On semi-hereditary orders integral over a commutative valuation ring

Hidetoshi Marubayashi

Department of Mathematics, Naruto University of Education, Naruto, Japan

Received 13 February 2001
Available online 10 April 2004
Communicated by Susan Montgomery

Abstract

In this paper, we study semi-hereditary \( V \)-orders in a simple Artinian ring \( Q \) with finite dimension over its center, where \( V \) is a commutative valuation ring. Let \( S \) be a minimal \( V \)-overring of a semi-hereditary \( V \)-order \( R \). In Section 1, we investigate some relations between all maximal ideals of \( S \) and \( R \), and characterize the commutativities of idempotent maximal ideals of \( R \) in terms of orders of ideals. In Section 2, we show that there is a bijection between the set of all \( V \)-overrings of \( R \) and the set of all idempotent ideals which are finitely generated as left ideals. Any element in the latter set is characterized by four different types of cycles. In Section 3, we discuss the principalness of the Jacobson radical \( J(R) \). Some results in Sections 1–3 are used to derive the exact numbers of all semi-hereditary maximal \( V \)-orders containing \( R \) and of all \( V \)-overrings of \( R \), and to study the nilpotency of \( J(R) \) modulo \( J(V)R \). Some invariant properties of semi-hereditary \( V \)-orders are also given.

© 2004 Elsevier Inc. All rights reserved.

Introduction

This paper is concerned with semi-hereditary orders in a simple Artinian ring. Let \( Q \) be a simple Artinian ring with finite dimension over its center \( F \) and let \( R \) be a subring of \( Q \) with its center \( V \). We say that \( R \) is an order in \( Q \) if \( Q \) is the quotient ring of \( R \), i.e., \( Q = F \cdot R \) and \( F \) is the quotient field of \( V \). If every element in \( R \) is integral over \( V \), then we say that \( R \) is a \( V \)-order. We concentrate ourselves to the local theory, namely, \( V \) is a valuation ring of \( F \), though some of the results in this paper may be globalized. If
V is a discrete rank-one valuation ring, then semi-hereditary V-orders are nothing but hereditary orders. As we can see from [3,4], the idealizer theory and maximal ideals play important roles to study hereditary orders, which are still useful methods to study semi-hereditary orders. However, if we study semi-hereditary orders from the viewpoint of hereditary orders, then we encounter several obstructions such as: maximal ideals are not necessarily finitely generated; semi-hereditary maximal V-orders are not necessarily principal ideal rings (even being Bezout, i.e., any finitely generated one sided ideal is principal); the Jacobson radicals of semi-hereditary orders are not necessarily invertible, and so on. Much differences between hereditary and semi-hereditary orders occur in the case the Jacobson radical \( J(V) \) of \( V \) is idempotent. In the case \( J(V) \supset J(V)^2 \), we know that there are similarities in some stages between hereditary and semi-hereditary orders. Note that \( J(V) \) is principal if \( J(V) \supset J(V)^2 \).

In Section 1, we show that there is a bijection between the set of all \( V \)-overrings of a semi-hereditary \( V \)-order \( R \) and the set \( I(R) \) of all idempotent ideals which are finitely generated as left ideals (Proposition 1.2). Let \( S \) be a semi-hereditary \( V \)-order containing \( R \). Then we show that \( S/R \) is an Artinian \( R \)-module and so we can use the principle of mathematical induction to study any \( V \)-overrings of \( R \), i.e., any semi-hereditary \( V \)-order containing \( R \). This is the reason why we only study, in Section 1, some relations between maximal ideals of a minimal \( V \)-overrings of \( R \) and maximal ideals of \( R \) (Theorem 1.11).

Section 1 contains a characterization of the commutativities of idempotent maximal ideals in terms of left and right orders of ideals (Theorem 1.14).

In Section 2, we describe all elements in \( I(R) \) in terms of four different types of cycles—namely, the first and the second type of cycles, the open cycles, the left and right open cycles (Theorems 2.5)—and give the exact number of \( I(R) \) (Theorem 2.7). Section 3 is concerned with the principalness of the Jacobson radical \( J(R) \) of \( R \) (Theorem 3.3). Some results of Sections 1–3 are applied in Section 4 to give the exact number of all \( V \)-overrings of \( R \) (in particular, of all semi-hereditary maximal \( V \)-orders containing \( R \)) (Theorem 4.3) and to study the nilpotency of \( J(R) \) modulo \( J(V)R \). Furthermore, let \( J(R) = J_1(R) \cap \cdots \cap J_k(R) \), where each \( J_i(R) \) is an intersection of first (second) type of cycle. Then we show that \( J(V)R \) is a finite product of \( J_i(R) \) (1 ≤ i ≤ k) (Theorems 4.8 and 4.10). Let \( T \) be a semi-hereditary maximal \( V \)-order containing \( R \). Then \( J(T) \subseteq J(R) \) by [6, Theorem 1.5] and Theorem 4.7 is concerned with the nilpotency of \( J(R) \) modulo \( J(T) \). Section 4 also contains some invariant properties of semi-hereditary \( V \)-orders (Theorem 4.1).

We assume that the reader is familiar with elementary properties of orders, and we use the notation and some results on semi-hereditary orders in the book [10].

1. Minimal \( V \)-overrings of a semi-hereditary \( V \)-order

Throughout this paper, \( Q \) will be a simple Artinian ring with finite dimension over its center \( F \) and let \( R \) be a semi-hereditary order with its center \( V \) unless otherwise specified, i.e., \( F \cdot R = Q \). \( F \) is the quotient field of \( V \) and any finitely generated one-sided ideal of \( R \) is projective. We say that an order \( R \) is a \( V \)-order in \( Q \) if every element in \( R \) is integral over \( V \). We also assume, throughout this paper, that \( R \) is a semi-hereditary
$V$-order in $Q$ and $V$ is a valuation ring. Let $A$ and $B$ be subsets of $Q$. We will use the following notation: $(A : B)_l = \{q \in Q \mid qB \subseteq A\}$, $(A : B)_r = \{q \in Q \mid Bq \subseteq A\}$. For an $R$-ideal $A$ (see [10, pp. 2–3] for the definition), we will denote $O_l(A) = (A : A)_l$, a left order of $A$ and $O_r(A) = (A : A)_r$, a right order of $A$. We say that $A$ is invertible if $(R : A)_l A = R = (R : A)_r$. If $A$ is invertible, then $(R : A)_l = (R : A)_r = A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.

Let $S$ be a $V$-order containing $R$. Then we say that $S$ is a $V$-overring of $R$. Note that any overring of $R$ is again semi-hereditary (see [10, Remark, p. 9]). The aim of this section is to study some relations between the set of all maximal ideals of any $V$-overring of $R$ and the set of all maximal ideals of $R$.

If $T$ is an order in $Q$, then we denote by $J(T)$ its Jacobson radical, $Z(T)$ the center of $T$. Let $M$ be a vector space over a field $K$. Then we denote by $[M : K]$ the dimension of $M$ over $K$.

The following is essentially due to [10, Lemma 20.19], in which there is a careless mistake. So we give a complete proof of the lemma.

**Lemma 1.1.** Let $M \subseteq N$ be right $R$-submodules of $Q$ such that $M J(V)^n \subseteq N$ for some natural number $n$. Then $M/N$ is an Artinian right $R$-module. Furthermore, if $N$ is finitely generated, then so is $M$.

**Proof.** Set $M_i = M J(V)^i + N$, a right $R$-submodule with $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = N$ and $M_i J(V) \subseteq M_{i+1}$ for all $i \ (0 \leq i \leq n - 1)$. Since $[M_i : V] \leq [Q : F]$ by [10, Lemma 7.2], where $M_i = M_i / M_{i+1}$ and $V = V/J(V)$, we have $M/N$ is an Artinian right $R$-module. The last assertion is clear. □

It is easily seen that there is a maximal $V$-overring $T$ of $R$ by Zorn’s lemma and $J(V)T \subseteq J(T) \subseteq J(R)$ by [6, Theorem 1.5]. So $[T/R : V] \leq [Q : F]$ by [10, Lemma 7.2]. Thus a $V$-overring’s series between $T$ and $R$, such as $T = T_0 \supseteq \cdots \supseteq T_n = R$, where $T_i$ is a $V$-overring, is finite and $n \leq [Q : F]$. We denote by $l(R)$ the maximal length of $V$-overring’s series of $R$, which is called the length of $V$-overrings of $R$ and, throughout this paper, we often use the induction on it. To study any $V$-overring of $R$, it essentially suffices to study a minimal $V$-overring $S$ of $R$ by induction on $l(R)$, i.e., there are no overrings of $R$ properly contained in $S$. Similarly, there is a minimal semi-hereditary $V$-order which is contained in $R$. We see from [14, 13.8.14] that $M \cap V = J(V)$ for any maximal ideal $M$ of $R$ and so $J(R) \cap V = J(V)$.

Set $I(R) = \{A \mid A$ is a proper idempotent ideal and is finitely generated as a left ideal$\}$ and $O(R) = \{S \mid S$ is a $V$-overring of $R$ with $S \neq R\}$, though we consider $R$ as a $V$-overring of $R$. The following is crucial in study of $V$-overrings of $R$.

**Proposition 1.2.** There is a bijection between $I(R)$ and $O(R)$ given by $A \mapsto O_r(A)$ and $S \mapsto (R : S)$, where $A \in I(R)$ and $S \in O(R)$. In particular, any $V$-order containing $R$ is finitely generated as a left and right $R$-module.

**Proof.** By [10, Lemma 20.20], it suffices to prove that $O(R) = \{S \mid S$ is an overring of $R$ such that $S$ is finitely generated as a right $R$-module$\}$. Let $S$ be a $V$-overring of $R$. Then
\[ J(V)S \subseteq J(S) \subseteq J(R) \] by [6, Theorem 1.5]. So \( S \) is finitely generated as a left and right \( R \)-module by Lemma 1.1. Conversely, let \( S \) be an overring of \( R \) such that \( S \) is finitely generated as a right \( R \)-module. Then \( S \oplus S' = \bigoplus R \), finite copies of \( R \), where \( S' \) is a right \( R \)-module. Let \( e \) be the projection from \( S \oplus S' \) to \( S \). Then \( \text{End}(S_R) \cong eM_n(R)e \) and \( S \) is embeddable to \( \text{End}(S_R) \) naturally as a ring, where \( M_n(R) \) is the \( n \times n \) matrix ring over \( R \). Hence \( S \) is integral over \( V \), i.e., \( S \) is a \( V \)-order in \( Q \), because \( M_n(R) \) is integral over \( V \) by [14, (13.8.12)]. Since \( V = Z(R) \subseteq Z(S) \) and \( S \) is integral over \( V \), we have \( V = Z(S) \). Now the last assertion is clear from the proof.

**Corollary 1.3.** Let \( A \) be a proper ideal of \( R \).

1. If \( J(V) \supset J(V)^2 \) and \( A \supset J(V)^n \) for some natural number \( n \), then \( A \) is finitely generated as left and right ideals.
2. If \( A \) is finitely generated as a left ideal and \( A = A^2 \), then \( A \supset J(V) \) and \( R/A \) is Artinian as both left and right \( R \)-modules.

**Proof.** (1) If \( J(V) \supset J(V)^2 \), then \( J(V) \) is principal. So the statement follows from Lemma 1.1.

(2) \( A = (R : S)_l \) for some \( S \in \mathcal{O}(R) \) and so \( SJ(V) \subseteq R \) by Proposition 1.2 and [6, Theorem 1.5]. Thus \( J(V) \subseteq A \) and \( R/A \) is Artinian by Lemma 1.1.

**Lemma 1.4.** Let \( A \) and \( B \) be proper ideals of \( R \) such that \( A \) and \( B \) are both finitely generated as left ideals.

1. \( AB \) is finitely generated as left ideal.
2. Assume that \( A = A^2 \) and \( B = B^2 \). Then
   (a) \( A = B \) if and only if \( O_r(A) = O_r(B) \);
   (b) \( O_r(A) \supset R \) and \( O_r(A) \neq O_l(A) \).

**Proof.** (1) Write \( A = Ra_1 + \cdots + Ra_n \) and \( B = Rb_1 + \cdots + Rb_m \) for some \( a_i \in A \) and \( b_j \in B \). Then \( AB = Ab_1 + \cdots + Ab_m = \sum_{i,j} Ra_ib_j \), finitely generated as a left ideal.

(2) (a) This follows from Proposition 1.2.

(b) That \( O_r(A) \supset R \) follows from Proposition 1.2. Since \( A = A^2 \), \( (R : A)_l = O_r(A) \).

Thus, if \( O_r(A) = O_l(A) \), then \( A \) is finitely generated as a right ideal by Proposition 1.2, and \( A = AO_r(A) = AO_l(A) = A(R : A)_l = O_l(A) \) by [10, Lemma 1.5], a contradiction.

In the case of idempotent maximal ideals, we have

**Lemma 1.5.** Let \( M \) be an idempotent maximal ideal of \( R \). Then \( M \) is finitely generated as left ideal if and only if \( O_r(M) \supset R \).

**Proof.** If \( M \) is finitely generated as left ideal, then \( O_r(M) \supset R \) by Lemma 1.4. Conversely, if \( S = O_r(M) \supset R \), then \( S \) is a \( V \)-order in \( Q \) by [6, Corollary 1.3] with \( (R : S)_l = M \). Hence \( M \) is finitely generated as a left ideal by Proposition 1.2.
Let $M$ be a right $R$-module. Then we denote by $r_R(M) = \{ r \in R \mid Mr = 0 \}$ the right annihilator of $M$ and by $\dim_R M$ the Goldie dimension of $M$ or $\dim M$ if there are no confusion.

Let $S$ be an overring of $R$. Then we know from Proposition 1.2 that $S$ is a minimal $V$-overring of $R$ if and only if $O_r(K) = S = O_l(L)$ for some idempotent maximal ideals $K$ and $L$ such that $K$ is finitely generated as left ideal and $L$ is finitely generated as a right ideal. Furthermore, if $S$ is a $V$-overring of $R$, then $S$ is finitely generated as both left and right $R$-modules; so it is projective. We use these facts throughout the paper.

**Lemma 1.6.** Let $K$ and $L$ be distinct idempotent maximal ideals of $R$ such that $S = O_r(K) = O_l(L) \supset R$ and let $M$ be a maximal ideal of $R$ different from $K$ and $L$.

1. $M' = MS = SM$ is a maximal ideal of $S$ with $M' \cap R = M$ and $\dim R/M = \dim S/M'$.
2. Let $U$ be a simple right $R$-module with $UM = 0$. Then $U$ is a simple right $S$-module with $UM' = 0$.
3. $O_r(M') = O_r(M)S$, $O_l(M') = SO_l(M)$, $(S : M')_r = (R : M)_rS$ and $(S : M')_l = S(R : M)_l$.
4. $M = M^2$ if and only if $M' = M^{\prime 2}$.

**Proof.** (1) Assume that $MS = S$ and write $1 = \sum m_is_i$, where $m_i \in M$ and $s_i \in S$. Then $L = (\sum m_is_i)L \subseteq \sum m_iL \subseteq M$, a contradiction. Hence $S \supset MS$ and $MS \cap R = M$. Let $M_1$ be maximal right ideal of $R$ with $M_1 \supseteq M$. Then it follows that $M_1S$ is a maximal right ideal of $S$ with $M_1S \cap R = M_1$, because $LS = S$ [10, Lemma 1.5] and $SL = L$. Note that $M_1S + K = S$, otherwise $K \subseteq M_1S \cap R = M_1$ and so $R = K + M \subseteq M_1$, a contradiction. Thus $S = M_1S + K = M_1S + R$ and so $R/M_1 \cong S/M_1S$. Hence $M' = r_S(S/M_1S)$ is a maximal ideal of $S$ with $M = M' \cap R$. Thus $MS \subseteq M'$. Now, let $U = R/M_1$, a simple right $R$-module with $UM = 0$ and $R/M \cong U \oplus \cdots \oplus U$, $m$ copies of $U$, where $m = \dim R/M$. Then $S/MS \cong R/M \otimes_R S \cong U \oplus \cdots \oplus U$, $m$ copies of $U$, since $U \otimes S \cong U$, i.e., $S/MS$ is a semi-simple right $R$-module with $(S/MS)M' = 0$ and so $MS \subseteq M'$. Hence $M' = MS$ and similarly $M' = SM$. It is now clear from the proof that $\dim R/M = \dim S/M'$.

(2) is clear from the proof of (1).

(3) It is clear that $O_r(M') \supseteq O_r(M)S$. To prove the converse inclusion, let $x \in O_r(M')$. Then $Mx \subseteq M'x \subseteq MS$ and so $MxL \subseteq MLS = ML \subseteq M$. Thus $xL \subseteq O_r(M)$ and $xL \subseteq O_r(M)S$ since $LS = S$. Hence $O_r(M') \supseteq O_r(M)S$. The other statements are also proved in a similar way.

(4) If $M = M^2$, then $M' = M^{\prime 2}$ by (1). Conversely, assume that $M' = M^{\prime 2}$ and that $M \supset M^2$. Then $M/M^2$ is copies of $U$ and so $M/M^2 \otimes S \neq 0$. Hence $M' \supset M^{\prime 2}$, a contradiction. Hence $M = M^2$ follows. \qed

**Lemma 1.7.** Let $M_1, \ldots, M_n$ be distinct maximal ideals of $R$ and let $A = M_1 \cap \cdots \cap M_n$. Then $A$ is idempotent if and only if $M_i$ is idempotent and $M_iM_j = M_jM_i$ for all $i, j$. In this case $A = M_1 \cdots M_n$. 


Proof. First assume that \( A \) is idempotent. Set \( A_{ij} = M_i \cap M_j \). Then \( \overline{A}_{ij} = A_{ij}/A \) is idempotent, because \( \overline{R} \) is a semi-simple Artinian ring. Thus \( A_{ij} = A_{ij}^2 + A = A_{ij}^2 \) since \( A = A^2 \). This implies that \( M_iM_j = A_{ij} = M_jM_i \), because \( A_{ij} = A_{ij}^2 \subseteq M_iM_j \). It is now clear that \( M_i \) are all idempotent. Conversely assume that \( M_i = M_i^2 \) and \( M_iM_j = M_jM_i \) for all \( i, j \). If \( n = 2 \), then \( A = M_1 \cap M_2 = M_1M_2 = M_2M_1 \) and so \( A \) is idempotent. By induction on \( n \), we may assume that \( B = M_2 \cap \cdots \cap M_n = M_2 \cdots M_n \) and is idempotent. Hence \( A = M_1 \cap B = (M_1 \cap B)(M_1 + B) \subseteq BM_1 + M_1B = M_1 \cdots M_n \subseteq A \). So \( A = M_1 \cdots M_n \) and is idempotent. \( \square \)

It is well known that any maximal ideal of hereditary Noetherian prime rings is either idempotent or invertible (see [3, Proposition 2.2]). The following shows that this property holds in the case of semi-hereditary orders.

**Proposition 1.8.** Any maximal ideal of \( R \) is either invertible or idempotent. In particular, if \( J(V) = J(V)^2 \) then any maximal ideal is idempotent.

**Proof.** Case 1. \( J(V) \supset J(V)^2 \). By Corollary 1.3, any maximal ideal is projective as both right and left \( R \)-modules. So the statement is proved by the similar way as in [3, Proposition 2.2].

Case 2. \( J(V) = J(V)^2 \). First assume that \( R \) is a semi-hereditary maximal \( V \)-order, then \( J(R) = J(V)R \) and is idempotent by [8, Corollary 1]. Thus any maximal ideal is idempotent by Lemma 1.7. Now assume that any maximal ideal of a semi-hereditary \( V \)-order strictly containing \( R \) is idempotent and that \( R \) is not a maximal \( V \)-order. By Proposition 1.2 and its right version, we can find idempotent maximal ideals \( K \) and \( L \) of \( R \) such that \( O_r(K) = S = O_r(L) \supset R \). Let \( M \) be any maximal ideal of \( R \) which is different from \( K \) and \( L \). Then \( M' = MS = SM \) is a maximal ideal of \( S \) with \( M' \cap R = M \) by Lemma 1.6. By assumption, \( M' \) is idempotent and so is \( M \) by Lemma 1.6. This completes the proof. \( \square \)

**Remark.** It follows from Proposition 1.8 that \( J(V) \supset J(V)^2 \) if there is a maximal ideal \( M \) of \( R \) with \( M \supseteq M^2 \); equivalently, \( M \) is invertible.

**Lemma 1.9.** Let \( M \) and \( N \) be distinct idempotent maximal ideals of \( R \). Suppose that \( M \) is finitely generated as left ideal.

1. \( O_r(M) \neq O_r(N) \) if and only if \( M \cap N = NM \).
2. If \( O_r(M) \neq O_r(N) \) and \( O_r(N) = O_r(M) \supset R \), then \( M \cap N \supset (M \cap N)^2 = MN \), which is idempotent.

**Proof.** (1) Set \( S = O_r(M) \), a \( V \)-order by Proposition 1.2. First assume that \( S \neq O_r(N) \). Then, by Proposition 1.2, \( S = O_r(L) \) for some idempotent maximal ideal \( L \) different from \( N \) such that \( L \) is finitely generated as a right ideal. Thus \( N' = SN = NS \) is a maximal ideal of \( S \) by Lemma 1.6. Assume that \( M \cap N \supset NM \). Since \( N(N \cap M) \subseteq NM \), it follows that \( (M \cap N)/NM \) is copies of the simple left \( R \)-module \( U \) with \( NU = 0 \). Thus \( S \otimes (M \cap N)/NM \neq 0 \) by Lemma 1.6(2). However, \( S \otimes (N \cap M) \cong S(N \cap M) = \)
\[ N' \cap SM = N' \cap S = N' \text{ and } S \otimes NM \cong SNM = N', \text{ because } SM = S. \text{ Thus } S \otimes (N \cap M)/NM = 0, \text{ a contradiction. Hence } M \cap N = NM \text{ follows. Conversely assume that } M \cap N = NM \text{ and } O_r(M) = O_l(N). \text{ Then } M \cap N = NM \text{ is an ideal of } S. \text{ Thus we have } M \supset M \cap N = S(M \cap N) = S \cap N = N, \text{ a contradiction. Hence } O_r(M) \neq O_l(N) \text{ follows.}

(2) By (1), we have \((MN)^2 = MNMN \supseteq M^2N^2 = MN\), showing that \(MN\) is idempotent and \(MN \supseteq (M \cap N)^2 \supseteq (MN)^2 = MN\). Hence \((M \cap N)^2 = MN\). Let \(T = O_r(N) = O_l(M)\) and assume that \(M \cap N = MN\). Then \(MN = TMNT = T(M \cap N)T = MT \cap TN = T\), a contradiction. Hence \(M \cap N \supset MN\) follows.

The following remark is a right version of Lemma 1.9.

**Remark.** Let \(M\) and \(N\) be distinct idempotent maximal ideals of \(R\). Suppose that \(N\) is finitely generated as a right ideal. Then

(1) \(O_r(M) \neq O_l(N)\) if and only if \(M \cap N = NM\).

(2) If \(O_r(M) \neq O_l(N)\) and \(O_r(N) = O_l(M) \supset R\), then \(M \cap N \supset (M \cap N)^2 = MN\) and \(MN\) is idempotent.

Let \(K\) be a semi-maximal right ideal of a semi-hereditary \(V\)-order \(S\) with \(SK = S\). Then 
\(R = I_S(K) = \{s \in S \mid sK \subseteq K\}\), the *idealizer of \(K\) in \(S\)* is again a semi-hereditary \(V\)-order (see [10, Theorem 20.17]).

The following is a necessary and sufficient conditions for \(S\) to be a minimal \(V\)-overring of \(R\) and the proof is more or less known in hereditary case. But we will give a complete proof.

**Lemma 1.10.** Let \(S\) be a semi-hereditary \(V\)-order in \(Q\) and let \(K = \bigcap K_i\) be a semi-maximal right ideal of \(S\) with \(SK = S\), where \(K_i\) are maximal right ideals of \(S\). Set \(R = I_S(K)\). The following are equivalent:

(1) \(S\) is a minimal \(V\)-overring of \(R\).

(2) \(K\) is a maximal ideal of \(R\).

(3) \(S/K_i \cong S/K_j\) for all \(i, j\).

**Proof.** First note that \(S = O_r(K)\) and \(K = K^2\) by [10, Theorem 20.17] and \(K = (R : S)_l\) follows, because \(SK = S\). Hence \(K\) is finitely generated as left ideal by Proposition 1.2.

(1) \(\Rightarrow\) (2). This follows from [10, Lemma 20.21].

(2) \(\Rightarrow\) (3). By [10, Theorem 20.3], \(S/K_i\) is a uniserial right \(R\)-module of length 2, namely, \(S \supset R + K_i \supset K_i\). Since \(K\) is a maximal ideal of \(R\), it follows that \(r_R((R + K_i)/K_i) = K\) and that \((R + K_i)/K_i \cong (R + K_j)/K_j\) for all \(i, j\). Hence

\[ S/K_i \cong (R + K_i)/K_i \otimes S \cong (R + K_j)/K_j \otimes S \cong S/K_j. \]
(3) ⇒ (1). By [10, Proposition 20.1], \( R/K \cong \text{End}_S(S/K) \), a simple Artinian ring by assumption. So \( K \) is a maximal ideal of \( R \). Hence \( S \) is a minimal \( V \)-overring of \( R \) by Proposition 1.2. □

We are now in a position to prove the following theorem which provides us with necessary information about maximal ideals of a minimal \( V \)-overring of \( R \) and of \( R \) itself.

**Theorem 1.11.** Let \( K \) and \( L \) be distinct idempotent maximal ideals of a semi-hereditary \( V \)-order \( R \) in \( Q \) such that \( O_r(K) = S = O_l(L) \supset R \) and let \( M \) be a maximal ideal of \( R \) different from \( K \) and \( L \). The following hold:

1. \( LK \) is a maximal ideal of \( S \) with \( \dim_S S/LK = \dim R/K + \dim R/L \).
2. \( M' = MS = SM \) and is a maximal ideal of \( S \) with \( \dim_S S/M' = \dim R/M \).
3. \( \{LK, M' = MS | M \text{ ranges over all maximal ideals of } R \text{ different from } K \text{ and } L \} \) is the set of all maximal ideals of \( S \).
4. (a) \( M = M^2 \) if and only if \( M' = M^2 \).
   (b) \( M \) is invertible if and only if \( M' \) is invertible.
   (c) \( M \) is finitely generated as left ideal if and only if \( M' \) is finitely generated as left ideal of \( S \).
   (d) \( O_r(M) = R \) if and only if \( O_r(M') = S \).
5. Let \( N \) be a maximal ideal of \( R \) different from \( K, L, \) and \( M \). Then \( O_r(M) = O_l(N) \) if and only if \( O_r(M') = O_l(N') \), where \( N' = NS = SN \).
6. \( K \) is finitely generated as a right ideal of \( R \) if and only if \( LK \) is a finitely generated as a right ideal of \( S \).
7. \( L \) is finitely generated as a left ideal if and only if \( LK \) is finitely generated as a left ideal of \( S \).
8. \( LK \) is invertible if and only if \( O_r(L) = O_l(K) \supset R \).
9. Assume that \( M \) and \( LK \) are idempotent.
   (a) \( O_r(M) = O_l(K) \) if and only if \( O_r(M') = O_l(LK) \).
   (b) \( O_r(L) = O_l(M) \) if and only if \( O_r(LK) = O_l(M') \).

**Proof.** (1) By [10, Theorem 20.22] and Lemma 1.5, \( R = I_S(K) \) and \( S/K \) is a semi-simple right \( S \)-module with \( SK = S \). So we can write \( K = \bigcap K_i \), where \( K_i \) are maximal right ideals of \( S \) such that \( S/K_i \cong S/K_j \) for all \( i, j \) and \( r_R((R + K_i)/K_j) = K \) by Lemma 1.10 and its proof. Since \( R + K_i \supset R \supset S = SL \), we have \( r_R(S/(R + K_i)) = L \), because \( S/(R + K_i) \) is a simple right \( R \)-module. Since \( KS = K \) and \( SL = L \), \( LK \) is an ideal of \( S \). Set \( M^* = r_S(S/K_1) \), a maximal ideal of \( S \) with \( M^* \supset LK \), because \( SLK \subseteq (R + K_1)K \subseteq K_1 \). We claim that \( KM^* \subseteq LK \). If this is true, then \( M^* = SM^* \subseteq SLK = LK \) and so \( M^* = LK \), a maximal ideal of \( S \). To prove the claim, set \( U = S/(R + K_1) \). Then \( R/L \cong U \oplus \cdots \oplus U \), finite copies of \( U \) and we have the formulas:

\[
K/LK \cong R/L \otimes_R K \cong (U \oplus \cdots \oplus U) \otimes_R K \cong S/K_1 \oplus \cdots \oplus S/K_1, \quad (\ast)
\]

because \( (R + K_1)K = K + K_1SK = K + K_1 = K_1 \). Hence \( KM^* \subseteq LK \) since \( M^* = r_S(S/K_1) \).
Let $k = \dim R/K$ and $l = \dim R/L$. Then we see from the formulas (⋆) that $l = \dim S(K/LK)$. From $R/K \cong (R + K_1)/K_1 \oplus \cdots \oplus (R + K_1)/K_1$, $k$ copies, we have $S/K \cong R/K \otimes S \cong S/K_1 \oplus \cdots \oplus S/K_1$, $k$ copies of $S/K_1$. This means that $k = \dim_S S/K$. Hence $\dim_S S/LK = \dim_S S/K + \dim_S K/LK = k + l = \dim R/K + \dim R/L$.

(2) This is a part of Lemma 1.6.

(3) Let $N'$ be any maximal ideal of $S$ and let $W$ be a simple right $S$-module with $WN' = 0$. If $\text{Hom}_S(S/K, W) = 0$, then $W$ is a simple right $R$-module by [10, Theorem 20.3] and so $N = r_R(W)$ is a maximal ideal of $R$. Since $W$ is not isomorphic to $S/(R + K_1)$ and $(R + K_1)/K_1$, $N \neq K$ and $N \neq L$. Thus $NS = SN$ and is a maximal ideal of $S$ by Lemma 1.6. Hence $N' = NS = SN$ follows. If $\text{Hom}_S(S/K, W) \neq 0$, then $W \cong S/K$ and so $N' = LK$ by (1) and its proof. Therefore $\{LK, M' = MS = S \mid M \}$ ranges over all maximal ideals different from $K$ and $L$, is the set of all maximal ideals of $S$.

(4)(a) This is a part of Lemma 1.6.

(b) By Lemma 1.6, $(S : M')_l = S(R : M)_l$ and $(S : M')_r = (R : M)_r S$. Hence $M'$ is invertible if $M$ is invertible. Conversely assume that $M'$ is invertible and $M$ is not invertible. Then $M$ is idempotent by Proposition 1.8 and so is $M'$ by Lemma 1.6, a contradiction. Hence $M$ must be invertible.

(c) Since any invertible ideal is finitely generated, we may assume that $M$ is idempotent by (4)(b). If $M$ is finitely generated as a left ideal, then it is clear that $M' = SM$ is finitely generated as a left ideal of $S$. Assume that $M'$ is finitely generated as a left ideal and that $M$ is not finitely generated as left ideal, then $R = O_r(M)$ by Lemma 1.5 and then, by Lemma 1.6, $O_r(M') = S$, which implies that $M'$ is not finitely generated as left ideal of $S$, a contradiction. Hence $M$ is finitely generated as left ideal.

(d) This follows from Lemmas 1.5, Proposition 1.8 and (a)–(c) (note that $O_r(A) = R = O_l(A)$ if an ideal $A$ is invertible).

(5) Assume that $O_r(M) = O_l(N)$. Then, by Lemma 1.6, $O_r(M') = O_r(M)S = O_l(N)S \subseteq O_l(N')$, since $O_l(N')$ is an $S$-bimodule. Similarly we have $O_r(M') \supseteq O_l(N')$ and hence $O_r(M') = O_l(N')$. Conversely assume that $O_r(M') = O_l(N')$. If $O_r(M') = S$, then $O_r(M) = R = O_l(N)$ by (d) and its left version. If $O_r(M') \supseteq S$, then $M' = M^2$ and $M'$ is finitely generated as left ideal of $S$ by Proposition 1.8 and Lemma 1.5. Thus $M = M^2$ and $M$ is finitely generated as a left ideal. We will show that $O_r(M) \neq O_l(L)$ and $O_r(M) \neq O_l(K)$. If $O_r(M) = O_l(L)$, then $M = K$ by Lemma 1.4. This is a contradiction. If $O_r(M) = O_l(K)$, then we claim that $O_r(M') = O_l(LK)$. If this is true, then $N' = LK$, a contradiction. To prove the claim, let $x \in O_r(M')$, then $MxL \subseteq M'xL \subseteq M'L = ML \subseteq M$ and so $xL \subseteq O_r(M) = O_l(K)$, which implies $xLK \subseteq K$. If $LK$ is invertible, then $x \in (S : LK) = S = O_l(LK)$. If $LK$ is idempotent, then $xLK \subseteq LK \subseteq LK$, which implies $x \in O_l(LK)$. Conversely, let $y \in O_l(LK)$, i.e., $yLK \subseteq LK \subseteq K$. So $yL \subseteq O_l(K) = O_r(M)$ and $yL \subseteq L \subseteq M$. Thus $M'y \subseteq M'yLS = S \subseteq L'$ and so $y \in O_r(M')$, proving the claim. Hence we can find an idempotent maximal ideal $M_1$ different from $K$ and $L$ with $O_r(M) = O_l(M_1)$, which implies $M'_1 = N'$ and hence $M_1 = M'_1 \cap R = N' \cap R = N$ by Lemma 1.6, proving $O_r(M) = O_l(N)$.

(6) Assume that $K$ is finitely generated as a right ideal of $R$, then $LK$ is finitely generated as a right ideal of $R$ by Lemma 1.4 and so $LK$ is finitely generated as a right ideal of $S$. Conversely assume that $LK$ is finitely generated as a right ideal of $S$. Then it
is finitely generated as a right ideal of \( R \), because \( S \) is finitely generated as both left and right \( R \)-modules. Since \( K \supseteq LK \), it follows from Lemma 1.1 that \( K \) is finitely generated as a right ideal of \( R \).

(7) This is similar to (6).

(8) Assume that \( LK \) is invertible. Then \( K \) is finitely generated as a right ideal of \( R \), and \( L \) is finitely generated as left ideal of \( R \). Now assume that \( O_L(L) \neq O_K(K) \). Then \( LK \) is idempotent by Lemma 1.9, a contradiction. Hence \( O_L(L) = O_K(K) \), which properly contains \( R \) by Lemma 1.5. Conversely assume that \( O_L(L) = O_K(K) \supseteq R \). Then \( K \) and \( L \) is a cycle in the sense of [3]. Thus \( I = K \cap L \) is invertible (see the proof of [3, Proposition 2.5]). Since \( LK \) is a maximal ideal of \( S \), it follows from Proposition 1.8 that either \( LK \) is invertible or \( LK \) is idempotent. If \( LK \) is idempotent, then \( LK \supseteq I^2 \supseteq (LK)^2 = LK \) imply that \( I^2 = LK \) and is idempotent, a contradiction. Hence \( LK \) is invertible.

(9)(a) Assume that \( O_L(M) = O_K(K) \). Then we have proved \( O_L(M') = O_L(LK) \) (see the proof of (5)). Conversely assume that \( O_L(M') = O_K(LK) \). If \( O_L(M') = S \), then \( O_L(M) = R = O_L(K) \) by (4)(d), (6), and Lemma 1.5. If \( O_L(M') \supseteq S \), then \( O_L(M) \supseteq R \) and we have \( O_L(M) = O_L(M_1) \) for some idempotent maximal ideal \( M_1 \) of \( R \) which is different from \( L \), otherwise \( M = K \), a contradiction. If \( M_1 \neq K \), then \( O_L(M') = O_L(M_1') \), where \( M_1' = M_1S \) and so \( M_1' \supseteq LK \). Thus \( LK = LK \cap R = M_1' \cap R = M_1 \), a contradiction. Hence \( M_1 = K \) and so \( O_L(M) = O_L(K) \) follows.

(9)(b) This is similar to (9)(a). \( \square \)

In the remainder of this section, we shall study the commutativity of idempotent maximal ideals.

**Lemma 1.12.** Let \( K \) and \( L \) be distinct idempotent maximal ideals such that \( O_L(K) = S = O_L(L) \). Assume that \( LK \) is idempotent. Then \( K \cap L = KL \).

**Proof.** Assume that \( K \cap L \supseteq KL \). Then \( (K \cap L)/KL \cong U \oplus \cdots \oplus U \), finite copies of a simple left \( R \)-module \( U \) with \( KU = 0 \). However, \( L(K \cap L) = LK \cap L^2 \subseteq LK = (LK)^2 = LK \) show that \( LU = 0 \), a contradiction. Hence \( K \cap L = KL \). \( \square \)

**Proposition 1.13.** Let \( R \) be a semihereditary \( V \)-order in \( Q \). Then \( R \) is maximal if and only if \( O_L(M) = R = O_L(M) \) for any maximal ideal \( M \) of \( R \).

**Proof.** Assume that \( R \) is maximal and let \( M \) be a maximal ideal of \( R \). Then \( O_L(M) \) is a \( V \)-order containing \( R \) by [6, Corollary 1.3] and thus \( O_L(M) = R \). Similarly, \( O_L(M) = R \). To prove the if part, assume that \( R \) is not maximal and let \( S \) be a \( V \)-order with \( S \supseteq R \). We may assume that \( S \) is a minimal \( V \)-overring of \( R \). Set \( M = (R : S) \), an idempotent maximal ideal of \( R \) with \( O_L(M) = S \) by Proposition 1.2. This is a contradiction. \( \square \)

The following is another main result in this section, frequently used in Sections 2–4.

**Theorem 1.14.** Let \( M \) and \( N \) be distinct idempotent maximal ideals of a semihereditary \( V \)-order \( R \) in \( Q \).
It follows from Proposition 1.8 and Theorem 1.11 that 

\[ \text{LM} \cap S \text{ ideal of } \text{finite copies of } a \text{ simple left } R \]

[118x261]L(M \cap N = NM \text{ if and only if } O_1(M) \neq O_1(N).]

(3) Assume that \( M \) is finitely generated as both left and right ideals. Then \( MN = M \cap N = NM \) if and only if \( O_r(M) \neq O_r(N) \).

Proof. First note that we may assume that \( \text{J}(V) = J(V)^2 \) in (1) and (2) by Corollary 1.3.

(1) Assume that \( R \) is maximal. Then, by [8, Corollary 1], \( J(V)R = J(R) \). So the assertion follows from Lemma 1.7. Thus we may assume that \( R \) is not maximal and that \( M'N' = M' \cap N' = N'M' \) for any maximal ideals \( M' \) and \( N' \) of any semi-hereditary \( V \)-order strictly containing \( R \) with \( M' \) not finitely generated as left and right ideals.

There are idempotent maximal ideals \( K \) and \( L \) such that \( S = O_r(K) = O_r(L) \supset R \) by Proposition 1.2. We know from the assumption on \( M \) and Lemma 1.5 that \( M \neq K \) and \( M \neq L \). If \( N = K \), then \( M \cap N = MN \) by Lemma 1.9, because \( O_r(N) \supset R = O_r(M) \).

Assume that \( M \cap N \supset NM \). Then \( (M \cap N)/NM \cong U \oplus \cdots \oplus U \), finite copies of a simple left \( R \)-module \( U \) with \( NU = 0 \). By Theorem 1.11, we see that \( M' \) is not finitely generated as left and right ideals and that \( LN \) is a maximal ideal of \( S \), where \( M' = MS = SM \).

Thus we have \( M' \cap LN = LNM' = LNM \) by the assumption and so \( L(M \cap N) = LM \cap LN \subseteq M' \cap LN = LNM \subseteq NM \). This implies that \( LU = 0 \), a contradiction. Hence \( M \cap N = NM \) follows. If \( N = L \), then we have \( MN = M \cap N = NM \) in a similar way by using the remark to Lemma 1.9.

If \( N \neq K, N \neq L \) and assume that \( M \cap N \supset MN \). Then \( (M \cap N)/MN \) is finite copies of a simple right \( R \)-module \( U \) with \( UN = 0 \). It follows from Lemma 1.6(2) that \( U \cong U \oplus S \).

However, by our assumption, \( (M \cap N)S = M' \cap N' = M'N' = MNS \) and thus \( U \oplus S = 0 \), a contradiction. Hence \( M \cap N = MN \) and similarly \( M \cap N = NM \) follows.

(2) Assume that \( M \cap N = NM \), then \( O_r(M) \neq O_r(N) \) by Lemma 1.9. Conversely assume that \( O_r(M) \neq O_r(N) \). We only need to prove that \( M \cap N = MN \) by Lemma 1.9.

Set \( S = O_r(M) \supset R \). Then \( S = O_r(L) \) for some maximal ideal \( L \) of \( R \) with \( N \neq L \). It follows from Proposition 1.8 and Theorem 1.11 that \( LM \) is an idempotent maximal ideal of \( S \) and that \( LM \) is not finitely generated as a right ideal of \( S \). We claim that \( LM \cap N' = LMN' \), where \( N' = NS = SN \). Assuming for the moment that the claim is true, suppose, by way of contradiction, that \( M \cap N \supset MN \). Then \( (M \cap N)/MN \) is finite copies of a simple left \( R \)-module \( U \) with \( MU = 0 \). However, the claim implies that \( L(M \cap N) = LM \cap LN \subseteq LM \cap N' = LMN' = LMN \subseteq MN \), showing that \( LU = 0 \), a contradiction. Hence \( M \cap N = MN \) follows.

To prove the claim, first assume that \( S \) is maximal, i.e., \( 1 = l(R) \), the length of \( V \)-overrings of \( R \), then the claim follows from Lemma 1.7, because \( J(V)S = J(S) \).

Thus we may assume that the statement in (2) holds in any semi-hereditary \( V \)-order strictly containing \( R \). If \( LM \) is not finitely generated as left ideal of \( S \), then the claim follows from (1). Assume that \( LM \) is finitely generated as left ideal of \( S \). If \( O_r(LM) \neq O_r(N') \), then the claim follows from our assumption.

If \( O_r(LM) = O_r(N') \), then the claim follows from Lemma 1.12, because \( N'LM \) is an idempotent maximal ideal of \( O_r(LM) \).
(3) If \( N \) is not finitely generated as either a right ideal or a left ideal, then the statement follows from (1) or (2), its right version and Lemma 1.5. If \( N \) is finitely generated as left and right ideals, then the statement follows from Lemma 1.9. \( \square \)

We close this section with the following remark.

**Remark.**

(1) If \( J(V) \supset J(V)^2 \), then any maximal ideal of \( R \) is finitely generated as both left and right ideals by Corollary 1.3.

(2) If \( J(V) = J(V)^2 \), then any maximal ideal of a semi-hereditary maximal \( V \)-order is not finitely generated as both left and right ideals by Lemma 1.5 and Propositions 1.8, 1.13.

(3) The following is one of the simplest examples of a semi-hereditary \( V \)-order which has a maximal ideal finitely generated as left ideal but not as a right ideal: Let \( S = M_2(V) \), the \( 2 \times 2 \) matrix ring over \( V \) with \( J(V) = J(V)^2 \) and let

\[
R = \begin{pmatrix} V & J(V) \\ V & V \end{pmatrix}
\]

be a semi-hereditary \( V \)-order by [11, Example 3.12]. The maximal ideal

\[
M = \begin{pmatrix} J(V) & J(V) \\ V & V \end{pmatrix}
\]

of \( R \) is finitely generated as left ideal but not as a right ideal, because \( O_r(M) = S \) and \( O_l(M) = R \).

2. Idempotent ideals containing \( J(V)R \)

In this section, we will describe all idempotent ideals of \( R \) containing \( J(V)R \) by using the following five different types of cycles. Let \( M_1, \ldots, M_n \) be distinct idempotent maximal ideals of \( R \) \((n \geq 2)\) satisfying \( O_r(M_i) = O_l(M_{i+1}) \supset R \) \((1 \leq i < n-1)\). Under these conditions, we classify them into the following five types:

(a) \( O_r(M_n) = O_l(M_1) \supset R \), a first-type cycle.
(b) \( O_r(M_n) = R = O_l(M_1) \), a second-type cycle.
(c) \( O_r(M_1) \supset R \), \( O_r(M_{n-1}) \supset R \) and \( O_r(M_n) \neq O_l(M_1) \), an open cycle.
(d) \( O_r(M_1) \supset R = O_l(M_1) \), a right open cycle.
(e) \( O_l(M_1) \supset R = O_r(M_n) \), a left open cycle.

Note that an idempotent maximal ideal \( M \) is finitely generated as a left ideal if and only if \( O_r(M) \supset R \) by Lemma 1.5. If a maximal ideal \( M \) is invertible, then it is considered as a (trivial) first-type cycle, because \( O_r(M) = R = O_l(M) \) and \( M \) is finitely generated as both left and right ideals. If a maximal ideal \( M \) is not finitely generated as both left and right ideals, then it is considered as a (trivial) second-type cycle, because \( M = M^2 \) and \( O_r(M) = R = O_l(M) \) by Lemma 1.5 and Proposition 1.8. In the case of one of (b), (d), and (e), we know that \( J(V) = J(V)^2 \).
Let \( F(R) = \{ A : R\text{-ideals } | J(V)^n R \subseteq A \subseteq (R : J(V)^n R) \} \) for some natural number \( n \) and let \( D(R) = \{ A \in F(R) | A \text{ is invertible} \} \). We start this section with the following proposition, whose proofs are almost similar to the classical theory, i.e., \( V \) is a discrete rank-one valuation ring.

**Proposition 2.1.**

1. Let \( M_1, \ldots, M_m \) be a union of first-type cycles. Then \( I = M_1 \cap \cdots \cap M_m \) is invertible.
2. Any two cycles of first (second) type coincide or disjoint.
3. Let \( A \in F(R) \) with \( A \subseteq R \). Then \( A \) is invertible if and only if \( A \) is an intersection of first-type cycles.
4. Let \( M_1, \ldots, M_n \) be one of cycles (a)–(e) and \( A = M_1 \cap \cdots \cap M_n \). Then \( M_{i+1}A = AM_i \) for all \( i \) (1 \( \leq i \leq n \), \( n+1 = 1 \)).
5. Assume that \( J(V) \supset J(V)^2 \). Then
   a. Any maximal ideal belongs to a cycle of the first type.
   b. \( D(R) \) is a free Abelian group generated by maximal invertible ideals of \( R \).
6. Assume that \( J(V) = J(V)^2 \). Then
   a. Any maximal ideal belongs to a cycle of the second type.
   b. \( D(R) = [R] \).

**Proof.** (1) This is proved as in [3, Proposition 2.5].

(2) This is clear from Lemma 1.4(2)(a).

(3) Let \( A \in F(R) \) with \( A \subseteq R \). Then we see from Lemma 1.1 that \( R/A \) is an Artinian ring, and \( A \) is invertible by either the assumption or (1). Hence, by [3, Proposition 2.4], there are maximal ideals \( M_1, \ldots, M_n \) which make a first-type cycle containing \( A \). We know from (1) that \( B = M_1 \cap \cdots \cap M_n \) is invertible with \( B \supseteq A \). Now the statements easily follow.

(4) By Lemmas 1.9 and 1.12, we have \( M_j \cap M_i = M_j M_i \) for any \( i \) (1 \( \leq i \leq n \)) if \( j \neq i + 1 \) (\( j = n \Rightarrow i + 1 = 1 \)). Hence \( M_{i+1}A = M_{i+1}M_1 \cap \cdots \cap M_{i+1}M_i \cap \cdots \cap M_{i+1}M_n = M_1 \cap \cdots \cap M_{i-1} \cap M_{i+1}M_i \cap M_{i+1} \cap \cdots \cap M_n \), and similarly, we have \( AM_i = M_1 \cap \cdots \cap M_{i-1} \cap M_{i+1}M_i \cap M_{i+1} \cap \cdots \cap M_n \). Hence \( M_{i+1}A = AM_i \) follows.

(5) Let \( A \) be any element of \( D(R) \). Then \( J(V)^n RA \subseteq R \) for some natural number \( n \). It follows, as in [3, Proposition 2.1], that \( J(V)^n RA \) and \( J(V)^n R \) are finite products of maximal invertible ideals. Hence \( D(R) \) is a free Abelian group generated by maximal invertible ideals of \( R \) by (2) and (3) (see the proof of [3, Proposition 2.8]).

(6)(a) This also follows as in [3, Proposition 2.4], since \( R/J(V)R \) is Artinian and \( J(V)R \) is idempotent.

(b) is also trivial, because \( J(V)R \) is idempotent and \( R/J(V)R \) is Artinian. \( \square \)

The following is essentially due to [6, Proposition 1.2].

**Lemma 2.2.** Assume that \( J(V) \supset J(V)^2 \), say, \( J(V) = \pi V \) and let \( x \in Q \). If, for a fixed natural number \( k \), \( \pi^k x^n \) is integral over \( V \) for all natural number \( n \), then \( x \) is integral over \( V \).
Proof. Since \( x \) is algebraic over \( F \), the subring \( F[x] \) of \( Q \), generated by \( F \) and \( x \), is of finite dimension over \( F \). Let \( S = F[x]/J(F[x]) \cong F_1 \oplus F_2 \oplus \cdots \oplus F_n \), where \( F_i \) are fields. Since \( F[x] \) is Artinian, \( J(F[x]) \) is nilpotent and so \( J(F[x]) \cap F = 0 \). This implies that \( F_i \) are all finite field extensions of \( F \). Let \( \bar{x} = \bar{x}_1 + \cdots + \bar{x}_n \) be the image of \( x \) in \( S \). We want to prove that \( \bar{x}_i \) is integral over \( V \). Let \( S_i \) be the integral closure of \( V \) in \( F_i \). Then \( S = S_i \cap \cdots \cap S_n \), where \( S_j \) are all valuation rings of \( F_i \) with \( S_j \cap F = V \) (see [2, (13.3)(b)]). Assume that \( \bar{x}_i \notin S_j \) for some \( j \). Then there is a natural number \( n \) with \( \bar{x}_i^n \in \pi^k S_j \). On the other hand, \( \pi^k x_i^n + 1 \) is integral over \( V \) by the assumption, i.e., \( \pi^k x_i^n + 1 \in W_j \). Thus \( \pi^k W_j = \pi^k W_j \cdot W_j \ni \bar{x}_i^n \pi^k x_i^n + 1 = \pi^k \bar{x}_i \). It follows that \( \bar{x}_i \in W_j \), a contradiction. Hence \( \bar{x}_i \in S_i \) for all \( i \), showing that \( \bar{x} \) is integral over \( V \). So there exists a monic polynomial \( h(t) \in F[t] \) with \( h(\bar{x}) \in J(F[x]) \). But \( J(F[x]) \) is nilpotent. Hence \( h(\bar{x})^l = 0 \) for some natural number \( l \) and so \( \bar{x} \) is integral over \( V \).

Let \( R \) be a maximal \( V \)-order and let \( V \) be a discrete rank-one valuation ring. Then it is well known that \( R \) is a principal ideal ring such that \( J(R) \) is the unique maximal ideal, i.e., \( R \) is local and that \( F(R) \) is a free Abelian group generated by \( J(R) \) [17, Theorem 19.3]. As it is seen in [15, Section 4], if rank \( V \geq 2 \), then a semi-hereditary maximal \( V \)-order \( R \) is not necessarily a principal ideal ring (even being Bezout). However, the following shows that \( F(R) \) is a free Abelian group, though \( R \) is not necessarily local.

Proposition 2.3. Let \( R \) be a semi-hereditary maximal \( V \)-order. Assume that \( J(V) \supset J(V)^2 \). Then \( F(R) = D(R) \) and is a free Abelian group generated by maximal ideals.

Proof. By Proposition 2.1(5)(b), it suffices to prove that any element \( A \in F(R) \) with \( A \subseteq R \) is invertible. Let \( S = O_t(A) \) and let \( x \in S \). Then \( x^n \in S \) for all natural number \( n \) since \( S \) is a ring. Let \( J(V) = \pi V \). Since \( \pi^k \in A \) for some \( k \) by assumption, we have \( \pi^k x^n \in x^n A \subseteq A \subseteq R \) for all \( n \). Hence \( x \) is integral over \( V \) by Lemma 2.2. Furthermore, \( V = Z(R) \subseteq Z(S) \), which is integral over \( V \) and so \( V = Z(S) \). This shows that \( S \) is a \( V \)-order containing \( R \). Hence \( S = R \) follows. Since \( A \) is finitely generated as left and right ideals by Corollary 1.3, \( A \) is projective. Hence \( A(R : A)_{ij} = O_t(A) = R \) by [10, Lemma 1.5] and similarly \( (R : A)_{ji} = A = R \), proving that \( A \) is invertible.

In the case of \( J(V) \supset J(V)^2 \), the following is due to [4, Theorem 1.3]. However, we cannot use their techniques in some parts in the case of \( J(V) = J(V)^2 \). So we give the complete proof.

Lemma 2.4. Let \( M_1, \ldots, M_n \) be distinct idempotent maximal ideals of \( R \) which form one of the cycles (b)–(e). Set \( A = M_1 \cap \cdots \cap M_n \),

1. \( A(R : A)_i = M_i \) and \( (R : A)_i A = M_n \).
2. \( A = M_1 \cdots M_n \).
3. \( A^n = M_1 \cdots M_n \) and is idempotent. In particular, if \( M_n \) is finitely generated as left ideal, then so is \( A^n \).
4. \( A'(R : A)_i = M_i \cdots M_n \) and \( (R : A)_i A' = M_n \cdots M_{n-i+1} \) for any \( i \) (1 \( \leq i \leq n \)).
5. \( A \cup A^2 \cup \cdots \cup A^n = A^{n+1} \).
Proof. (1) It is clear from Proposition 2.1 that $A$ is not invertible. Thus either $A(R : A)_v \neq R$ or $(R : A)_v A \neq R$. If $n = 1$, then the assertion is clear. So we may assume that $n \geq 2$.

For any $i$ ($2 \leq i \leq n$), let $S = O_i(M_{i-1}) = O_i(M_i)$ and $B = M_1 \cdots M_{i-2}M_{i+1} \cdots M_n$ ($B = R$ if $n = 2$). Since $M_iS = S$ and $M_{i-1}S = M_{i-1}$, we have $A(R : A)_v A R \supseteq AS \supseteq BM_{i-1}M_iS = BM_{i-1}$. This implies that $A(R : A)_v$ is not contained in $M_i$ for any $i$ ($2 \leq i \leq n$). Thus either $A(R : A)_v = R$ or $A(R : A)_v = M_1$. If $A(R : A)_v = R$, then $O_i(A^n) = R$ for all natural number $m$. But $A^i$ is idempotent for some $k$. This is clear if $J(V)$ is idempotent and is shown as in [3, Proposition 4.3] if $J(V) \supset J(V)^2$. It follows that $R = O_i(A^n) \supseteq O_i(M_n) \supset R$, a contradiction. Hence $A(R : A)_v = M_1$ and similarly $(R : A)_v A = M_n$ follows.

(2) If $n = 2$, then it follows from Lemma 1.9, its remark, Theorem 1.11(8), and Lemma 1.12 that $A = M_1M_2$. We may assume that $B = M_2 \cap \cdots \cap M_n = M_2 \cdots M_n$ by induction on $n$. Furthermore assume that $A \supseteq M_1 \cdots M_n$. Since $M_iA \subseteq M_iB = M_1 \cdots M_n$, $M_iA$ is finite copies of a simple left $R$-module $U$ with $M_1U = 0$. Set $S = O_i(M_1) = O_i(M_2)$. Then $M_2M_1$ is an idempotent maximal ideal of $S$ and $M_2M_1$, $M_2$, $\cdots$, $M_n$ satisfy all assumptions in the proposition by Theorem 1.11, where $M_j = M_jS = SM_j$ ($3 \leq j \leq n$). So $M_2M_1 \cap M_1 \cap \cdots \cap M_n = M_2M_1M_1 \cdots M_1 = M_2M_1M_3 \cdots M_n$ by induction hypothesis. Since $M_2M_1$ is idempotent, $M_1M_2 = M_1 \cap M_2 \supseteq M_2M_1$ by Lemma 1.12. Combining these properties, we have $M_2A \subseteq M_2M_1 \cap M_2 \cap M_3 \cap \cdots \cap M_n \subseteq M_2M_1 \cap M_1 \cap \cdots \cap M_n = M_2M_1M_3 \cdots M_n \subseteq M_1M_2 \cdots M_n$. This implies that $M_2U = 0$, a contradiction. Hence $A = M_1M_2 \cdots M_n$ follows.

(3) If $n = 2$, then, by the proof of (2), $A = M_1M_2$ and $M_2M_1$ is idempotent. So $A^2 = M_2M_1$ follows. Let $C = M_nM_{n-1} \cdots M_1$. Since $M_nM_{n-1}$ is idempotent, $M_nM_{n-1} = M_nM_{n-1}M_n$. Thus, by Theorem 1.14 and induction on $n$, we have:

$$C^2 = (M_nM_{n-1} \cdots M_1)(M_nM_{n-1} \cdots M_1) = (M_nM_{n-1}M_nM_{n-2} \cdots M_1)(M_{n-1} \cdots M_1) = M_n(M_{n-1} \cdots M_1)(M_{n-1} \cdots M_1) = M_nM_{n-1} \cdots M_1 = C,$$

so $C$ is idempotent. It follows that $C = C^n \subseteq A^n \subseteq M_n \cdots M_1 = C$ and so $C = A^n$. The last statement follows from Lemma 1.4.

(4) This easily follows from (1) and Proposition 2.1(4).

(5) By (3), $A^n$ is idempotent and so $A^n = A^{n+1}$. Assume that $A^i = A^{i+1}$ for some $i$ ($1 \leq i \leq n - 1$). Then, by (4), $A_j \cdots M_1 = A^i((R : A)_v)^i = AM_j \cdots M_1 \subseteq M_n$, a contradiction. Hence $A^i \supset A^{i+1}$ for any $i$ ($1 \leq i \leq n - 1$). \hfill $\square$

We are now ready to prove the following theorem which describes all elements in $\mathcal{I}(R)$.

**Theorem 2.5.** Let $A$ be an ideal of a semi-hereditary $V$-order $R$ in $Q$. Then $A \in \mathcal{I}(R)$ if and only if

(1) any maximal ideal containing $A$ is an element in $\mathcal{I}(R)$ and
Proof. First assume that $A \in \mathcal{I}(R)$. Then (1) follows from Lemma 1.1, Corollary 1.3, and Proposition 1.8. Since $R = R/A$ is Artinian, we can classify the set of all maximal ideals $M_{ij}$ containing $A$ and $m_i = 1$ or if $m_i \geq 2$, then $M_{i1}, \ldots, M_{im_i}$ is either an open cycle or a right open cycle $(1 \leq i \leq l)$.

(2) $A = (M_{1m_1} \cdots M_{11}) \cdots (M_{im_i} \cdots M_{11})$, where $M_{ij}$ $(1 \leq i \leq l, 1 \leq j \leq m_i)$ are all maximal ideals containing $A$, and $m_i = 1$ or if $m_i \geq 2$, then $M_{i1}, \ldots, M_{im_i}$ is either an open cycle or a right open cycle $(1 \leq i \leq l)$, $M_{i1}, \ldots, M_{im_i}$ and $M_{j1}, \ldots, M_{jm_j}$ are disjoint $(i \neq j)$.

Proof. Assume that $M_{ij}$ contains $A$, and $m_i = 1$ or if $m_i \geq 2$, then $M_{i1}, \ldots, M_{im_i}$ is either an open cycle or a right open cycle $(1 \leq i \leq l)$, $M_{i1}, \ldots, M_{im_i}$ and $M_{j1}, \ldots, M_{jm_j}$ are disjoint $(i \neq j)$. Then $M_{ij}$ is finitely generated as a left ideal by Lemma 1.4 and is idempotent by Lemma 2.4 and Theorem 1.14. Thus $C = C_1 \cdots C_l$ is idempotent containing $A$. Since $R$ is Artinian with $J(R) = B/A$, we have $A \subset C = C^n \subset B^n \subset A$ for some natural number $n$, proving (2). Conversely assume that (1) and (2) hold. Then $A$ is finitely generated as a left ideal by Lemma 1.4 and is idempotent by Lemma 2.4 and Theorem 1.14. \qed

Corollary 2.6. Assume that $J(V) = J(V)^2$.

(1) Let $A$ be an idempotent with $J(V) \subseteq A$. Then $A$ is idempotent if and only if $A = (M_{1m_1} \cdots M_{11}) \cdots (M_{im_i} \cdots M_{11})$, where $M_{ij}$ $(1 \leq i \leq l, 1 \leq j \leq m_i)$ are all maximal ideals containing $A$, and $m_i = 1$ or if $m_i \geq 2$, then each $M_{i1}, \ldots, M_{im_i}$ is one of the cycles (b)–(e).

(2) $J(R)$ is eventually idempotent, i.e., $J(R)^n$ is idempotent for some $n$.

Proof. (1) is proved as in Theorem 2.5.

(2) follows from Theorem 1.14, Proposition 2.1, Lemma 2.4, and (1). \qed

The following is clear from Theorems 1.14, 2.5, and Corollary 2.6.

Theorem 2.7. Let $R$ be a semi-hereditary $V$-order in $Q$ and let $M_{11}, \ldots, M_{n(i)}$ all be cycles of the first (second) type of $R$ $(1 \leq i \leq k)$.

(1) Assume that $J(V) \supset J(V)^2$. Then the number of $\mathcal{I}(R)$ is $(2^n(1) - 1) \cdots (2^n(k) - 1) - 1$. In particular, let $C_{i(i(i))} = M_{i(j(i)) - 1} \cdots M_{i(n(i))} \cdots M_{i(j(i)) + 1}$, where $1 \leq i \leq k, 1 \leq j(i) \leq n(i)$ $(C_{j(i)}) = R$ if $n(i) = 1$ for any $i, j(i), j(i) + 1 = n(i)$ if $j(i) = 1$ and $j(i) + 1 = 1$ if $j(i) = n(i))$. Then $|C_{j(n(i))} \cdots C_{j(i)} | 1 \leq j(i) \leq n(i)$ is the set of all minimal elements in $\mathcal{I}(R)$ and the number of minimal elements in $\mathcal{I}(R)$ is $n(1) \cdots n(k)$, if $R$ is not maximal (note that if $R$ is maximal, then $\mathcal{I}(R) = \phi$ by Proposition 1.13).

(2) Assume that $J(V) = J(V)^2$.

(a) The number of $\mathcal{I}(R)$ is $2^n(1) + \cdots + n(k) - k - 1$. In particular, let $C_i = M_{n(i)} - 1 \cdots M_{11}$,
where $C_i = R$ if $n(i) = 1$. Then $C_1 \cdots C_k$ is the unique minimal element in $\mathcal{I}(R)$.

(b) The number of idempotent ideals containing $J(V)R$ is $2^{n(1)+\cdots+n(k)} - 1$.

We close this section with the following lemma used in Section 4.

**Lemma 2.8.** Let $M_1, \ldots, M_n$ be a first-type cycle, $J = M_1 \cap \cdots \cap M_n$ and $A = M_n \cdots M_1$.

(1) $J^n \subseteq A$ and $J^{n-1}$ is not contained in $A$.

(2) $J^mA = M_n \cdots M_1A^{j+1}$ for any natural number $m$, where $m = ln + r$ with $0 \leq r < n$, and $J^mA = A^{j+1}$ if $r = 0$.

**Proof.** (1) It is clear that $J^n \subseteq A$. To prove that $J^{n-1}$ is not contained in $A$, set $S = O_r(M_1) = O_r(M_2)$. We shall prove this by induction on $n$. If $n = 2$, assume that $J \subseteq A$, i.e., $J = A$. Since $M_2M_1$ is an ideal of $S$, we have $M_2M_1 = SJS = S$; a contradiction, because $SM_1 = S = M_2S$. So we may assume that $n \geq 3$. Let $J' = M_2M_1 \cap M_3 \cap \cdots \cap M_n$, where $M'_j = M_jS = SM_j (3 \leq j \leq n)$. It follows from Theorem 1.11 that $M_2M_1, M_3, \ldots, M_n$ is a first-type cycle. Now we assume that $J^{n-1} \subseteq A$, then $A \supseteq J^{n-1} \supseteq J^{n-2}J$ because $J' = J' \cap R = M_2M_1 \cap M_3 \cap \cdots \cap M_n \subseteq J$. It follows that $SJS = S(M_1 \cap M_2 \cap M_3 \cap \cdots \cap M_n)S = M'_1 \cap \cdots \cap M'_n = M'_3 \cdots M'_n = SM_3 \cdots M_n$ by Lemma 2.4. Thus $A \supseteq J^{n-2}JS = J^{n-2}M_3 \cdots M_n$. By multiplying $O_l(M_n), \ldots, O_l(M_3)$, one by one, to the inequality from right side, we have $A \supseteq J^{n-2}$, because $M_iO_l(M_j) = O_l(M_j)M_i$ for any $i$ with $i \neq j$, $i \neq j - 1$, and $3 \leq j \leq n$, $M_jO_l(M_j) = O_l(M_j)$ and $M_{j-1}O_l(M_j) = M_{j-1}$. This contradicts the induction hypothesis on $n$, because $A = M'_n \cdots M'_3M_1M_2$. Hence $J^{n-1}$ is not contained in $A$.

(2) For any $i (1 \leq j \leq n)$, $M_iM_{j+1}M_i = M_{j+1}M_i$, because $M_{j+1}M_i$ is idempotent by Lemma 1.9 $(i = n \Rightarrow i + 1 = 1)$. So $M_iA = A$ for any $j (2 \leq j \leq n)$ by Theorem 1.14. Hence $JA = M_1A \cap M_2A \cap \cdots \cap M_nA = M_1A$. We may assume that $J^mA = M_1 \cdots M_1A^{j+1}$, where $m = ln + r$, $0 \leq r < n$ and so, by Proposition 2.1(4), we have $J^{m+1}A = JM_1 \cdots M_1A^{j+1} = M_{r+1} \cdots M_2A^{j+1} = M_{r+1} \cdots M_1A^{j+1}$. If $r + 1 < n$, then we are done. If $r + 1 = n$, then $J^{m+1}A = A^{j+2}$, completing the proof. □

3. The principalness of Jacobson radical

In this section, we shall study the principalness of the Jacobson radical $J(R)$ of a semi-hereditary V-order $R$. If $J(V) = J(V)^2$, then $J(R)$ is eventually idempotent by Corollary 2.6, so that $J(R)$ is not principal. Furthermore, since $[R/J(R) : \mathfrak{P}] \leq [Q : F]$, $R$ is semi-local; a ring $S$ is semi-local if $S/J(S)$ is a semi-simple Artinian ring.

**Lemma 3.1.** Let $R_i$ be semi-local orders in a simple Artinian ring such that $R_i$ are Morita equivalent through $A (i = 1, 2)$, that is, $A$ is a pregenerator of the category of left $R_1$-modules and $R_2 = \text{Hom}_{R_1}(A, A)$. Then $R_1/J(R_1) \cong \text{Hom}_{R_2}(\mathcal{A}, \mathcal{A})$, where $\mathcal{A} = A/AJ(R_2)$.
Proof. Let $B$ be a left $R_2$- and right $R_1$-module with $BA = R_2$ and $AB = R_1$. Since $J(R_1) = AJ(R_2)B$, we have $J(R_1)A = AJ(R_2)$. Define $\varphi : R_1 \mapsto \text{Hom}_{R_2}(\overline{A}, \overline{A})$, given by $\varphi(r)(a + AJ(R_2)B) = [ra + AJ(R_2)]$, where $r \in R_1$ and $a \in A$. It is clear that $\varphi(r) \in \text{Hom}_{R_2}(\overline{A}, \overline{A})$ and $\varphi$ is a ring homomorphism. Assume that $\varphi(r) = 0$, equivalently $r A \subseteq AJ(R_2)$ if and only if $r \in AJ(R_2)B = J(R_1)$, proving that $\text{Ker} \varphi = J(R_1)$. To prove that $\varphi$ is epimorphism, let $\bar{f} \in \text{Hom}_{R_2}(\overline{A}, \overline{A})$. Then, since $A$ is a projective right $R_2$-module, there exists an $f \in \text{Hom}_{R_2}(A, A)$ with $\eta f = \bar{f}$, where $\eta$ is the natural mapping from $A$ to $\overline{A}$. Thus $f = \lambda_r$, a left multiplication of $r$ for some $r \in R_1$. Hence $\varphi(r) = \bar{f}$.

Let $M_1, \ldots, M_n$ be a cycle of the first (second) type of $R$ with $\dim R/M_i = m_i$. Then we write $(M_1, \ldots, M_n) = (m_1, \ldots, m_n)$ to show implicitly the dimension of $R/M_i$ as a right $R$-module and call it the form of $M_1, \ldots, M_n$. Let $M_{11}, \ldots, M_{1n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)}$ be the set of all maximal ideals of a semi-hereditary $V$-order $R$ such that $M_{11}, \ldots, M_{n(i)}$ is a cycle of the first (second) type of $R$ with $\dim R/M_i = m_i$ ($1 \leq i \leq k$, $1 \leq j \leq n(i)$). We say that $M_{11}, \ldots, M_{1n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)}$ is the series of maximal ideals of $R$ and $(M_{11}, \ldots, M_{1n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)}) = (m_{11}, \ldots, m_{1n(1)}, \ldots, m_{k1}, \ldots, m_{kn(k)})$ is the form of $R$. □

**Proposition 3.2.** Let $M_1, \ldots, M_n$ be a cycle of the first type of $R$ with its form $(M_1, \ldots, M_n) = (m_1, \ldots, m_n)$.

1. $J = M_1 \cap \cdots \cap M_n$ is principal as left and right ideals if and only if $m_1 = \cdots = m_n$.
2. $J^n$ is principal.

**Proof.** (1) First assume that $J = aR = Ra$ for some $a \in J$. By Proposition 2.1(4), $M_{i+1} = aM_i a^{-1}$. Hence $R/M_{i+1} = aRa^{-1}/aM_ia^{-1} \cong R/M_i$ and so $m_{i+1} = m_i$ follows ($1 \leq i \leq n - 1$). Conversely, assume that $m = m_1 = \cdots = m_n$. Since $J$ is invertible by Proposition 2.1(1), $\varphi : \overline{R} = R/J(J) \cong \text{Hom}_{R_2}(\overline{J}, \overline{J})$ by Lemma 3.1, where $\overline{J} = J/J(J(R))$. Let $\{N_k : \text{maximal ideals} | N_k \neq M_i \}$ for all $i$, $1 \leq k \leq l$. Set $M^*_i = \bigcap \{M_i \cap \cap N_j_k \mid 1 \leq i \leq k \leq l \}$ for each $i$ and $N^*_k = \bigcap \{M_i \cap \cap N_j \mid 1 \leq j \leq l, j \neq k \}$ for each $k$ ($1 \leq k \leq l$). Then it is not hard to see that $\overline{R} = \bigoplus_{i=1}^n M^*_i / J(J(R)) \oplus \bigoplus_{k=1}^l N^*_k / J(J(R))$ is the decomposition of $\overline{R}$ into the simple components and that $\overline{J} = \bigoplus_{i=1}^n J M^*_i / J(J(R)) \oplus \bigoplus_{k=1}^l J N^*_k / J(J(R))$ is the decomposition of $\overline{J}$ into the homogeneous components with $r(k)(J M^*_i / J(J(R))) = M_i$. Note that $M^*_i / (M^*_i \cap M_i) \cong (M^*_i + M_i) / M_i = R/M_i$. Write $J M^*_j / J(J(R)) \cong U_i \oplus \cdots \oplus U_i, \ i \ copies$ of a simple right $R$-module $U_i$. By the property of $M_{i+1}J = JM_j$ and Theorem 1.14, we have:

- $M^*_{i+1} J M^*_i = J M^*_i M^*_i$ is not contained in $J J(J)$,
- $M^*_j J M^*_i = J M^*_i M^*_i \subseteq J J(J)$ for any $j \neq i + 1$, and
- $N^*_k J M^*_i = J N^*_k M^*_i \subseteq J J(J)$ for any $N_k$.

These imply that $\varphi(M^*_i(J(J(R)))$ is the simple component of $\text{Hom}_{R_2}(\overline{J}, \overline{J})$ which operates on the homogeneous component $J M^*_i / J(J(R))$. Hence $m = m_{i+1} = l_i$ and
thus \( \dim R/M_i = \dim JM_i/JJ(R) \) follows. Similarly, we have that \( JN_k/JJ(R) \) is the homogeneous component of \( JJ(R) \) annihilated by \( N_k \) for any \( k \) and \( \dim R/N_k = \dim \varphi(N_k/JJ(R)) = \dim JN_k/JJ(R) \). Hence \( R/JJ(R) \cong J/JJ(R) \) as a right \( R \)-module and so \( J \) is principal as a right ideal by Nakayama’s Lemma. Similarly, \( J \) is principal as a left ideal; i.e., \( J = aR = Rb \) for some \( a, b \in J \). Since \( a \) and \( b \) are regular elements, it follows that \( J = aR = Ra \) (see the proof of [5, p. 37]).

(2) We replace \( \bar{J} \) by \( J_n/J_nJ(R) \) in (1). Since \( MiJ_n = J_nM_i \) and \( NkJ_n = J_nN_k \) for all \( i, k \), the proof is similar to one in (1).

The following is the main result in this section and is used for the conjugacy theorem on semi-hereditary maximal \( V \)-orders containing \( R \).

**Theorem 3.3.** Let \( R \) be a semi-hereditary \( V \)-order with \( M_{11}, \ldots, M_{1n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)} \) as the series of maximal ideals of \( R \). Set \( J_i(R) = M_{1i} \cap \cdots \cap M_{n(i)i} \) (1 \( \leq i \leq k \)). Assume that \( J(V) \supset J(V)^2 \). The following are equivalent:

1. \( J(R) \) is principal.
2. \( J_i(R) \) is principal for each \( i \).
3. The form of \( R \) is

\[
(M_{11}, \ldots, M_{1n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)}) = (m_1, m_1, \ldots, m_k, \ldots, m_k).
\]

**Proof.** (2) \( \Leftrightarrow \) (3). This follows from Proposition 3.2.

(2) \( \Rightarrow \) (1). Since \( J_i(R) \) are comaximal, it follows that \( J(R) = J_1(R) \cap \cdots \cap J_k(R) = J_1(R) \cdots J_k(R) \) and so \( J(R) \) is principal.

(1) \( \Rightarrow \) (3). For any fixed \( i \) (1 \( \leq i \leq k \)), we have \( M_{ij+1}J(R) = J(R)M_{ij} \) for any \( j \) (1 \( \leq j \leq n(i) \), \( j + 1 = 1 \) if \( j = n(i) \)) by Theorem 1.14 and Proposition 2.1. So if \( J(R) \) is principal, then \( m_{ij} = \dim R/M_{ij} = \dim R/M_{ij+1} = m_{ij+1} \) as in Proposition 3.2.

**Corollary 3.4.** Under the same assumptions and notations as in Theorem 3.3, \( J_i(R)^{n(i)} \) is principal for any \( i \) and \( J(R)^n \) is principal, where \( n = \operatorname{lcm}(n(1), \ldots, n(k)) \).

**Corollary 3.5.** Let \( R \) be a minimal semi-hereditary \( V \)-order and \( J(V) \supset J(V)^2 \). Then \( J(R) \) is principal.

**Proof.** Let \( M \) be any maximal ideal of \( R \). If \( \dim R/M > 1 \), then we can find a semi-hereditary \( V \)-order strictly being contained in \( R \) by [10, Theorem 20.17], a contradiction. Hence \( \dim R/M = 1 \) for any maximal ideal \( M \) of \( R \) and therefore \( J(R) \) is principal by Theorem 3.3.

**Corollary 3.6.** Let \( R \) be a semi-hereditary \( V \)-order and assume that \( J(V) \supset J(V)^2 \). The following are equivalent:

1. \( R \) is maximal.
2. Any maximal ideal is principal.
(3) Any ideal in \( F(R) \) is principal.

**Proof.** (1) \( \Rightarrow \) (2). By Proposition 2.3, any maximal ideal is invertible, that is, a trivial first-type cycle. Hence it is principal by Theorem 3.3.

(2) \( \Rightarrow \) (3). This follows from Proposition 2.3.

(3) \( \Rightarrow \) (1). This follows from Proposition 1.13. \( \square \)

We close this section with the following remark.

**Remark.** Let \( R \) be a semi-hereditary maximal \( V \)-order in \( Q \). Assume that \( J(V) \supset J(V)^2 \). Then any ideal in \( F(R) \) is principal. However, \( R \) is not necessarily Bezout as it is seen from the example: Let \( U \) be any proper over ring of \( V \) with \( J(V) \supset J(V)^2 \). Then

\[
   R = \begin{pmatrix}
   V & J(U) \\
   U & V
   \end{pmatrix}
\]

is a semi-hereditary maximal \( V \)-order but not Bezout by [15, Theorem 4.7].

### 4. Overrings and the nilpotency of the Jacobson radical

In this section, we shall apply some results of Sections 1–3 for studying \( V \)-overrings of a semi-hereditary \( V \)-order \( R \) and for the nilpotency of \( J(R) \) modulo \( J(V)R \). We start with Henselization to get some invariant properties of \( R \), in particular, the invariant of the division part of \( R/M \) for any maximal ideal \( M \).

Let \( V^h \) be the Henselization of \( V \) with its quotient field \( F^h = F \otimes V^h \) and \( Q^h = Q \otimes V^h = M_N(D^h) \), the \( N \times N \) matrix ring over \( D^h \), where \( D^h \) is the division ring with \( Z(D^h) = F^h \). There exists the unique invariant valuation ring \( \Delta^h \) of \( D^h \) with \( Z(\Delta^h) = V^h \) [10, Corollary 8.3]. In particular, the matrix size \( N \) of \( Q^h \) is unique up to \( V \) and \( Q \). Let \( M \) be a maximal ideal of a semi-hereditary \( V \)-order \( R \). Then \( R/M \cong M_n(D) \), where \( D \) is a division ring. We say that \( D \) is the division part of \( M \).

**Theorem 4.1** (Invariantness). Let \( R \) be a semi-hereditary \( V \)-order in \( Q \) and let \( S \) be a \( V \)-overring of \( R \).

1. The number of cycles of the first (second) type of \( R \) is equal to the number of cycles of the first (second) type of \( S \).
2. \( D(R) \cong D(S) \).
3. \( \dim R/J(R) = N \), the matrix size of \( Q^h \), that is, \( \dim R/J(R) \) is unique up to \( V \) and \( Q \).
4. Let \( M \) be any maximal ideal of \( R \) with \( \dim R/M = n \). Then \( R/M \cong M_n(\Delta^h) \), where \( \Delta^h = \Delta^h/J(\Delta^h) \), that is, the division part of \( M \) is unique up to \( V \) and \( Q \).

**Proof.** (1) There are a finite length of \( V \)-overrings \( S_i \) of \( R \) with \( S = S_0 \supset S_1 \supset \cdots \supset S_m = R \) such that \( S_i \) is a minimal \( V \)-overring of \( S_i \) (1 \( \leq i \leq m \)). So it is enough to prove
the statement in the case $S$ is a minimal $V$-order of $R$. Then this immediately follows from Theorem 1.11.

(2) This follows from (1) and Proposition 2.1.

(3) It follows from [10, Lemma 4.4 and Theorem A.18] that $R / J(R) \cong R^h / J(R^h)$ and the orthogonal idempotents in $R^h / J(R^h)$ can be lifted to orthogonal idempotents in $R^h$. So $\dim R / J(R) = \dim R^h / J(R^h) = \dim Q^h = \mathcal{N}$ (note that $R^h$ is also semi-hereditary by [6, Theorem 3.4]).

(4) Since $R / M \cong R^h / M^h$ by the proof of [10, Lemma 4.4], we may assume that $V = V^h$ and $R = J(R)$. Then we may assume that $R = (\Delta_{ij})$ by [6, Theorem 2.4], where $\Delta_{ij}$ are non-zero submodules of $Q(\Delta)$, the quotient ring of $\Delta$, with $\Delta_{ii} = \Delta$, an invariant valuation ring with $Z(\Delta) = V$ for any $i$. We see from the proof of [6, Proposition 2.7] that $J(\Delta)R \subseteq J(R)$ and so $M = (\Delta_{ij}')$, where $\Delta_{ij}'$ are $\Delta$-submodules of $\Delta_{ij}$ with $\Delta_{ij}' \supseteq J(\Delta) \Delta_{ij}$. In particular, $\Delta \supseteq \Delta_{ii}' \supseteq J(\Delta)$. If $\Delta = \Delta_{ii}'$ for all $i$, then $M \equiv 1$ and so $M = R$, a contradiction. Thus $\Delta_{ii}' = J(\Delta)$ for some $i$. Then we have $e_{ii}R_{ei} \cong \overline{\Delta} = \Delta / J(\Delta)$, where $R = R / M$ and $e_{ii}$ is the matrix unit in $R$ whose $(i, i)$ entry is 1 and the other entries are all 0. Hence the division part of $R / M$ is isomorphic to $\overline{\Delta} = \Delta / J(\Delta)$. \hfill \Box

As we have noted in the paragraph before Proposition 1.2, there exists a minimal semi-hereditary $V$-order which is contained in $R$. The following shows that any minimal semi-hereditary $V$-orders are characterized in terms of the forms.

**Proposition 4.2.** Let $T$ be a semi-hereditary maximal $V$-order in $Q$ with its form $(M_1, \ldots, M_k) = (m_1, \ldots, m_k)$.

(1) Any minimal semi-hereditary $V$-order in $Q$ which contained in $T$ has of the form $(M_{11}, \ldots, M_{1m_1}, \ldots, M_{k1}, \ldots, M_{km_k}) = (1, 1, \ldots, 1, 1, 1)$.

(2) Assume that $J(V) \supset J(V)^2$ and let $R_0$ be a minimal semi-hereditary $V$-order in $Q$ with $T \supseteq R_0$. Then there are $m_1 \cdots m_k$ semi-hereditary maximal $V$-orders in $Q$ containing $R_0$ and they are all conjugate (see [8, Corollary 2]).

**Proof.** (1) First, note that each $M_i$ is a trivial cycle by Proposition 1.13. Let $R_0$ be a minimal semi-hereditary $V$-order with $T \supseteq R_0$. Then there are $V$-overrings $T_i$ of $R_0$ such that $T = T_0 \supset T_1 \supset \cdots \supset T_n = R_0$ and $T_{i-1}$ is a minimal $V$-overring of $T_i$ ($1 \leq i \leq n$). By Proposition 1.2 and its left version, there are idempotent ideals $K_1$ and $L_1$ of $T_1$ such that $O_i(K_1) = T_0 = O_i(L_1)$, $K_1 = (T_1 : T_0)_i$, and $L_1 = (T_1 : T_0)_i$. Since $T_0$ is a minimal $V$-overring of $T_1$, it follows from [10, Lemma 20.21] that $K_1$ and $L_1$ are maximal ideals of $R_1$. So, by Theorem 1.11(1), $L_1K_1$ is a maximal ideal of $T_0$, say, $M_1 = L_1K_1$ such that $\dim T_0 / L_1$ $K_1 = \dim T_1 / K_1 + \dim T_1 / L_1$. Set $N_{11} = K_1$, $N_{12} = L_1$ and $N_i = M_i \cap T_1$, $2 \leq i \leq k$. Then it is easily checked from Theorem 1.11, (2), (3), (5), and (9), that $N_{11}, N_{12}, N_2, \ldots, N_k$ is the series of maximal ideals of $T_1$ with its form $(m_{11}, m_{12}, m_2, \ldots, m_k)$, where $m_{11} = \dim T_1 / K_1$ and $m_{12} = \dim T_1 / L_1$. Continuing this process, we can prove that $R_0$ has of the form stated.

(2) Assume that $R_0$ has of the form as in (1). Then it follows from Theorem 2.7 and Proposition 1.2 that there are $m_1 \cdots m_k$ semi-hereditary maximal $V$-orders contained $R_0$, which are obtained in the following way: Set $I_i = M_{im_i}M_{im_{i-1}} \cdots M_{i2}$ ($1 \leq i \leq k$) and
\[ I = I_1 \cdots I_k, \ \text{a minimal element in } \mathcal{I}(R_0), \ i.e., \ \mathcal{O}_r(I) \text{ is a semi-hereditary maximal } V\text{-order containing } R_0. \ \text{For any } i \text{ and } j_i \ (1 \leq i \leq k \text{ and } 1 \leq j_i \leq m_i), \ \text{we set}
\]
\[ L_i = M_{i m_i-1} \cdots M_i M_{i j_i+1}
\]
and \( L = L_1 \cdots L_k \). Then \( \mathcal{O}_r(L) \) is any semi-hereditary maximal \( V\)-order in \( Q \) containing \( R_0 \). By Proposition 3.2, \( J_i(R_0) = M_{i1} \cap \cdots \cap M_{i m_i} \) is principal, say, \( J_i(R_0) = a_i R_0 = R_0 a_i \) for some \( a_i \in J_i(R_0) \ (1 \leq i \leq k) \). It follows from Theorem 1.14 and Proposition 2.1 that
\[
M_{ij+1} J_i(R_0) = J_i(R_0) M_{ij} \quad (1 \leq j \leq m_i, \ m_i + 1 = 1) \quad \text{and}
\]
\[ M_{ij} J_i(R_0) = J_i(R_0) M_{ij} \quad (i \neq i, \ 1 \leq j \leq m_i).
\]
Hence there is a natural number \( n_i \) such that \( L_i J_i(R_0)^{n_i} = J_i(R_0)^{n_i} I_i \) and \( J_i(R_0)^{n_i} = a R_0 = R_0 a \), where \( a = a_i^{n_1} \cdots a_k^{n_k} \), because \( J_i(R_0) J_j(R_0) = J_j(R_0) J_i(R_0) \) for any \( i, j \). Since \( J_i J_i(R_0) = J_i(R_0)I_j \) for \( i \neq j \), we have
\[
L = L_1 \cdots L_k = (J_1(R_0)^{n_1} I_1 J(R_0)^{-n_1}) \cdots (J_k(R_0)^{n_k} I_k J(R_0)^{-n_k})
\]
\[ = J_1(R_0)^{n_1} \cdots J_k(R_0)^{n_k} \cdot I_1 \cdots I_k J_1(R_0)^{-n_1} \cdots J_k(R_0)^{-n_k} = a R_0 a^{-1}.
\]
Thus \( \mathcal{O}_r(L) = \mathcal{O}_r(a R_0 a^{-1}) = a \mathcal{O}_r(I) a^{-1} \). Therefore any semi-hereditary maximal \( V\)-order containing \( R_0 \) are conjugate. \( \square \)

In [7,8], Kauta has investigated \( V\)-overrings of \( R \) inside a Bezout \( V\)-order. We will obtain in the following theorem the exact number of \( V\)-overrings of \( R \) in terms of the maximal ideals series of \( R \).

**Theorem 4.3.** Let \( R \) be a semi-hereditary \( V\)-order in \( Q \) but not maximal with the maximal ideals series \( M_{11}, \ldots, M_{1 n(1)}, \ldots, M_{k1}, \ldots, M_{kn(k)} \).

(1) Assume that \( J(V) \supset J(V)^2 \). Then
(a) The number of \( V\)-overrings of \( R \) is \( (2^{n(1)} - 1) \cdots (2^{n(k)} - 1) \).
(b) [8, Corollary 2] The number of semi-hereditary maximal \( V\)-orders containing \( R \) is \( n(1) \cdots n(k) \), they are all conjugate and \( R = \bigcap T_i \), where \( T_i \) are all semi-hereditary maximal \( V\)-orders containing \( R \) \((1 \leq i \leq n(1) \cdots n(k))\).

(2) Assume that \( J(V) = J(V)^2 \). Then
(a) The number of \( V\)-overrings of \( R \) is \( 2^{n(1)+\cdots+n(k)-k} \).
(b) [8, Corollary 2] There is a unique semi-hereditary maximal \( V\)-order containing \( R \).

**Proof.** (1)(a) This follows from Theorem 2.7 and Proposition 1.2 (note that \( R \) is considered as a \( V\)-overring of \( R \)).

(b) Let \( A_i \ (1 \leq i \leq n(1) \cdots n(k)) \) be the set of all minimal elements in \( \mathcal{I}(R) \) (see Theorem 2.7). Then \( T_i = \mathcal{O}_r(A_i) \) are all semi-hereditary maximal \( V\)-orders containing \( R \) by Proposition 1.2. Thus the number of semi-hereditary maximal \( V\)-orders containing
Lemma 4.4. Let $n > 1$. Then $A_t \subseteq A_i \subseteq R$ for all $i$. It follows from the construction of $A_t$ that $R = \sum A_i$ and so $t \in R$. Hence $R = \cap T_i$. Since $R$ contains a minimal semi-hereditary $V$-order, $T_i$ are all conjugate by Proposition 4.2.

(2) This follows from Theorem 2.7 and Proposition 1.2. □

Proof. (1) and (2). It follows from Lemma 2.4 and Theorem 1.14 that $AB$ is idempotent and that $AB = BA$. Thus (1) and (2) follow from the proof of [9, Lemma 3.2].

(3) Let $M$ be a maximal ideal with $O_r(M_{n-1}) = O_l(M_n)$. Then $\{M_{n-1} \cdots M_1, NT = TN \mid N$ are any maximal ideals of $R$ with $N \neq M_i$ $(1 \leq i \leq n)\}$ is the set of all maximal ideals of $T$.

(4) If $M_1, \ldots, M_n$ is a first-type cycle, then $M_n \cdots M_1$ is an invertible ideal of $T$.

Let $T$ be any semi-hereditary maximal $V$-order containing $R$. Kauta showed that $J(T) \subseteq J(R)$ [6, Theorem 1.5]. Is there a natural number $n$ with $J(R)^n \subseteq J(T)$? In the remainder of this section, we shall give more detailed informations on this question and study the nilpotency of $J(R)$ modulo $J(V)R$. The following lemma is easily proved by induction on $k$.

Lemma 4.4. Let $A_i$ and $B_i$ $(1 \leq i \leq k)$ be ideals of a ring $S$ such that $A_iA_j = A_jA_i$, $B_iB_j = B_jB_i,$ and $B_i + B_j = S$ for any $i \neq j$. Assume that $A_1^n \cdots A_k^n \subseteq B_1 \cdots B_k$ and that $A_i \supseteq B_i$ for each $i$. Then $A_i^n \subseteq B_i$ for all $i$.

Lemma 4.5. Let $M_1, \ldots, M_{n-1}$ be a (right) open cycle of $R$, and $T = O_r(A)$, where $A = M_{n-1} \cdots M_1$.

(1) $T = O_r(M_1) \cdots O_r(M_{n-1})$.

(2) Let $N_1, \ldots, N_{m-1}$ be another (right) open cycle and $B = N_{m-1} \cdots N_1$. Then $O_r(AB) = O_r(B)O_r(A) = O_r(A)O_r(B)$.

(3) Let $M_n$ be a maximal ideal with $O_r(M_{n-1}) = O_l(M_n)$. Then $\{M_n, M_{n-1} \cdots M_1, \sum_i N_i \mid N$ are any maximal ideals of $R$ with $N \neq M_i$ $(1 \leq i \leq n)\}$ is the set of all maximal ideals of $T$.

(4) If $M_1, \ldots, M_n$ is a first-type cycle, then $M_n \cdots M_1$ is an invertible ideal of $T$.

Proof. (1) and (2). It follows from Lemma 2.4 and Theorem 1.14 that $A$ and $B$ are idempotent and that $AB = BA$. Thus (1) and (2) follow from the proof of [9, Lemma 3.2].

(3) If $n = 2$, then the statement follows from Theorem 1.11. Thus we may assume that $n > 2$. Then $M_{n-1} \cdots M_1, M_nS = SM_n, NS = SN \mid N$ are all maximal ideals with $N \neq M_j$ $(1 \leq j \leq n)$ is the set of all maximal ideals of $S = O_r(M_{n-2} \cdots M_1)$ and that $O_r(M_{n-1} \cdots M_1) = O_l(M_nS) \supseteq S$ by induction hypothesis on $n$. Hence the statement follows from Theorem 1.11.

(4) As in (3), let $S = O_r(M_{n-2} \cdots M_1)$. Then, by induction on $n$, we may assume that $M_{n-1} \cdots M_1, M_nS$ is a first-type cycle and hence $M_nSM_{n-1} \cdots M_1 = M_nM_{n-1} \cdots M_1$ is an invertible ideal of $T$ by Theorem 1.11(8). □

Lemma 4.6. Let $R$ be a semi-hereditary $V$-order with its form

$$\begin{align*}
(M_{1,1}, \ldots, M_{1n(1)}, \ldots, M_{k,1}, \ldots, M_{kn(k)}) &= (m_{1,1}, \ldots, m_{1n(1)}, \ldots, m_{k,1}, \ldots, m_{kn(k)}), \\
t_i &= O_r(M_{in(i)-1} \cdots M_{i1}), \text{ and let } A_i = M_{in(i)} \cdots M_{i1} (1 \leq i \leq k).
\end{align*}$$
Proof. (1) This follows from Proposition 1.2, Theorems 1.14, 2.7, and Lemma 4.5.

(2) First note that \{A_i, M_j T_i = T_i M_j \mid j \neq i, 1 \leq l \leq n(j)\} is the set of all maximal ideals of \(T_i\) by Lemma 4.5. Let \(S = T_1 \cdots T_{k-1}\). Then we may assume that \(A_1 S, \ldots, A_{k-1} S, M_{k1} S, \ldots, M_{kn(k)} S\) is the maximal ideals series of \(S\) such that \(A_i S\) are trivial cycles with \(A_i S = S A_i\) and \(M_{k1} S, \ldots, M_{kn(k)} S\) is a cycle of the first (second) type with \(M_{k1} S = SM_{kj}\) for any \(j\). Then we may assume that \(O_r(B_k S) = O_r(B_k S(T) = T)\). It is clear that \(O_r(B_k S) \supseteq O_r(B_k S)\), because \(B_k S = SB_k\) and is idempotent. Then we may assume that \(x \in O_r(B_k S)\), that is, \(B_k x \subseteq B_k S\). There are idempotent ideals \(C_i\) of \(R\) for \(T_i = O_r(C_i)\) by Proposition 1.2. By multiplying \(C_i\) on both sides of \(B_k x \subseteq B_k S\), we have \(B_k x C_1 \cdots C_{k-1} \subseteq R\), which implies \(x C_1 \cdots C_{k-1} \subseteq O_r(B_k)\). Hence \(x \in O_r(B_k)\), because \(C_i T_i = T_i\). Therefore \(A_i T_i T_j = T_i T_j\) for all \(i, j\) and \(A_k S = A_k T_i S = A_k T_i\) are the set of all maximal ideals of \(T\) by Lemma 4.5. Since \(T\) is a maximal, it follows from Propositions 1.13, 2.1 and Theorem 1.14 that \(J(T) = A_1 T \cap \cdots \cap A_k T = (A_1 T)(\cdots)(A_k T) = A_1 \cdots A_k\).

Theorem 4.7. Let \(R\) be a semi-hereditary \(V\)-order in \(Q\) with its maximal ideals series \(M_{n(1)}, \ldots, M_{n(m)}\) and let \(T\) be a semi-hereditary \(V\)-order containing \(R\). Set \(J_i(R) = M_{n(i)} \cap \cdots \cap M_{n(m)}\).

1. \(J_1(R) n(1) \cdots J_k(R) n(k) \subseteq J(T)\) and \(J_1(R) n(1) \cdots J_k(R) n(k)\) is not contained in \(J(T)\) if \(l_1 + \cdots + l_k < n(1) + \cdots + n(k)\).

2. \(J(R) n - 1 \subseteq J(T)\) and \(J(R) n - 1\) is not contained in \(J(T)\) if \(n = \max\{n(1), \ldots, n(m)\}\).

Proof. (1) Let \(T_i = O_r(M_{n(i)-1} \cdots M_{n1})\). Then we may assume that \(T = T_1 \cdots T_k\) by Proposition 1.2, Theorem 2.7, and Lemma 4.5. Thus \(J(T) = A_1 \cdots A_k\) by Lemma 4.6, where \(A_i = M_{n(i)} \cdots M_{n1}\). Hence we have \(J_1(R) n(1) \cdots J_k(R) n(k) \subseteq J(T)\) by Lemmas 2.4 and 2.8.

Assume that \(J_1(R) l_1 \cdots J_k(R) l_k \subseteq J(T)\) for some \(l_i\) with \(l_1 + \cdots + l_k < n(1) + \cdots + n(k)\). By Lemma 4.4. \(J_i(R) n(1) \subseteq A_i\) and so \(l_i \geq n(i)\) for all \(i\) by Lemma 2.4 or 2.8, a contradiction.

(2) By Theorem 1.14 and the comaximality of \(J_i(R)\) and \(J_j(R)\), we have \(J(R) = J_1(R) \cdots J_k(R) = J_1(R) \cdots J_j(R)\) and \(J_i(R) J_j(R) = J_i(R) J_j(R)\) for all \(i, j\). Hence \(J(R) n - 1 \subseteq J(T)\) by (1). Assume that \(J(R) n - 1 \subseteq J(T)\), then it follows from Lemma 4.4 that \(J_i(R) n - 1 \subseteq A_i\) for all \(i\), which is impossible by Lemma 2.4 or 2.8.

Theorem 4.8. Let \(R\) be a semi-hereditary \(V\)-order in \(Q\) with its form

\[
(M_{n1}, \ldots, M_{n(m-1)}, M_{n1}, \ldots, M_{n(k-1)}) = (m_{n1}, \ldots, m_{n(m-1)}, m_{n1}, \ldots, m_{n(k-1)}).
\]

Set \(J_i(R) = M_{n1} \cap \cdots \cap M_{n(i)}\) (1 \(\leq i \leq k\)). Assume that \(J(V) = J(V)^2\).
(1) $J(V)^n = J_1(R)^{n(1)} \cdots J_k(R)^{n(k)} = (M_{i_1(1)} \cdots M_{i_1(1)} \cdots M_{k(n)} \cdots M_{k(n)})$ and $J_1(R)^{l_1} \cdots J_k(R)^{l_k} \supset J(V)^n$ if $l_1 + \cdots + l_k < n(1) + \cdots + n(k)$.

(2) $J(V)R = J(R)^n$ and $J(R)^{n-1} \supset J(V)R$, where $n = \max\{n(1), \ldots, n(k)\}$.

**Proof.** (1) It follows from Theorem 1.14 and Lemma 2.4 that $J(R)^{n(i)} = M_{i_1(1)} \cdots M_{i_1(1)} \cdots M_{n(i)}$ is idempotent and $J_i(R)J_j(R) = J_i(R)J_j(R)$ for all $i, j$. So $A = J_1(R)^{n(1)} \cdots J_k(R)^{n(k)}$ is idempotent with $J(R) \supset A \supset J(V)R$. Hence $A = J(V)R$, because $R = R/J(V)R$ is an Artinian ring with $J(R) = J(R)/J(V)R$.

Assume that $J_1(R)^{l_1} \cdots J_k(R)^{l_k} = J_1(R)^{n(1)} \cdots J_k(R)^{n(k)}$ for some $l_i$ with $l_1 + \cdots + l_k < n(1) + \cdots + n(k)$. Then $J_i(R)^{l_i} \leq J_i(R)^{n(i)}$ for all $i$ by Lemma 4.4 and so $l_i \geq n_i$ by Lemma 2.4, a contradiction.

(2) It follows from (1) that $J(V)R \subseteq J(R)^n = (J_1(R)^{n(1)} \cdots J_k(R)^{n(k)}) \subseteq J_1(R)^{n(1)} \cdots J_k(R)^{n(k)} = J(V)R$. Hence $J(V)R = J(R)^n$. If $J(V)R = J(R)^{n-1} = J_1(R)^{n-1} \cdots J_k(R)^{n-1}$, we may assume that $n = n(1)$. Then we have by Lemma 4.4 that $J_1(R)^{n(1)-1} \subseteq J_1(R)^{n(1)}$, which is impossible by Lemma 2.4. □

The following is due to Kauta in the case either $V$ is Henselian or $J(V) = J(V)^2$.

**Lemma 4.9.** Let $T$ be a semi-hereditary maximal V-order in $Q$. Then there is a natural number $e$ such that $J(V)T = J(T)^e$. If $J(V) = J(V)^2$ then $e = 1$.

**Proof.** If $J(V) = J(V)^2$ then the lemma follows from [8, Corollary 1]. By [8, Theorem 1], $T^h = T \otimes_V V^h$ is also a semi-hereditary maximal $V$-order and so there is a natural number $e$ such that $J(T)^h = J(V)^h T^h$ by [6, Corollary 2.8]. Hence, by using [10, Lemma 11.6], we have $J(T)^e = J(T)^e \otimes V^h \cap T = (J(T) \otimes V^h)^e \cap T = J(T^h)^e \cap T = J(V)^h T^h \cap T = (J(V) \otimes V^h)(T \otimes V^h) \cap T = J(V)T \otimes V^h \cap T = J(V)T$, since $J(T^h) = J(T) \otimes V^h$ and $J(V^h) = J(V) \otimes V^h$. □

We call $e$ the ramification index of $T$ over $V$.

**Theorem 4.10.** Let $R$ be a semi-hereditary V-order in $Q$ with its maximal ideals series $M_1, \ldots, M_{i_1(1)}, \ldots, M_{k_1}, \ldots, M_{i_n(k)}$ and let $T$ be any semi-hereditary maximal V-order in $Q$ containing $R$ with ramification index $e$, that is, $J(V)T = J(T)^e$. Set $J_i(R) = M_{i_1(1)} \cap \cdots \cap M_{i_n(i)}$ and assume that $J_i(R) \supset J_i(R) = J_1(R)^{m_1} \cdots J_k(R)^{m_k}$ for some nonnegative integers $m_i$ by Proposition 2.1 and so we have $J_1(R)^{m_1} \cdots J_k(R)^{m_k} T = J(V)T = A_1^e \cdots A_k^e$.  

\[ \begin{align*} \end{align*} \]
It follows from Lemma 4.4 that \( J_i(R) \subseteq A_i^e \) for all \( i \). Assume that \( m_i < e \) for some \( i \), and set \( m_i = n(i)x_i + r_i \), where \( 0 \leq x_i < e \) and \( 0 \leq r_i < n(i) \). Then \( J_i(R)^{m_i}A_i \subseteq A_i^{e+1} \) implies, by Lemma 2.8, that \( J_i(R)^{x_i}A_i^{r_i+1} \subseteq A_i^{e+1} \) and so \( J_i(R)^{r_i} \subseteq A_i \) follows, because \( A_i \) is an invertible ideal of \( T_i \) by Lemma 4.5. This is a contradiction with Lemma 2.8.

Thus \( m_i \geq e(n(i)) \) for all \( i \). Assume that \( m_i > e(n(i)) \) for some \( i \), say \( i = 1 \), and let \( l_i = m_i - e(n(i)) \). Since \( J_1(R)^{n(i)}A_i = A_i^e \) by Lemma 2.8, we have \( J_1(R)^{n(i)}A_i^{l_i} = T_i \). Hence (\( \ast \)) implies that \( J_1(R)^{l_1} \cdots J_k(R)^{l_k} \subseteq T_i \) by using Lemma 4.6. Write 1 = \( \sum x_iy_i \), where \( x_i \in J_1(R) \cdots J_k(R) \) and \( y_i \in T_i \). Since \( T_i = O_r(M_{i1}) \cdots O_r(M_{i(n-i)}) = O_r(M_{i1}) \cdots O_r(M_{i(n)}) = O_r(M_{i1}) \cdots O_r(M_{i(k)}) \), we have \( y_iB \subseteq R \). Thus \( B \subseteq J_1(R)^{l_1} \cdots J_k(R)^{l_k} \subseteq M_{11} \), a contradiction. Therefore \( m_i = e(n(i)) \) for all \( i \) and \( J(V)R = J(R)^n \) follows.

Corollary 4.11. Under the same notation and assumption as in Theorem 4.10, \( J(V)R = J(R)^n \) for some natural number \( n \) if and only if \( e(n(1)) = e(n(2)) = \cdots = e(n(k)) \).

We end this section with some examples to demonstrate some of the various phenomena we have discussed.

(1) Any Bezout \( V \)-orders are maximal and conjugate (see [10, Theorems 16.15, 17.3, and 17.5]). However, any semi-hereditary maximal \( V \)-orders are not necessarily conjugate. More precisely: if \( \operatorname{rank} V = \infty \), then there are infinite non-conjugate semi-hereditary maximal \( V \)-orders in \( M_2(F) \). To show this, let \( W \supseteq U \) be proper overrings of \( V \). Then

\[
T = O_r(M_{i1}) \cdots O_r(M_{i(n-i)-1}) = O_r(M_{i2}) \cdots O_r(M_{i(n)}) = O_r(M_{i1}) \cdots O_r(M_{i2}),
\]

\[T = O_r(B), \quad B = (M_{i1(1)} \cdots M_{i1(2)}) \cdots (M_{i(1)} \cdots M_{i(2)}), \]

we have \( y_iB \subseteq R \). Thus \( B \subseteq J_1(R)^{k_1} \cdots J_k(R)^{k_k} \subseteq M_{11} \), a contradiction. Therefore \( m_i = e(n(i)) \) for all \( i \) and \( J(V)R = J(R)^{n(1)} \cdots J_k(R)^{n(k)} \) follows. \( \Box \)

(2) If \( V \) is a discrete rank-one valuation ring, then any hereditary \( V \)-order has only one cycle of the first type ([17, Theorem 18.3] and Theorem 4.1), and so \( D(R) \cong \mathbb{Z} \), the ring of integers. However, if the rank of \( V \) is more than two, then there exists a semi-hereditary \( V \)-order which has more than two cycles of the first (second) type, according to \( J(V) \supset J(V)^2(J(V) = J(V)^2) \). Let \( W \) be a proper overring of \( V \). Set

\[
T = \begin{pmatrix} V & V & J(W) & J(W) \\ V & V & J(W) & J(W) \\ W & W & V & V \\ W & W & V & V \end{pmatrix}
\]

and \( K = \begin{pmatrix} J(V) & J(V) & J(W) & J(W) \\ J(V) & J(V) & J(W) & J(W) \\ W & W & J(V) & J(V) \\ W & W & J(V) & J(V) \end{pmatrix}. \)
$T$ is a semi-hereditary maximal $V$-order and $K$ is a semi-maximal right ideal of $T$ with $TK = T$ (see [6, Theorem 2.4 and Proposition 2.7]). Then

$$R = \mathfrak{I}_T(K) = \begin{pmatrix} V & J(V) & J(W) \\ V & V & J(W) \\ W & W & V & J(V) \end{pmatrix}$$

is a semi-hereditary $V$-order with maximal ideals series $M_{11}$, $M_{12}$, $M_{21}$, $M_{22}$, where

$$M_{11} = \begin{pmatrix} J(V) & J(V) & J(W) & J(W) \\ V & V & J(W) & J(W) \\ W & W & V & J(V) \\ W & W & V & V \end{pmatrix}, \quad M_{12} = \begin{pmatrix} V & J(V) & J(W) & J(W) \\ V & J(V) & J(W) & J(W) \\ W & W & V & J(V) \\ W & W & V & V \end{pmatrix}.$$

$$M_{21} = \begin{pmatrix} V & J(V) & J(W) & J(W) \\ V & V & J(W) & J(W) \\ W & W & J(V) & J(V) \\ W & W & V & V \end{pmatrix}, \quad M_{22} = \begin{pmatrix} V & J(V) & J(W) & J(W) \\ V & V & J(W) & J(W) \\ W & W & J(V) & J(V) \\ W & W & V & J(V) \end{pmatrix}.$$

Hence $R$ has two cycles of the first (second) type, according to $J(V) \supset J(V)^2$ ($J(V) = J(V)^2$). Since its form

$$(M_{11}, M_{12}, M_{21}, M_{22}) = (1, 1, 1, 1),$$

$J(R)$ is principal by Theorem 3.3 if $J(V) \supset J(V)^2$. Furthermore, set $J_i(R) = M_{11} \cap M_{i2}$ ($i = 1, 2$). Then $J(V)R = J_1(R)^2J_2(R)^2 = (J_1(R)J_2(R))^2 = (J_1(R) \cap J_2(R))^2 = J(R)^2$ by Theorem 4.10, because the ramification index of $T$ over $V$ is one.

The Jacobson radical $J(R)$ of any semi-hereditary $V$-order $R$ is not necessarily principal as it will be seen in the next example.

(3) Let

$$R = \begin{pmatrix} V & J(V) & J(V) \\ V & V & V \\ V & V & V \end{pmatrix}$$

be a semi-hereditary $V$-order with its form $(M_{11}, M_{12}) = (1, 2)$, where

$$M_{11} = \begin{pmatrix} J(V) & J(V) & J(V) \\ V & V & V \\ V & V & V \end{pmatrix} \quad \text{and} \quad M_{12} = \begin{pmatrix} V & J(V) & J(V) \\ V & J(V) & J(V) \\ V & J(V) & J(V) \end{pmatrix}.$$

So $J(R) = M_{11} \cap M_{12}$ is not principal.
(4) If $V$ is a discrete rank-one valuation ring then, for any hereditary $V$-order $R$, $J(V)R = J(R)^n$ for some natural number $n$. However, this not necessarily hold in case rank $V \geq 2$. To show this, let

$$R = \begin{pmatrix} V & J(V) & J(W) \\ W & V & J(W) \end{pmatrix}$$

be a semi-hereditary $V$-order with its form $(M_{11}, M_{12}, M_{21}) = (1, 1, 1)$, where $W$ is any proper overring of $V$ and

$$M_{11} = \begin{pmatrix} J(V) & J(V) & J(W) \\ V & V & J(W) \end{pmatrix}, \quad M_{12} = \begin{pmatrix} V & J(V) & J(W) \\ V & J(V) & J(W) \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} V & J(V) & J(W) \\ W & W & J(V) \end{pmatrix}.$$ 

So, by Corollary 4.11, $J(V)R \neq J(R)^n$ for any natural number $n$ if $J(V) \supset J(V)^2$.

(5)(a) $T = \begin{pmatrix} V & J(W) \\ W & V \end{pmatrix}$ is a simplest example of semi-hereditary maximal $V$-order that is not Bezout and whose ramification index over $V$ is one, where $W$ is any proper overring of $V$ by [15, Theorem 4.7 and Proposition 4.8].

(b) To give another example of semi-hereditary maximal $V$-order whose ramification index over $V$ is one, we recall Example 4 in [13]: Let $K$ be a field of $\text{char}(K) = p \neq 2$ and let $D = K[t]_tK[t]$, the localization of $K[t]$ at maximal ideal $tK[t]$, where $K[t]$ is the polynomial ring over $K$ in an indeterminate $t$. Set $F = Q(K[t])$, the quotient field of $K[t]$. Let $\sigma$ be an automorphism of $F$ defined by $\sigma(t) = -t$ and $\sigma(k) = k$ for any $k \in K$. Set $S = D[x]_x, D[x]$, the localization of $D[x]$ at $xD[x]$, where $D[x]$ is the polynomial ring over $D$ in an indeterminate $x$ and define the map $\varphi: S \to F$, $\varphi(s) = f(a)c(a)^{-1}$, where $s = f(x)c(x)^{-1}(f(x) \in D[x], c(x) \in D[x] \setminus xD[x])$. Then $\varphi$ is a ring epimorphism. Set $R = \varphi^{-1}(D)$, the complete inverse image of $D$ by $\varphi$. Then $R$ is a valuation ring with $\text{Spec}(R) = \{J(R) = J(D)R = tR, P_0 = \text{Ker} \varphi, (0)\}$. We extend $\sigma$ to an automorphism of $Q(F[x])$, the quotient field of $F[x]$ as follows: For any $f(x) = a_0x^n + \cdots + a_0 \in F[x], \sigma(f(x)) = \sigma(a_0)x^n + \cdots + \sigma(a_0)$ so that $\sigma^2 = 1, \sigma(S) = S$, and $\sigma(R) = R$. Let $G = \langle \sigma \rangle$, a group of order 2. Then skew group ring $R * G$ is a semi-hereditary $V$-order but not Prüfer in a simple Artinian ring $Q(R * G)$ with maximal ideals $M = (1 - \sigma)R + J(R) * G$, $N = (1 + \sigma)R + J(R) * G$ and $V = Z(R * G) = R^G = \{r \in R | \sigma(r) = r\}$.
by [13, Lemma 2.6, Example 4, and Remark to Example 4]. It is not hard to see from the
definition of $\sigma$ that

$$V = \varphi^{-1} \left( \frac{K[t^2]}{t^2, K[t^2]} \right) \quad \text{and} \quad J(V) = t^2 V,$$

that is, $J(V) \supset J(V)^2$. Furthermore, we have $M^2 = M$, $N^2 = N$, so that $M$ and $N$
must be the cycle of the first type by Proposition 2.1. Hence $O_t(M) = T = O_t(N)$
is a semi-hereditary maximal $V$-order. Let $e$ be the ramification index of $T$ over $V$.
Then it follows from Theorem 4.10 that $t^2(R \ast G) = J(V)(R \ast G) = (J(R \ast G)^2)^e = (J(R \ast G)^2)^e = t^{2e}R \ast G$, so that $e = 1$, because $J(R) = tR$, and $J(R) \ast G = J(R \ast G)$
by [12, Theorem 2.9] and [16, Theorem 4.2].

(c) To give a semi-hereditary maximal $V$-order but not Bezout whose ramification
index over $V$ is $n > 1$, let $W$ be a complete rank-one valuation ring of a field $K$
such that $W = W/J(W)$ is finite, let $D$ be a division ring with $Z(D) = K$ with $[D : K] = n^2$
and let $\Delta$ be an invariant valuation ring with $Z(\Delta) = W$. Then $J(W)\Delta = J(\Delta)^n$, that
is, $n$ is the ramification index of $\Delta$ over $W$ and $\overline{\Delta} : W = n$ by [17, Theorem 14.3],
where $\overline{\Delta} = \Delta/J(\Delta)$. As in (b), let $S = D[x_1, D[x_1]$ and the map $\varphi : S \to D$ defined by
$\varphi(s) = f(\phi)c(\phi)^{-1}$, where $s = (f(x)c(x)^{-1} \in S(f(x) \in D[x_1), c(x) \in D[x_1), xD[x_1])$. Then
$R = \varphi^{-1}(\Delta)$ is a total valuation ring with Spec$(R) = [J(R) = J(\Delta)R, P_0 = \ker \varphi, (0)]$ by
[1, Proposition 3.4]. Similarly, $V = \varphi^{-1}(W)$ is a valuation ring with Spec$(V) = [J(V) = J(W)R, p_0 = \ker \varphi \cap K[x_1, E[x_1], (0)]$. It is not hard to see that $Z(R) = V$ and that $R$
is an invariant valuation ring. So $R$ is a $V$-order by [10, Corollary 8.6]. Set

$$T = \begin{pmatrix} R & J(S) \\ S & R \end{pmatrix},$$

a semi-hereditary maximal $V$-order with $J(T) = J(R)T$ by [15, Theorem 4.7 and
Proposition 4.8]. Hence $J(V)T = J(W)VT = J(W)\Delta T = J(\Delta)^nT = J(R)^nT = J(T)^n$;
that is, the ramification index of $T$ over $V$ is $n$.

Acknowledgment

The author expresses his gratitude to the referee for his/her valuable comments on the
previous version which makes the paper readable.

References

[1] H.H. Brungs, H. Marubayashi, A. Ueda, A classification of primary ideals of Dubrovin valuation rings,