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# Simple Flat Extensions*

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Our aim here is to give a structure theorem for flat extensions of a commutative noetherian ring $R$—that is, those $R$-algebras which are flat when viewed as $R$-modules—which are obtained, essentially, by adjoining a single element to $R$. Such an extension $S$ is best described by an exact sequence of $R$-homomorphisms

$$(*) \qquad 0 \to I \to R[X] \to S \to 0;$$

$S$ will be characterized in terms of the ideal $I$ of the polynomial ring $R[X]$.

Here is the main result.

STRUCTURE THEOREM. *$S$ as in (*) is a flat extension of $R$ if and only if $I$ is a projective ideal of $R[X]$, and the ideal of $R$ generated by the coefficients of the polynomials in $I$ [the so-called content of $I$, notation : $c(I)$] is generated by an idempotent element of $R$. Moreover, if $c(I) = R$, then $S$ is $R$-projective if and only if $S$ is integral over $R$.*

The proof will only use basic commutative algebra and is self-contained except for a light invocation of [1].

*Proof of Theorem.* Assume first that $S$ is $R$-flat and that $I \neq (0)$. With $J = c(I)$ tensor (*) with $R/J$ to get

$$0 \to I/JI \to R/J[X] \to S/JS \to 0$$

and $I = JI$ as $I \subseteq JR[X]$, by the definition of content above. In particular the last equality says that $J = J^2$ which implies, as it is well known, $J = Re$ for some idempotent $e$. Using the decomposition $R = Re \oplus R(1 - e)$, we reduce the question of projectivity of $I$ to the case when $c(I) = R$. Let $M$ be

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a prime ideal in $R[X]$ and $P = M \cap R$ its projection in $R$. By localizing at $P$ we may assume that $R$ is local and, without danger of confusion still, denote by $P$ its maximal ideal. Tensoring (*) with $R/P$ we get

$$(**) \qquad 0 \to I/PI \to R/P[X] \to S/PS \to 0$$

and $I/PI$ is generated by one element as an $R[X]$-module, being an ideal of the principal ideal domain $R/P[X]$. This is clearly enough to ensure that $I_M$ is a principal ideal of $R[X]_M$ ; thus $I$ is $R[X]$-projective.

For the converse, as above, we first reduce to the case when $c(I) = R$. We need the following lemmas which can be traced to Bourbaki [1].

LEMMA 1. *Let $\phi : R \to S$ be a homomorphism of commutative rings and $E$ an $S$-module. Then $E$, via $\phi$, is flat over $R$ if and only if, for all prime ideals $M \subset S$, $E_M$ is flat over $R_P$, $P = \phi^{-1}(M)$.*

LEMMA 2. *Let $R$ and $S$ be two noetherian local rings, with maximal ideals $\underline{m}, \underline{n}$; $k = R/\underline{m}$. If $\phi : R \to S$ is a local homomorphism (i.e., $\phi(\underline{m}) \subseteq \underline{n}$) and $E$ is a finitely generated $S$-module then $E$, via $\phi$, is $R$-flat if and only if* $\operatorname{Tor}_1^R(k, E) = 0$.

To complete the proof let $M$ (for Lemma 1) be a prime ideal of $S$, $N$ its inverse image in $R[X]$, and $P = N \cap R$. Localizing $R[X]$ at $N$ we get that $I_N = (f)$ where $f$ can be taken as a polynomial where not all of its coefficients are in $P$. Thus the monomorphism of $R[X]_N$ induced by multiplication by $f$ remains monic when one takes the coefficients mod $P$, that is, when we tensor the exact sequence

$$0 \to R[X]_N \xrightarrow{f} R[X]_N \to S_M \to 0$$

by $R/P$. Thus $\operatorname{Tor}_1^R(R/P, S_M) = 0$ and by Lemma 2 $S$ is $R$-flat.

Now consider the second statement of the theorem. If $S$ is $R$-projective, (**) says that $S/PS$ is finitely generated as an $R$-module for each maximal ideal $P$. Writing $S \oplus E = \oplus \sum R_\alpha$ (free direct sum of copies $R_\alpha$ of $R$), to show that $S$ is finitely generated it is enough to prove that $S$ only has coordinates in finitely many $R_\alpha$'s. Now pass to the total ring of quotients of $R$, which is a semilocal ring and the statement follows (see [2] for a treatment of these questions). The converse is well-known and the proof of the theorem is complete.

*Remark.* It would be interesting to have a similar result for non-noetherian rings, at least for those commutative rings for which finitely generated flat modules are projective.

## References

1. N. Bourbaki, "Algèbre Commutative," Chap. III, Hermann, Paris, 1961.
2. W. V. Vasconcelos, Projective modules of finite rank, *Proc. Amer. Math. Soc.* 22 (1969), 430–433.