$ are of the form (1), where $\{P_n(x)\}_{n=0}^\infty$ are the classical orthogonal polynomials in question and $\{Q_n(x)\}_{n=0}^\infty$ are explicitly known polynomials. If $\boldsymbol{L} = \boldsymbol{L}(x)$ denotes the linear differential or difference operator of the second order, having the classical orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ as eigenfunctions with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, then the differential or difference operator is of the form $\boldsymbol{L} + \mu\boldsymbol{A}$, where $\boldsymbol{A} = \boldsymbol{A}(x)$ is a linear differential or difference operator, possibly of infinite order.

Recently, in [9] some more general results were proved. These results will be discussed later.

In this paper we take a different point of view putting the perturbations directly to the polynomials instead of to the inner product or moment functional.

Let $\mathfrak{P}$ be the space of all polynomials and $\mathfrak{P}_n$ be the space of all polynomials of degree $\leqslant n$ $(n=0,1,2,\ldots)$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials with $\deg[P_n(x)]=n$ for each $n=0,1,2,\ldots,$ which are eigenfunctions of a linear operator $\boldsymbol{L}=\boldsymbol{L}(x)$ mapping $\mathfrak{P}_n$ into $\mathfrak{P}_n$ with eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, which are not all zero. Hence,

$$\boldsymbol{L}(x)P_n(x) = \lambda_n P_n(x), \quad n = 0,1,2,\ldots .$$

We consider a system of polynomials (1), where $\mu$ is a real parameter and $\{Q_n(x)\}_{n=0}^{\infty}$ denotes a system of polynomials with $\deg[Q_n(x)] \leqslant n$ for each $n = 0,1,2,\ldots$ .

We look for a linear operator $\boldsymbol{A}$ mapping $\mathfrak{P}_n$ into $\mathfrak{P}_n$ and numbers $\{\alpha_n\}_{n=0}^{\infty}$ such that the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ (linear perturbations of $\{P_n(x)\}_{n=0}^{\infty}$) are eigenfunctions of a linear operator $\boldsymbol{L}+\mu\boldsymbol{A}$ (a linear perturbation of $\boldsymbol{L}$) with eigenvalues $\{\lambda_n + \mu\alpha_n\}_{n=0}^{\infty}$ (linear perturbations of $\{\lambda_n\}_{n=0}^{\infty}$), i.e.

$$[(\boldsymbol{L} - \lambda_n\boldsymbol{I}) + \mu(\boldsymbol{A} - \alpha_n\boldsymbol{I})]P_n^{\mu}(x) = 0, \quad n = 0,1,2,\ldots . \tag{4}$$

Here $\boldsymbol{I}$ denotes the identity operator. We derive conditions for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ such that the operator $\boldsymbol{A}$ mapping $\mathfrak{P}_n$ into $\mathfrak{P}_n$ and the numbers $\{\alpha_n\}_{n=0}^{\infty}$ exist.

## 2. A necessary condition for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$

If we insert (1) into (4) and consider both sides to be polynomials in $\mu$, comparing the coefficients of equal powers of $\mu$ we obtain the following two systems of equations:

$$(\boldsymbol{L} - \lambda_n\boldsymbol{I})Q_n(x) + (\boldsymbol{A} - \alpha_n\boldsymbol{I})P_n(x) = 0, \quad n = 0,1,2,\ldots , \tag{5}$$

$$(\boldsymbol{A} - \alpha_n\boldsymbol{I})Q_n(x) = 0, \quad n = 0,1,2,\ldots . \tag{6}$$

We expand the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ in terms of $\{P_n(x)\}_{n=0}^{\infty}$:

$$Q_n(x) = \sum_{k=0}^{n} q_{n,k}P_k(x), \quad n = 0,1,2,\ldots \tag{7}$$

and find

$$(\boldsymbol{L} - \lambda_n\boldsymbol{I})Q_n(x) = -\sum_{k=0}^{n}(\lambda_n - \lambda_k)q_{n,k}P_k(x), \quad n = 0,1,2,\ldots ,$$

From (5) and (6) we deduce that any operator $\boldsymbol{A}$ mapping $\mathfrak{P}_n$ into $\mathfrak{P}_n$ and any numbers $\alpha_n$ satisfy (4) if and only if

$$\boldsymbol{A}P_n(x) = \alpha_n P_n(x) + \sum_{k=0}^{n}(\lambda_n - \lambda_k)q_{n,k}P_k(x), \quad n = 0,1,2,\ldots , \tag{8}$$

$$\sum_{k=0}^{n} q_{n,k} A P_k(x) = \alpha_n \sum_{k=0}^{n} q_{n,k} P_k(x), \quad n = 0, 1, 2, \ldots . \tag{9}$$

A necessary condition for the existence of an operator $A$ and the numbers $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (8) and (9) is

$$\sum_{k=0}^{n} q_{n,k} \left[ \alpha_k P_k(x) + \sum_{j=0}^{k} (\lambda_k - \lambda_j) q_{k,j} P_j(x) \right] = \alpha_n \sum_{k=0}^{n} q_{n,k} P_k(x), \quad n = 0, 1, 2, \ldots ,$$

or

$$\sum_{k=0}^{n} q_{n,k} [\alpha_n - \alpha_k] P_k(x) = \sum_{k=0}^{n} q_{n,k} \sum_{j=0}^{k} (\lambda_k - \lambda_j) q_{k,j} P_j(x), \quad n = 0, 1, 2, \ldots ,$$

Interchanging the summations we find

$$\sum_{k=0}^{n} q_{n,k} [\alpha_n - \alpha_k] P_k(x) = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} (\lambda_j - \lambda_k) q_{n,j} q_{j,k} \right) P_k(x), \; n = 0, 1, 2, \ldots . \tag{10}$$

As the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ form a basis for $\mathfrak{P}$ and since (10) is trivial for $n = 0$ we conclude:

**Theorem 2.1.** *A necessary condition for the existence of a linear operator $A$ and numbers $\{\alpha_n\}_{n=0}^{\infty}$, satisfying (8) and (9) is that for all $n = 1, 2, 3, \ldots$ and $k = 0, 1, \ldots, n-1$*

$$q_{n,k}[\alpha_n - \alpha_k] = \sum_{j=k+1}^{n} (\lambda_j - \lambda_k) q_{n,j} q_{j,k}. \tag{11}$$

**Example 2.2.** Let $P_n(x) = x^n$ for $n = 0, 1, 2, \ldots$ and let the operator $L$ be given by

$$L(x^n) = LP_n(x) = \frac{1}{x} \int_0^x P(t)\,\mathrm{d}t = \frac{x^n}{n+1}.$$

Hence, $\lambda_n = 1/(n+1)$. If we take

$$Q_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \quad n = 0, 1, 2, \ldots ,$$

then $q_{n,k} = 1/k!$ for all $n \in \{0, 1, 2, \ldots\}$ and all $k \in \{0, 1, 2, \ldots, n\}$. By using (11) for $k = n-1$ and the fact that $q_{n,n-1} \neq 0$ we obtain

$$\alpha_n - \alpha_{n-1} = (\lambda_n - \lambda_{n-1}) q_{n,n},$$

which can be summed up to

$$\alpha_n - \alpha_k = \sum_{j=k+1}^{n} (\lambda_j - \lambda_{j-1}) q_{j,j}.$$

Insertion into (11) would lead to

$$\frac{1}{k!}\sum_{j=k+1}^{n}\left(\frac{1}{j+1}-\frac{1}{j}\right)\frac{1}{j!}=\frac{1}{k!}\sum_{j=k+1}^{n}\left(\frac{1}{j+1}-\frac{1}{k+1}\right)\frac{1}{j!}$$

or

$$\frac{1}{k+1}\sum_{j=k+2}^{n}\frac{k+1-j}{j\cdot j!}=0,$$

which is clearly false for $n \geqslant k+2$. It follows that there cannot exist a linear operator $A$ and numbers $\alpha_n$ $(n\in\{0,1,2,\ldots\})$ such that the polynomials

$$x^n+\mu\sum_{k=0}^{n}\frac{x^k}{k!}$$

are eigenfunctions of $L+\mu A$.

## 3. Difference or differential operators

### 3.1. Notations

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of polynomials with $\deg[P_n(x)]=n$ for each $n=0,1,2,\ldots$ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers with $\lambda_0=0$ and $\{\lambda_n\}_{n=1}^{\infty}$ not all equal to zero such that $\{P_n(x)\}_{n=0}^{\infty}$ is a polynomial set of solutions of

$$L(x)y(x)\equiv\sum_{i=1}^{\infty}l_i(x)\mathfrak{D}_x^i y(x)=\lambda_n y(x). \tag{12}$$

Here $\{l_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials with $\deg[l_i(x)]\leqslant i$ for all $i=1,2,3,\ldots$. $\mathfrak{D}_x y(x)$ may be read as the derivative $Dy(x)=\mathrm{d}y(x)/\mathrm{d}x$, the forward difference $\Delta y(x)=y(x+1)-y(x)$ or backward difference $\nabla y(x)=y(x)-y(x-1)$ and $\mathfrak{D}_x^i y(x)=\mathfrak{D}_x(\mathfrak{D}_x^{i-1}y(x))$.

Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a set of polynomials with $\deg[Q_n(x)]\leqslant n$ for each $n=0,1,2,\ldots$ and $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ be the set of polynomials given by

$$P_n^{\mu}(x)=P_n(x)+\mu Q_n(x),\quad n=0,1,2,\ldots,\mu\in\mathbb{R}. \tag{13}$$

We look for an operator $A$ of the form

$$A(x)y(x)\equiv\sum_{i=1}^{\infty}a_i(x)\mathfrak{D}_x^i y(x), \tag{14}$$

where $\{a_i(x)\}_{i=1}^{\infty}$ is a sequence of polynomials with $\deg[a_i(x)]\leqslant i$ for all $i=1,2,3,\ldots$, and for a sequence of real numbers $\{\alpha_n\}_{n=0}^{\infty}$ with $\alpha_0=0$ such that

$$(L+\mu A)P_n^{\mu}(x)=(\lambda_n+\mu\alpha_n)P_n^{\mu}(x),\quad n=0,1,2,\ldots. \tag{15}$$

We will use the following lemma (see, for instance, [16]).

**Lemma 3.1.** *Let $\{p_n(x)\}_{n=0}^{\infty}$ be an arbitrary set of polynomials with* $\deg[p_n(x)] = n$ *for each* $n = 0, 1, 2, \ldots$ *and let* $\{\lambda_n\}_{n=0}^{\infty}$ *be an arbitrary sequence of constants with* $\lambda_0 = 0$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *not all equal to zero. Then there exists a unique sequence* $\{l_i(x)\}_{i=1}^{\infty}$ *of polynomials with* $\deg[l_i(x)] \leqslant i$ *for all* $i = 1, 2, 3, \ldots$ *such that*

$$\sum_{i=1}^{\infty} l_i(x)\mathfrak{D}_x^i y(x) = \lambda_n y(x)$$

*has* $\{p_n(x)\}_{n=0}^{\infty}$ *as a polynomial set of solutions. Moreover, if* $l_i(x) = k_i x^i +$ *lower-order terms for* $i = 1, 2, 3, \ldots,$ *then*

$$nk_1 + n(n-1)k_2 + \cdots + n!k_n = \lambda_n, \quad n = 1, 2, 3, \ldots .$$

**Definition 3.2.** Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be as in Section 3.1 and let (7) hold. We call the set of polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ given by (13) *a linear perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ of the class $m$ ($m \in \{0, 1, 2, \ldots\}$) when the following conditions are satisfied:

(i) if $n \leqslant m$ then $q_{n,k} = 0$ for all $k$ with $0 \leqslant k \leqslant n$,
(ii) if $n > m$ then $q_{n,n} \neq 0, q_{n,n-1} \neq 0$ and $q_{n,k} = 0$ for all $k$ with $0 \leqslant k < m$. (16)

We now state and prove our main result.

**Theorem 3.3.** *Let* $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ *be a linear perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ *of the class* $m$*. Then a necessary and sufficient condition for the existence of an operator* $A$ *of the form* (14) *and real numbers* $\alpha_n$ *such that* (15) *holds, is*

$$q_{n,k} \sum_{j=k+1}^{n} (\lambda_j - \lambda_{j-1})q_{j,j} = \sum_{j=k+1}^{n} (\lambda_j - \lambda_k)q_{n,j}q_{j,k}, \tag{17}$$

*for all* $n \in \{1, 2, 3, \ldots\}, k \in \{0, 1, \ldots, n-1\}$*. If* $m = 0$*, then the real numbers* $\{\alpha_n\}_{n=1}^{\infty}$ *and the operator* $A$ *are uniquely determined. If* $m > 0$*, then* $\alpha_1, \ldots, \alpha_m$ *can be chosen arbitrarily and the operator* $A$ *is uniquely determined when* $\alpha_1, \ldots, \alpha_m$ *are chosen.*

**Proof.** (i) *Condition* (17) *is necessary.*

In Theorem 2.1 we derived the necessary condition (11) for the existence of the operator $A$ and the numbers $\{\alpha_n\}_{n=0}^{\infty}$ such that (15) holds.

It is clear that the condition (11) is trivially satisfied for $n \leqslant m$. If $n > m$ then $q_{n,n-1} \neq 0$ and (11) with $k = n-1$ yields

$$\alpha_n - \alpha_{n-1} = (\lambda_n - \lambda_{n-1})q_{n,n}.$$

Hence, by summing up

$$\alpha_n - \alpha_k = \sum_{j=k+1}^{n} (\lambda_j - \lambda_{j-1})q_{j,j} \quad \text{for } m \leqslant k \leqslant n-1.$$

If we insert this in (11) we obtain (17).

(ii) *Condition* (17) *is sufficient*. Let condition (17) be satisfied. We will construct a linear operator $A$ and real numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that (8) and (9) are both fulfilled. First, if $m > 0$, the numbers $\alpha_1, \ldots, \alpha_m$ are chosen arbitrarily and for $n > m$ we define

$$\alpha_n = \alpha_m + \sum_{j=m+1}^{n} (\lambda_j - \lambda_{j-1}) q_{j,j}, \quad n = m+1, m+2, \ldots . \tag{18}$$

Further, let $A$ be the operator of the form (14), uniquely determined by Lemma 3.1, such that

$$AP_n(x) = \alpha_n P_n(x), \quad n = 0, 1, \ldots, m, \tag{19}$$

$$AQ_n(x) = \alpha_n Q_n(x), \quad n = m+1, m+2, \ldots . \tag{20}$$

Note that $\deg[P_n(x)]=n$, $n=0,1,\ldots,m$ and by (16) $\deg[Q_n(x)]=n$, $n=m+1,m+2,\ldots$. Then it follows that the system of equations (9) is satisfied for all $n \in \{0,1,2,\ldots\}$ and (8) for all $n \in \{0,1,2,\ldots,m\}$. So it remains to prove that the system of equations (8) is satisfied for all $n \in \{m+1, m+2, \ldots\}$. We prove this by induction. By (16) we have

$$Q_n(x) = \sum_{k=m}^{n} q_{n,k} P_k(x), \quad n = m+1, m+2, \ldots$$

with $q_{n,n} \neq 0$, $q_{n,n-1} \neq 0$.

Take first $n = m+1$. Eq. (20) leads to

$$q_{m+1,m+1} AP_{m+1}(x) + q_{m+1,m} AP_m(x) = \alpha_{m+1}(q_{m+1,m+1} P_{m+1}(x) + q_{m+1,m} P_m(x)),$$

hence,

$$q_{m+1,m+1} AP_{m+1}(x) = \alpha_{m+1} q_{m+1,m+1} P_{m+1}(x) + q_{m+1,m}(\alpha_{m+1} - \alpha_m) P_m(x)$$

and by (18)

$$AP_{m+1}(x) = \alpha_{m+1} P_{m+1}(x) + q_{m+1,m}(\lambda_{m+1} - \lambda_m) P_m(x),$$

which is (8) for $n = m+1$.

Now let (8) hold for all $n$ with $m+1 \leqslant n \leqslant m+t$. We prove (8) for $n = m+t+1$. Eq. (20) leads to

$$\begin{aligned}
q_{m+t+1,m+t+1}(A - \alpha_{m+t+1} I) P_{m+t+1}(x) &= -\sum_{k=0}^{t} q_{m+t+1,m+k}(A - \alpha_{m+t+1} I) P_{m+k}(x) \\
&= \sum_{k=0}^{t} q_{m+t+1,m+k}(\alpha_{m+t+1} - \alpha_{m+k}) P_{m+k}(x) \\
&\quad - \sum_{k=0}^{t} q_{m+t+1,m+k} \sum_{j=0}^{k-1} (\lambda_{m+k} - \lambda_{m+j}) q_{m+k,m+j} P_{m+j}(x) \\
&= \sum_{k=0}^{t} q_{m+t+1,m+k} \left( \sum_{j=k+1}^{t+1} (\lambda_{m+j} - \lambda_{m+j-1}) q_{m+j,m+j} \right) P_{m+k}(x)
\end{aligned}$$

$$- \sum_{k=0}^{t} q_{m+t+1,m+k} \sum_{j=0}^{k-1} (\lambda_{m+k} - \lambda_{m+j}) q_{m+k,m+j} P_{m+j}(x)$$

$$= \sum_{k=0}^{t} \left( \sum_{j=k+1}^{t+1} (\lambda_{m+j} - \lambda_{m+k}) q_{m+t+1,m+j} q_{m+j,m+k} \right) P_{m+k}(x)$$

$$- \sum_{j=0}^{t} q_{m+t+1,m+j} \sum_{k=0}^{j-1} (\lambda_{m+j} - \lambda_{m+k}) q_{m+j,m+k} P_{m+k}(x)$$

$$= \sum_{k=0}^{t} \left( \sum_{j=k+1}^{t+1} (\lambda_{m+j} - \lambda_{m+k}) q_{m+t+1,m+j} q_{m+j,m+k} \right) P_{m+k}(x)$$

$$- \sum_{k=0}^{t} \left( \sum_{j=k+1}^{t} (\lambda_{m+j} - \lambda_{m+k}) q_{m+t+1,m+j} q_{m+j,m+k} \right) P_{m+k}(x)$$

$$= \sum_{k=0}^{t} (\lambda_{m+t+1} - \lambda_{m+k}) q_{m+t+1,m+t+1} q_{m+t+1,m+k} P_{m+k}(x).$$

Thus,

$$AP_{m+t+1}(x) = \alpha_{m+t+1} P_{m+t+1}(x) + \sum_{k=0}^{t} (\lambda_{m+t+1} - \lambda_{m+k}) q_{m+t+1,m+k} P_{m+k}(x). \qquad \square$$

**Remark.** Formula (17) can also be written as

$$q_{n,k} \left[ \left( \sum_{j=k+1}^{n-1} (\lambda_j - \lambda_{j-1}) q_{j,j} \right) - (\lambda_{n-1} - \lambda_k) q_{n,n} \right] = \sum_{j=k+1}^{n-1} (\lambda_j - \lambda_k) q_{n,j} q_{j,k}. \tag{21}$$

We now treat two other kinds of perturbations with more specific requirements for the coefficients $q_{n,k}$. These perturbations are especially relevant in the case of symmetric polynomials $\{P_n(x)\}_{n=0}^{\infty}$.

**Definition 3.4.** Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be as in Section 3.1 and let (7) hold. We call the set of polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ given by

$$P_n^{\mu}(x) = P_n(x) + \mu Q_n(x), \quad n = 0, 1, \ldots, \mu \in \mathbb{R},$$

*a special linear perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ of the class $m$ ($m \in \{0, 1, 2, \ldots\}$) when the following conditions are satisfied:

(a) if $n \leqslant m$ then

$$q_{n,k} = 0 \quad \text{for all } k \text{ with } 0 \leqslant k \leqslant n;$$

(b) if $n > m$ then for all $t \in \{1,2,3,\ldots\}$

$$
\begin{aligned}
&\text{(i) } q_{m+2t-1,k} = 0 \text{ for all } k \text{ with } 0 \leqslant k \leqslant m+2t-2,\\
&\text{(ii) } q_{m+2t,m+2t} = q_{m+2t-1,m+2t-1} \neq 0,\\
&\text{(iii) } q_{m+2t,k} = 0 \text{ for all } k \text{ with } 0 \leqslant k \leqslant m-1,\\
&\text{(iv) } q_{m+2t,m+2k+1} = 0 \text{ for all } k \text{ with } 0 \leqslant k \leqslant t-1,\\
&\text{(v) } q_{m+2t,m+2t-2} \neq 0 .
\end{aligned}
\tag{22}
$$

**Theorem 3.5.** *Let $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ be a special linear perturbation of $\{P_n(x)\}_{n=0}^{\infty}$ of the class m. Then a necessary and sufficient condition for the existence of an operator A of the form* (14) *and real numbers $\alpha_n$ such that* (15) *holds, is*

$$
\begin{aligned}
&q_{m+2n,m+2k} \sum_{j=k+1}^{n} (\lambda_{m+2j} - \lambda_{m+2j-2}) q_{m+2j,m+2j}\\
&\quad = \sum_{j=k+1}^{n} (\lambda_{m+2j} - \lambda_{m+2k}) q_{m+2n,m+2j} q_{m+2j,m+2k},
\end{aligned}
\tag{23}
$$

*for all $n \in \{1,2,3,\ldots\}$, $k \in \{0,1,\ldots,n-1\}$. For all $t \in \{1,2,3,\ldots\}$ the numbers $\alpha_{m+2t-1}$ and, if $m > 0$, the numbers $\alpha_1,\ldots,\alpha_m$ can be chosen arbitrarily. The other values of $\alpha_n$ and the operator A are uniquely determined when these arbitrary numbers have been chosen.*

**Proof.** (i) *Condition* (23) *is necessary.* In Theorem 2.1 we derived the necessary condition (11) for the existence of the operator $A$ and the numbers $\{\alpha_n\}_{n=0}^{\infty}$ such that (15) holds.

It is clear that conditions (11) are trivially satisfied for $n \leqslant m$. If $j \geqslant 1$ then (22(i), (v)) and (11) with $n = m+2j$, $k = m+2j-2$ yield

$$\alpha_{m+2j} - \alpha_{m+2j-2} = (\lambda_{m+2j} - \lambda_{m+2j-2}) q_{m+2j,m+2j}$$

and hence by summing up

$$\alpha_{m+2n} - \alpha_{m+2k} = \sum_{j=k+1}^{n} (\lambda_{m+2j} - \lambda_{m+2j-2}) q_{m+2j,m+2j} \quad \text{for } m \leqslant k \leqslant n-1.$$

If we insert this on the left-hand side of (11) with $n := m+2n$ and $k := m+2k$ we obtain the left-hand side of (23). The right-hand side of (11) is

$$\sum_{j=2k+1}^{2n} (\lambda_{m+j} - \lambda_{m+2k}) q_{m+2n,m+j} q_{m+j,m+2k}.$$

By (22(iv)) the right-hand side of (23) follows.

(ii) *Condition* (23) *is sufficient.* Let condition (23) be satisfied. We will construct a linear operator $A$ and real numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that (8) and (9) are both fulfilled. First, the numbers $\alpha_{m+2t-1}$ ($t \in \{1,2,3,\ldots\}$) and, if $m > 0$, the numbers $\alpha_1,\ldots,\alpha_m$ are chosen arbitrarily. For $n = m+2t$ ($t \in \{1,2,3,\ldots\}$) we define the numbers $\alpha_n$ by

$$\alpha_{m+2t} = \alpha_m + \sum_{j=1}^{t} (\lambda_{m+2j} - \lambda_{m+2j-2}) q_{m+2j,m+2j}. \tag{24}$$

Further, let $A$ be the operator of the form (14), uniquely determined by Lemma 3.1, such that (19) and (20) hold. Then it follows that the system of equations (9) is satisfied for all $n \in \{0,1,2,\dots\}$, and (8) for all $n \in \{0,1,2,\dots,m\}$. So it remains to prove that the system of equations (8) is satisfied for all $n = m + s$ $(s \in \{1,2,\dots\})$. This is obvious for odd values of $s$, since by (22(i))

$$Q_{m+2t-1}(x) = q_{m+2t,m+2t}P_{m+2t-1}(x) \quad \text{for all } t \in \{1,2,\dots\}.$$

For even values of $s$ we prove this by induction.

Take first $n = m + 2$. Eq. (20) leads to

$$q_{m+2,m+2}AP_{m+2}(x) + q_{m-2,m}AP_m(x) = \alpha_{m-2}(q_{m+2,m-2}P_{m+2}(x) + q_{m+2,m}P_m(x)),$$

hence,

$$q_{m\cdot 2,m+2}AP_{m+2}(x) = \alpha_{m+2}q_{m+2,m+2}P_{m-2}(x) + q_{m\cdot 2,m}(\alpha_{m+2} - \alpha_m)P_m(x)$$

and by (24)

$$AP_{m-2}(x) = \alpha_{m+2}P_{m+2}(x) + q_{m+2,m}(\lambda_{m+2} - \lambda_m)P_m(x),$$

which is (8) for $n = m + 2$.

Now let (8) hold for all $n$ with $m + 2 \leqslant n \leqslant m + 2t$. We prove (8) for $n = m + 2t + 2$. Eq. (20) leads to

$$
\begin{aligned}
&q_{m+2t+2,m+2t-2}(A-\alpha_{m+2t+2}I)P_{m+2t+2}(x)\\
&\quad = -\sum_{k=0}^{t} q_{m+2t+2,m+2k}(A-\alpha_{m+2t+2}I)P_{m-2k}(x)\\
&\quad =\sum_{k-0}^{t} q_{m+2t+2,m\cdot\cdot 2k}(\alpha_{m+2t+2} - \alpha_{m+2k})P_{m-2k}(x)\\
&\qquad -\sum_{k=0}^{t} q_{m+2t\cdot 2,m-2k}\sum_{j\cdot 0}^{k-1}(\lambda_{m+2k} - \lambda_{m+2j})q_{m+2k,m+2j}P_{m+2j}(x)\\
&\quad =\sum_{k=0}^{t} q_{m+2t+2,m-2k}\left(\sum_{j=k+1}^{t\cdot 1}(\lambda_{m+2j} - \lambda_{m+2j-2})q_{m+2j,m+2j}\right)P_{m+2k}(x)\\
&\qquad -\sum_{k=0}^{t} q_{m+2t+2,m+2k}\sum_{j-0}^{k\cdot\cdot 1}(\lambda_{m-2k} - \lambda_{m+2j})q_{m+2k,m-2j}P_{m-2j}(x)\\
&\quad =\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t+1}(\lambda_{m+2j} - \lambda_{m+2k})q_{m+2t+2,m+2j}q_{m-2j,m+2k}\right)P_{m+2k}(x)\\
&\qquad -\sum_{j=0}^{t} q_{m+2t+2,m+2j}\sum_{k=0}^{j-1}(\lambda_{m+2j} - \lambda_{m\cdot\cdot 2k})q_{m+2j,m+2k}P_{m+2k}(x)
\end{aligned}
$$

$$=\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t+1}(\lambda_{m+2j}-\lambda_{m+2k})q_{m+2t+2,m+2j}q_{m+2j,m+2k}\right)P_{m+2k}(x)$$

$$-\sum_{k=0}^{t}\left(\sum_{j=k+1}^{t}(\lambda_{m+2j}-\lambda_{m+2k})q_{m+2t+2,m+2j}q_{m+2j,m+2k}\right)P_{m+2k}(x)$$

$$=\sum_{k=0}^{t}(\lambda_{m+2t+2}-\lambda_{m+2k})q_{m+2t+2,m+2t+2}q_{m+2t+2,m+2k}P_{m+2k}(x).$$

Thus,

$$AP_{m+2t+2}(x)=\alpha_{m+2t+2}P_{m+2t+2}(x)+\sum_{k=0}^{t}(\lambda_{m+2t+2}-\lambda_{m+2k})q_{m+2t+2,m+2k}P_{m+2k}(x),$$

which completes the proof. □

**Remark.** Formula (23) can also be written as

$$q_{m+2n,m+2k}\left[\left(\sum_{j=k+1}^{n-1}(\lambda_{m+2j}-\lambda_{m+2j-2})q_{m+2j,m+2j}\right)-(\lambda_{m+2n-2}-\lambda_{m+2k})q_{m+2n,m+2n}\right]$$

$$=\sum_{j=k+1}^{n-1}(\lambda_{m+2j}-\lambda_{m+2k})q_{m+2n,m+2j}q_{m+2j,m+2k}. \tag{25}$$

**Definition 3.6.** Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be as in Section 3.1 and let (7) hold. We call the set of polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ given by (13) *a symmetric linear perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ *of the class* $m$ ($m\in\{0,1,2,\ldots\}$) when the following conditions are satisfied:

(i) $q_{n,k}=0$ for all $k$ with $0\leqslant k<n$ and $n-k$ odd,
(ii) if $n\leqslant m$ then $q_{n,k}=0$ for all $k$ with $0\leqslant k\leqslant n$,
(iii) if $n=m+1$ then $q_{n,n}\neq 0$ and $q_{n,k}=0$ for all $k$ with $0\leqslant k<n$,
(iv) if $n>m+1$ then $q_{n,n}\neq 0, q_{n,n-2}\neq 0$ and $q_{n,k}=0$ for all $k$ with $0\leqslant k<m$. (26)

**Theorem 3.7.** *Let* $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ *be a symmetric linear perturbation of* $\{P_n(x)\}_{n=0}^{\infty}$ *of the class* $m$. *Then necessary and sufficient conditions for the existence of an operator* $A$ *of the form* (14) *and real numbers* $\alpha_n$ *such that* (15) *holds, are*

$$q_{m+2n,m+2k}\sum_{j=k+1}^{n}(\lambda_{m+2j}-\lambda_{m+2j-2})q_{m+2j,m+2j}$$

$$=\sum_{j=k+1}^{n}(\lambda_{m+2j}-\lambda_{m+2k})q_{m+2n,m+2j}q_{m+2j,m+2k}, \tag{27}$$

$$q_{m+2n+1,m+2k+1}\sum_{j=k+1}^{n}(\lambda_{m+2j+1}-\lambda_{m+2j-1})q_{m+2j-1,m+2j-1}$$

$$=\sum_{j=k+1}^{n}(\lambda_{m+2j+1}-\lambda_{m+2k-1})q_{m+2n+1,m+2j+1}q_{m+2j+1,m+2k+1}, \tag{28}$$

*for all* $n\in\{1,2,3,\ldots\}$, $k\in\{0,1,\ldots,n-1\}$. *The real numbers* $\alpha_1,\ldots,\alpha_{m+1}$ *can be chosen arbitrarily. The numbers* $\{\alpha_k\}_{k=m+2}^{\infty}$ *and the operator* $A$ *are uniquely determined when* $\alpha_1,\ldots,\alpha_{m+1}$ *have been chosen.*

**Proof.** (i) *Conditions* (27) *and* (28) *are necessary.* In Theorem 2.1 we derived the necessary condition (11) for the existence of the operator $A$ and the numbers $\{\alpha_n\}_{n=0}^{\infty}$ such that (15) holds.

It is clear that the conditions (11) are trivially satisfied for $n\leqslant m+1$. If $j\geqslant 1$ then by (26(iv)) $q_{m+2j,m+2j-2}\neq 0$ and the proof that (27) is necessary is the same as in Theorem 3.5. The proof that (28) is necessary is similar.

(ii) *Conditions* (27) *and* (28) *are sufficient.* Let conditions (27) and (28) be satisfied. We will construct a linear operator $A$ and real numbers $\{\alpha_n\}_{n=1}^{\infty}$ such that (8) and (9) are both fulfilled. First, the numbers $\alpha_1,\ldots,\alpha_{m+1}$ are chosen arbitrarily and for $n>m+1$ we define for $t\in\{1,2,3,\ldots\}$

$$\alpha_{m+2t}=\alpha_m+\sum_{j=1}^{t}(\lambda_{m+2j}-\lambda_{m+2j-2})q_{m+2j,m+2j},$$

$$\alpha_{m+2t+1}=\alpha_{m+1}+\sum_{j=1}^{t}(\lambda_{m+2j+1}-\lambda_{m+2j-1})q_{m+2j+1,m+2j+1}.$$

Further, let $A$ be the operator of the form (14), uniquely determined by Lemma 3.1, such that (19) and (20) hold. Then it follows that the system of equations (9) is satisfied for all $n\in\mathbb{N}$ and (8) for all $n\in\{0,1,2,\ldots,m\}$. So it remains to prove that the system of equations (8) is satisfied for all $n=m+s$ ($s\in\{1,2,3,\ldots\}$). For $s=1$ this is obvious, since by our assumptions for the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ we have

$$Q_{m+1}(x)=q_{m+1,m+1}P_{m+1}(x).$$

For even values of $s$ we prove it by induction using (27) exactly as in the proof given in Theorem 3.5. For odd values of $s$ the proof is similar by using (28). □

**Remark.** Formulae (27) and (28) can also be written as

$$q_{m+2n,m+2k}\left[\left(\sum_{j=k+1}^{n-1}(\lambda_{m+2j}-\lambda_{m+2j-2})q_{m+2j,m+2j}\right)-(\lambda_{m+2n-2}-\lambda_{m+2k})q_{m+2n,m+2n}\right]$$

$$=\sum_{j=k+1}^{n-1}(\lambda_{m+2j}-\lambda_{m+2k})q_{m+2n,m+2j}q_{m+2j,m+2k}, \tag{29}$$

$$q_{m+2n+1,m+2k+1} \times \left[ \left( \sum_{j=k+1}^{n-1} (\hat{\lambda}_{m+2j+1} - \hat{\lambda}_{m+2j-1}) q_{m+2j+1,m+2j+1} \right) - (\hat{\lambda}_{m+2n-1} - \hat{\lambda}_{m+2k+1}) q_{m+2n+1,m+2n+1} \right]$$

$$= \sum_{j=k+1}^{n-1} (\hat{\lambda}_{m+2j+1} - \hat{\lambda}_{m+2k+1}) q_{m+2n+1,m+2j+1} q_{m+2j+1,m+2k+1}. \tag{30}$$

## 4. Applications

### *4.1. Sobolev-type orthogonal polynomials*

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials relative to a quasi-definite moment functional $\sigma$, which satisfy a differential equation of the form (12). Let $\phi$ be the symmetric bilinear form of Sobolev-type defined by

$$\phi(p,q) = \langle \sigma, pq \rangle + \mu p^{(l)}(c) q^{(l)}(c),$$

where $\mu(\neq 0)$ and $c$ are real constants, $l \in \{0, 1, 2, \ldots\}$, $p$ and $q$ are any polynomials and the notations

$$p^{(l)}(x) = \mathfrak{D}_x^l p(x), \quad l > 0,$$

and

$$p_n^{(0)}(x) = p_n(x)$$

are used. If $\phi$ is quasi-definite then in the case $\mathfrak{D}_x = \mathrm{d}/\mathrm{d}x$ it is shown in [9] that if $P_n^{(l)}(c) \neq 0$ for all $n = l, l+1, l+2, \ldots$, then the corresponding orthogonal polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ satisfy a differential equation (possibly of infinite order) of the form (15), where $\alpha_1, \ldots, \alpha_l$ can be chosen arbitrarily and the operator $A$ is uniquely determined when $\alpha_1, \ldots, \alpha_l$ are chosen. We are now in a position to derive this and the corresponding result for differences directly from Theorem 3.3.

If we write

$$K_n^{(r,s)}(x,y) = \sum_{i=0}^{n} \frac{\mathfrak{D}_x^r P_i(x) \mathfrak{D}_y^s P_i(y)}{\langle \sigma, P_i^2(x) \rangle}, \quad n, r, s \in \{0, 1, 2, \ldots\},$$

then (see [1, 3]) the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ can be written as (13) with

$$Q_n(x) = K_{n-1}^{(l,l)}(c,c) P_n(x) - P_n^{(l)}(c) K_{n-1}^{(0,l)}(x,c),$$

hence,

$$q_{n,k} = -\frac{P_n^{(l)}(c) P_k^{(l)}(c)}{\langle \sigma, P_k^2(x) \rangle}, \quad 0 \leqslant k \leqslant n-1, \tag{31}$$

and

$$q_{n,n} = K_{n-1}^{(l,l)}(c,c). \tag{32}$$

It follows that if $P_n^{(l)}(c) \neq 0$ for all $n = l, l+1, l+2, \ldots$, then $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ is a linear perturbation of class $l$ of $\{P_n(x)\}_{n=0}^{\infty}$.

In fact, if we insert (31) and (32) in (21), after some cancellation it remains to show that for $n \in \{0,1,2,\ldots\}$, $k \in \{0,1,\ldots,n-2\}$, $l \in \{0,1,2,\ldots\}$

$$\sum_{j=k+1}^{n-1} (\lambda_j - \lambda_{j-1}) K_{j-1}^{(l,l)}(c,c) - (\lambda_{n-1} - \lambda_k) K_{n-1}^{(l,l)}(c,c) = -\sum_{j=k+1}^{n-1} (\lambda_j - \lambda_k) \frac{P_j^{(l)}(c) P_j^{(l)}(c)}{\langle \sigma, P_j^2(x) \rangle}.$$

The summation-by-parts formula

$$\sum_{j=p}^{i} a_j b_j = \sum_{j=p}^{i-1} \left( \sum_{k=p}^{j} a_k \right) (b_j - b_{j+1}) + b_i \sum_{k=p}^{i} a_k, \tag{33}$$

applied to the sum at the left-hand side with $i := n-1$, $p := k+1$, $a_j := \lambda_j - \lambda_{j-1}$, $b_j := K_{j-1}^{(l,l)}(c,c)$ yields

$$\begin{aligned} \text{left-hand side} &= -\sum_{j=k+1}^{n-2} (\lambda_j - \lambda_k) \frac{P_j^{(l)}(c) P_j^{(l)}(c)}{\langle \sigma, P_j^2(x) \rangle} + (\lambda_{n-1} - \lambda_k) K_{n-2}^{(l,l)}(c,c) - (\lambda_{n-1} - \lambda_k) K_{n-1}^{(l,l)}(c,c) \\ &= \text{right-hand side.} \end{aligned}$$

### *4.2. Even and odd orthogonal polynomials with Sobolev-type perturbation at* 0

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials relative to a quasi-definite moment functional $\sigma$, such that $P_{2k}(x)$ is an even function and $P_{2k-1}(x)$ is an odd function for all $k = 0,1,2,\ldots$ and let these polynomials satisfy a differential equation of the form (12). Let $\phi$ be the symmetric bilinear form of Sobolev-type defined by

$$\phi(p,q) = \langle \sigma, pq \rangle + \mu p^{(l)}(0) q^{(l)}(0),$$

where $\mu (\neq 0)$ is a real constant, $p^{(l)}(x) = \mathrm{d}^l p(x)/\mathrm{d}x^l$, $l \in \{0,1,2,\ldots\}$, and $p$ and $q$ are any polynomials. The corresponding orthogonal polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ can be written as (13) with

$$Q_n(x) = K_{n-1}^{(l,l)}(0,0) P_n(x) - P_n^{(l)}(0) K_{n-1}^{(0,l)}(x,0).$$

Hence,

$$q_{n,k} = -\frac{P_n^{(l)}(0) P_k^{(l)}(0)}{\langle \sigma, P_k^2(x) \rangle}, \quad 0 \leqslant k \leqslant n-1, \tag{34}$$

and

$$q_{n,n} = K_{n-1}^{(l,l)}(0,0). \tag{35}$$

In this case if $l > 0$ then $P_n^{(l)}(x) \equiv 0$ for $0 \leqslant n < l$. Further for $n \geqslant l$ we have $P_n^{(l)}(0) = 0$ if $n - l$ is odd. Let

$$P_n^{(l)}(0) \neq 0 \quad \text{for all } n \geqslant l \text{ if } n - l \text{ is even.} \tag{36}$$

Then it easily follows from (34) and (35) that $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ is a special linear perturbation of the class $l$ of $\{P_n(x)\}_{n=0}^{\infty}$. We show that condition (25) is satisfied for all $n \in \{0,1,2,\ldots\}$, $k \in \{0,1,\ldots,n-2\}$. If we insert (34) and (35) in (25) we obtain after some cancellation

$$\sum_{j=k+1}^{n-1} (\lambda_{l+2j} - \lambda_{l+2j-2})K_{l+2j-2}^{(l,l)}(0,0) - (\lambda_{l+2n-2} - \lambda_{l+2k})K_{l+2n-2}^{(l,l)}(0,0)$$

$$= - \sum_{j=k+1}^{n-1} (\lambda_{l+2j} - \lambda_{l+2k})\frac{P_{l+2j}^{(l)}(0)P_{l+2j}^{(l)}(0)}{\langle \sigma, P_{l+2j}^2(x)\rangle}.$$

By using the summation-by-parts formula (33) to the sum at the left-hand side this follows immediately.

*Conclusion.* We may conclude that in the case of the classical symmetric polynomials (Gegenbauer, Chebyshev, Legendre and Hermite) which all satisfy condition (36) and are eigenfunctions of a second-order differential operator, the polynomials orthogonal with respect to the classical weight function with a one-parameter discrete Sobolev-type perturbation in 0 are also eigenfunctions of a differential operator, possibly of infinite order.

### 4.3. *Even and odd polynomials with a symmetric perturbation*

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of orthogonal polynomials relative to a quasi-definite moment functional $\sigma$, such that $P_{2k}(x)$ is an even function and $P_{2k+1}(x)$ is an odd function for all $k = 0,1,2,\ldots$ and let these polynomials satisfy a differential equation of the form (12). We consider the polynomials $\{P_n^{\mu}(x)\}_{n=0}^{\infty}$ written as (13) with

$$Q_n(x) = (K_{n-1}^{(l,l)}(-c,-c) + K_{n-1}^{(l,l)}(c,c))P_n(x) - (P_n^{(l)}(-c)K_{n-1}^{(0,l)}(x,-c) + P_n^{(l)}(c)K_{n-1}^{(0,l)}(x,c)),$$

where $p^{(l)}(x) = \dfrac{\mathrm{d}^l p(x)}{\mathrm{d}x^l}$, $l \in \{0,1,2,\ldots\}$. Hence, for $0 \leqslant k \leqslant n-1$

$$q_{n,k} = -\frac{P_n^{(l)}(-c)P_k^{(l)}(-c) + P_n^{(l)}(c)P_k^{(l)}(c)}{\langle \sigma, P_k^2(x)\rangle}$$

and

$$q_{n,n} = K_{n-1}^{(l,l)}(-c,-c) + K_{n-1}^{(l,l)}(c,c).$$

In this case if $l > 0$ then $P_n^{(l)}(x) \equiv 0$ for $0 \leqslant n < l$. Let $P_n^{(l)}(c) \neq 0$ for all $n \geqslant l$. For all $n \geqslant l$ we have $P_n^{(l)}(-c) = -P_n^{(l)}(c)$ if $n-l$ is odd and let $P_n^{(l)}(-c) = P_n^{(l)}(c)$ if $n-l$ is even. Hence, for $n > k \geqslant l$

$$q_{n,k} = \begin{cases} -\dfrac{2P_n^{(l)}(c)P_k^{(l)}(c)}{\langle \sigma, P_k^2(x)\rangle} \neq 0, & \text{if } n-k \text{ is even,} \\ 0, & \text{if } n-k \text{ is odd,} \end{cases} \tag{37}$$

and

$$q_{n,n} = 2K_{n-1}^{(l,l)}(c,c) \neq 0. \tag{38}$$

It easily follows from (37) and (38) that $\{P_n^\mu(x)\}_{n=0}^\infty$ is a symmetric linear perturbation of the class $I$ of $\{P_n(x)\}_{n=0}^\infty$ and in a similar way as in the preceding cases we can show that (29) and (30) are satisfied.

## 5. Conclusions

In the papers [4–6, 8, 11] differential and difference operators (in some cases of finite order, in some other cases of infinite order) are constructed having certain families of orthogonal polynomials as eigenfunctions. All these orthogonal polynomials are linear perturbations of the classical orthogonal polynomials. The classical orthogonal polynomials are eigenfunctions of a differential or difference operator $L$ of the second order with eigenvalues $\{\lambda_n\}_{n=0}^\infty$. In the papers mentioned above tedious proofs were needed to show the existence of an operator of the form $L+\mu A$ having the linear perturbations of the classical orthogonal polynomials as eigenfunctions with eigenvalues of the form $\{\lambda_n + \mu\alpha_n\}_{n=0}^\infty$. By the results in this paper in all these cases and in many more such proofs have become superfluous.

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