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On the kth derivative of meromorphic functions with zeros of multiplicity at least $k+1$

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ABSTRACT

In this paper, we prove the following

Theorem. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$ $(k \geqslant 2)$, except possibly finitely many, and all of whose poles are multiple, except possibly finitely many, and let the function $a(z) = P(z)\exp(Q(z)) \not\equiv 0$, where P and Q are polynomials such that $\overline{\lim}_{r\to\infty}(\frac{T(r,a)}{T(r,f)} + \frac{T(r,f)}{T(r,a)}) = \infty$. Then the function $f^{(k)}(z) - a(z)$ has infinitely many zeros.*

1. Introduction

In his excellent paper [2], W.K. Hayman studied the value distribution of certain meromorphic functions and their derivatives under various conditions. Among other important results, he proved that if $f(z)$ is a transcendental meromorphic function in the plane, then either $f(z)$ assumes every finite value infinitely often, or every derivative of $f(z)$ assumes every finite nonzero value infinitely often. This result is known as "Hayman's alternative." Thereafter, the value distribution of derivatives of transcendental functions continued to be studied.

In [9], Wang and Fang proved the following result.

Theorem WF. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least 3, then for all integer numbers $k \geqslant 1$, $f^{(k)}$ assumes every finite nonzero value infinitely often.*

Then, in [1], Bergweiler and Pang proved

Theorem BP. *Let f be a transcendental meromorphic function and $R \not\equiv 0$ be a rational function. If all zeros and poles of f are multiple, except possibly finitely many, then $f' - R$ has infinitely many zeros.*

In this paper, we continue to study omitted functions of derivatives of meromorphic functions. As a result, we have the following theorem for functions of infinite order.

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Theorem 1. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$ of infinite order $\rho(f)$, and $a(z) = P(z)\exp(Q(z)) \not\equiv 0$, where P and Q are polynomials. Let also $k \geqslant 2$ be an integer. Suppose that*

(C_1) *all zeros of f have multiplicity at least $k+1$, except possibly finitely many, and*
(C_2) *all poles of f are multiple, except possibly finitely many.*

Then the function $f^{(k)}(z) - a(z)$ has infinitely many zeros.

For functions of finite order, we have the following result.

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function on $\mathbb{C}$ of finite order $\rho(f)$, and $a(z) = P(z)\exp(Q(z)) \not\equiv 0$, where P and Q are polynomials. Let also $k \geqslant 2$ be an integer. Suppose that*

(C_1) *all zeros of f have multiplicity at least $k+1$, except possibly finitely many, and*
(C_2) $\overline{\lim}_{r\to\infty}\left(\frac{T(r,a)}{T(r,f)} + \frac{T(r,f)}{T(r,a)}\right) = \infty$.

Then the function $f^{(k)}(z) - a(z)$ has infinitely many zeros. Moreover, in the case that $\rho(f) \notin N$, then the result holds with condition (C_2) only.

Remarks. (i) Note that condition (C_2) of Theorem 2 is equivalent to the following condition:

($\tilde{C}_2$) There are no $M_1, M_2 > 0$, such that $M_1 T(r,a) \leqslant T(r,f) \leqslant M_2(T(r,a)$ for large enough r.

(ii) Condition (C_2) of Theorem 2 is sharp; for example, $f(z) = \exp(z^2)$, $a(z) = \exp(z^2)^{(k)} + e^{z^2}$.
(iii) Condition (C_2) of Theorem 2 is automatically fulfilled if $\rho(f) = \infty$.

Notation. Let $\Delta(z_0, r) := \{z\colon\ |z - z_0| < r\}$, $C(z_0, r) := \{z\colon\ |z - z_0| = r\}$, $V(z_0, \theta_0, A) := \{z\colon\ |\arg(z - z_0) - \theta_0| < A\}$. Let D be a domain in $\mathbb{C}$ and let $\{f_n\}$ be a sequence of meromorphic functions in D. We write $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in D to indicate that $\{f_n\}$ converges spherically uniformly to the limit function f on compact subsets of D. If $\{f_n\}$ is analytic in D, we write $f_n \Rightarrow f$ in D. If S is the angular domain $V(z_0, \theta_0, A)$, $C \in \hat{\mathbb{C}}$ and $f(z)$ is analytic in S for large enough $|z|$, we write $f(z) \overset{\forall}{\Rightarrow} C$ in S to indicate that $f(z)$ tends uniformly to the constant $C \in \hat{\mathbb{C}}$ as $z \to \infty$ in S.

2. Auxiliary results for the proof of Theorem 1

Lemma 1. *Let $k \geqslant 1$ be an integer and let $\{f_n\}$ be a family of functions meromorphic on Δ, all of whose zeros have multiplicity at least $k+1$. If $a_n \to a$, $|a| < 1$, and $f_n^{\#}(a_n) \to \infty$, then there exist*

(i) *a subsequence of $\{f_n\}$ (which we still write as $\{f_n\}$);*
(ii) *points $z_n \to z_0$, $|z_0| < 1$;*
(iii) *positive numbers $\rho_n \to 0$ such that*
(iv) $g_n(\zeta) := \frac{f_n(z_n+\rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} g(\zeta)$ *in* $\mathbb{C}$,

where g is a nonconstant meromorphic function on $\mathbb{C}$, such that $g^{\#}(\zeta) \leqslant g^{\#}(0) = k+1$, and $\rho_n \leqslant \frac{M}{\sqrt[k+1]{f_n^{\#}(a_n)}}$, where M is a constant which is independent on n.

The innovation of this lemma, comparing it to Lemma 2 of [6], Lemma 1 of [5] (or comparing it to the original Zalcman Lemma, see [11] or [12]) is that given information about the rate of growth of the spherical derivatives of the members of the sequence $\{f_n\}$ on some compact subset of the unit disc, we get an estimation to the size of the ρ_n's in the vicinity of some point of nonnormality, and this helps to estimate $f_n(z_n + \rho_n\zeta)$ when the f_n's are known. For related issues, the reader is referred also to [7].

Proof. There exists $0 < r^* < 1$ such that $|a_n| < r^*$, $\forall n$. Take r, $r^* < r < 1$. Since $f_n^{\#}(a_n) \to \infty$, then

$$S_n := \frac{(1 - (\frac{|a_n|}{r})^2)^{k+1}|f_n'(a_n)|}{(1 - |\frac{a_n}{r}|^2)^{2k} + |f_n(a_n)|^2} \geqslant \left(1 - \left|\frac{a_n}{r}\right|^2\right)^{k+1} f_n^{\#}(a_n) \to \infty,$$

and thus $S_n > k+1$ (for large enough n, without loss of generality, for every n). By Lemma 1 in [6], there exists for each n a point z_n, $|z_n| < r$ and $0 < t_n < 1$ such that

$$\sup_{|z|<r}\frac{(1-|\frac{z}{r}|^2)^{k+1}t_n^{k+1}|f_n'(z)|}{(1-|\frac{z}{r}|^2)^{2k}t_n^{2k}+|f_n(z)|^2}=\frac{(1-|\frac{z_n}{r}|^2)^{k+1}t_n^{k+1}|f_n'(z_n)|}{(1-|\frac{z_n}{r}|^2)^{2k}t_n^{2k}+|f_n(z_n)|^2}=k+1. \tag{1}$$

In particular,

$$k+1\geqslant\frac{(1-|\frac{a_n}{r}|^2)^{k+1}t_n^{k+1}|f_n'(a_n)|}{(1-|\frac{a_n}{r}|^2)^{k+1}t_n^{2k}+|f_n(a_n)|^2}\geqslant\left(1-\left|\frac{a_n}{r}\right|^2\right)^{k+1}t_n^{k+1}f_n^{\#}(a_n) \tag{2}$$

and thus $t_n\to 0$.

Set $\rho_n=(1-|\frac{z_n}{r}|^2)t_n$, then $\rho_n=\frac{1-|\frac{z_n}{r}|^2}{1-|\frac{a_n}{r}|^2}(1-|\frac{a_n}{r}|^2)t_n$. By (2) we have $\rho_n\leqslant\frac{1}{1-(\frac{r^*}{r})^2}\frac{\sqrt[k+1]{k+1}}{\sqrt[k+1]{f_n^{\#}(a_n)}}\leqslant\frac{\mu}{\sqrt[k+1]{f_n^{\#}(a_n)}}$, where $\mu=\frac{\sqrt[3]{3}}{1-(\frac{r^*}{r})^2}$. Now we continue by following the proof of Lemma 2 in [6].

We have

$$\frac{\rho_n}{r-|z_n|}\to 0, \tag{3}$$

and then the functions $g_n(\zeta):=f_n(z_n+\rho_n\zeta)/\rho_n^k$ are defined for $|\zeta|\leqslant R_n$, where $R_n=\frac{r-|z_n|}{\rho_n}\to\infty$. A calculation yields

$$\frac{|g_n'(\zeta)|}{1+|g_n(\zeta)|^2}=\frac{(1-|z_n/r|^2)^{k+1}t_n^{k+1}|f_n'(z_n+\rho_n\zeta)|}{(1-|z_n/r|^2)^{2k}t_n^{2k}+|f_n(z_n+\rho_n\zeta)|^2}, \tag{4}$$

so by (1)

$$g_n^{\#}(0)=\frac{|g_n'(0)|}{1+|g_n(0)|^2}=k+1. \tag{5}$$

For $|\zeta|\leqslant R<R_n$, we have

$$|z_n|^2-2\rho_nR-\rho_n^2R^2\leqslant|z_n+\rho_n\zeta|^2\leqslant|z_n|^2+2\rho_nR+\rho_n^2R^2.$$

It follows from (3) that $(r^2-|z_n|^2)/(r^2-|z_n+\rho_n\zeta|^2)$ tends uniformly to 1 on compact subsets of $\mathbb{C}$.

Now fix R and let $\varepsilon>0$. Then for n sufficiently large, we have by (1) and (4)

$$g_n^{\#}(\zeta)=\frac{|g_n'(\zeta)|}{1+|g_n(\zeta)|^2}\leqslant\frac{(1+\varepsilon)(1-|(z_n+\rho_n\zeta)/r|^2)^{k+1}t_n^{k+1}|f_n'(z_n+\rho_n\zeta)|}{(1-|(z_n+\rho_n\zeta)/r|^2)^{2\alpha}t_n^{2\alpha}+|f_n(z_n+\rho_n\zeta)|^2}\leqslant(1+\varepsilon)(k+1). \tag{6}$$

Thus, by Marty's Theorem, $\{g_n\}$ is a normal family in $\mathbb{C}$. Taking a subsequence and renumbering, we may assume that the g_n converge locally uniformly on compacta to a limit function g. It is evident from (5) and (6) that $g^{\#}(0):=k+1$ (so that g is nonconstant) and $g^{\#}(\zeta)\leqslant k+1$ for all ζ. This completes the proof. □

Lemma 2. *Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n\to\infty$, such that for every $N>0$, $g^{\#}(z_n)>|z_n|^N$ if n is sufficiently large.*

Proof. Suppose this were not the case. Then there exist $N>0$ and $R>0$ such that for all z, $|z|\geqslant R$, we have $f^{\#}(z)<|z|^N$. So

$$\begin{aligned}S(r,f)&=\frac{1}{\pi}\iint\limits_{|z|<r}f^{\#}(z)^2\,d\sigma=\frac{1}{\pi}\iint\limits_{R\leqslant|z|<r}f^{\#}(z)^2\,d\sigma+O(1)\leqslant\frac{1}{\pi}\iint\limits_{R\leqslant|z|<r}|z|^{2N}\,d\sigma+O(1)=\frac{1}{\pi}\int_0^{2\pi}d\theta\int_0^{2\pi}\int_R^r t\cdot t^{2N}\,dt+O(1)\\&=\frac{1}{N+1}\left(r^{2N+2}-R^{2N+2}\right)+O(1)=\frac{1}{N+1}r^{2N+2}+O(1).\end{aligned}$$

By the definition of Ahlfors characteristic of g, we have

$$T(r,f)=\int_0^r\frac{S(t,f)}{t}\,dt\leqslant\frac{1}{(N+1)(2N+2)}r^{2N+2}+O(\log r).$$

Thus, $\rho(f)=\overline{\lim}_{r\to\infty}\frac{\log T(r,f)}{\log r}\leqslant 2N+2$, which contradicts the fact that f is of infinite order. □

Lemma 3. *Let f be a nonconstant meromorphic function of finite order on $\mathbb{C}$, all of whose zeros have multiplicity at least $k+1$. If $f^{(k)}(z)\neq 1$ on $\mathbb{C}$, then*

$$f(z)=\frac{1}{k!}\frac{(z-a)^{k+1}}{z-b}\quad\textit{for some } a,b\in\mathbb{C},\ a\neq b.$$

This lemma follows easily from Lemmas 6 and 8 in [9]; see also [4, Lemma 4].

Lemma 4. *Let $R(z) \not\equiv 0$ be a rational function. Then there exists $k > 0$, such that for large enough z, $|zR'(z)| \leqslant k|R(z)|$.*

This lemma is obvious.

3. Proof of Theorem 1

We assume by negation that the equation $f^{(k)}(z) = a(z)$ has finitely many zeros. This means that

$$\frac{f^{(k)}(z)}{a(z)} \neq 1 \tag{7}$$

for large enough z.

Set $F(z) = \frac{f(z)}{a(z)}$, and write $b(z) = \frac{1}{a(z)} = \frac{1}{P(z)} e^{-Q(z)} = P_1(z)e^{Q_1(z)}$, we have

$$F^{(j)}(z) = \big(f(z)b(z)\big)^{(j)} = \sum_{i=0}^{j} C_j^i f^i(z) b^{(j-i)}(z).$$

Computation yields

$$\begin{pmatrix} 0 & 0 & \dots & 0 & b(z) \\ 0 & 0 & \dots & b(z) & b'(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b(z) & C_k^1 b'(z) & \dots & C_k^{k-1} b^{(k-1)}(z) & b^{(k)}(z) \end{pmatrix} \begin{pmatrix} f^{(k)}(z) \\ f^{(k-1)}(z) \\ \vdots \\ f(z) \end{pmatrix} = \begin{pmatrix} F(z) \\ F'(z) \\ \vdots \\ F^{(k)}(z) \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} f^{(k)}(z) \\ f^{(k-1)}(z) \\ \vdots \\ f(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & b(z) \\ 0 & 0 & \dots & b(z) & b'(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b(z) & C_k^1 b'(z) & \dots & C_k^{k-1} b^{(k-1)}(z) & b^{(k)}(z) \end{pmatrix}^{-1} \begin{pmatrix} F(z) \\ F'(z) \\ \vdots \\ F^{(k)}(z) \end{pmatrix}$$
$$= \frac{(-1)^{[\frac{k+1}{2}]}}{b^{k+1}} \begin{pmatrix} 0 & 0 & \dots & 0 & b(z) \\ 0 & 0 & \dots & b(z) & b'(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b(z) & C_k^1 b'(z) & \dots & C_k^{k-1} b^{(k-1)}(z) & b^{(k)}(z) \end{pmatrix}^{*} \begin{pmatrix} F(z) \\ F'(z) \\ \vdots \\ F^{(k)}(z) \end{pmatrix}. \tag{9}$$

So we obtain

$$f^{(k)}(z)b(z) = \sum_{j=0}^{k} L_j(z) F^{(k-j)}(z), \tag{10}$$

where $L_0(z) \equiv 1$.

Observe that the $(1, k+1)$ element in the adjoint matrix in the right-hand side of (9) is $(-1)^{k+[\frac{k}{2}]}$, but $L_0(z) \equiv 1$ is also obvious from (8) and $L_j(z)$ is a polynomial of $b'(z)/b(z), \dots, b^{(j)}(z)/b(z)$ $(1 \leqslant j \leqslant k)$. Next we should calculate $b^{(j)}(z)/b(z)$. Since $b(z) = P_1(z)\exp(Q_1(z))$, we have $b^{(j)}(z) = \sum_{\ell=0}^{j} C_j^{\ell} P_1^{(\ell)}(z)(\exp(Q_1(z)))^{(j-\ell)}$, and $b^{(j)}(z)/b(z)$ is a polynomial of $P_1^{(\ell)}(z)/P_1(z)$ and $Q_1^{(\ell)}(z)$ $(\ell = 1, 2, \dots, j)$. Since $\rho(f) = \rho(F) = \infty$, then by Lemma 2, there exist points $\{z_n\}$, $z_n \to \infty$ such that for every $N > 0$,

$$F^{\#}(z_n) > |z_n|^N \quad \text{if } n \text{ is large enough.} \tag{11}$$

By Marty's Theorem, the family of meromorphic functions $\{F(z + z_n)\}$ is not normal at $z = 0$, hence it is not normal in Δ. Also, since $a(z)$ has only finitely many zeros and poles, all the zeros of $F(z + z_n)$ in Δ have multiplicity at least $k + 1$, and poles of which are multiple if n is sufficiently large. Thus, by Lemma 1 there exist points $\{z_n'\}$, $|z_n'| < r < 1$; positive numbers $\rho_n \to 0^+$,

$$\rho_n \leqslant \frac{M}{\sqrt[k+1]{F^{\#}(z_n)}}, \tag{12}$$

such that

$$g_n(\zeta) := \frac{F(z_n + z'_n + \rho_n\zeta)}{\rho_n^k} \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{in } \mathbb{C}, \tag{13}$$

where g is a nonconstant meromorphic function on $\mathbb{C}$, all zeros of which have multiplicity at least $k+1$ and all poles of which are multiple.

(In fact, we can also ensure that $z'_n \to 0$, but this is not needed.)

Given K, a compact subset of $\mathbb{C}$, by (7), (10) and (13), we have for $\zeta \in K$,

$$\begin{aligned} 1 &\neq \frac{f^{(k)}(z_n + z'_n + \rho_n\zeta)}{a(z_n + z'_n + \rho_n\zeta)} \\ &= F^{(k)}(z_n + z'_n + \rho_n\zeta) + L_1(z_n + z'_n + \rho_n\zeta)F^{(k-1)}(z_n + z'_n + \rho_n\zeta) + \cdots + L_k(z_n + z'_n + \rho_n\zeta)F(z_n + z'_n + \rho_n\zeta) \\ &= g_n^{(k)}(\zeta) + \rho_n L_1(z_n + z'_n + \rho_n\zeta)g_n^{(k-1)}(\zeta) + \cdots + \rho_n^k L_k(z_n + z'_n + \rho_n\zeta)g_n(\zeta) \end{aligned} \tag{14}$$

for sufficiently large n.

We show now that for $1 \leqslant j \leqslant k$,

$$\rho_n^j L_j(z_n + z'_n + \rho_n\zeta) \to 0 \quad \text{uniformly as } \zeta \to \infty \text{ in } \mathbb{C}. \tag{15}$$

We have by Lemma 4

$$\begin{aligned} &\frac{P_1^{(j)}(z_n + z'_n + \rho_n\zeta)}{P_1(z_n + z'_n + \rho_n\zeta)} = O\left(\frac{1}{z_n^j}\right), \\ &Q_1^{(j)}(z_n + z'_n + \rho_n\zeta) = O(z_n^{|Q|-j}). \end{aligned} \tag{16}$$

It follows by the structure of $L_j(z)$ and (16) that it suffices if

$$\rho_n^j |z_n|^{|Q|-j} \underset{n\to\infty}{\Longrightarrow} 0 \quad \text{for } 1 \leqslant j \leqslant k. \tag{17}$$

By (11) and (12), we have for every $N > 0$,

$$\rho_n^j |z_n|^{|Q|-j} \leqslant M|z_n|^{|Q|-j-\frac{jN}{k+1}} \quad (1 \leqslant j \leqslant k) \tag{18}$$

for large enough n.

On the other hand,

$$\max_{1\leqslant j\leqslant k}\left(|Q| - j - \frac{jN}{k+1}\right) = |Q| - 1 - \frac{N}{k+1},$$

so (18) implies that (17) holds and so (15) holds. Thus, we have

$$g_n^{(k)}(\zeta) + C_n L_1(z_n + z'_n + \rho_n\zeta)g_n^{(k-1)}(\zeta) + \cdots + \rho_n^k L_k(z_n + z'_n + \rho_n\zeta)g_n(\zeta) \Rightarrow g^{(k)}(\zeta)$$

in $\mathbb{C} \setminus \mathbb{P}$, where $\mathbb{P}$ is the set of poles of $g(\zeta)$ in $\mathbb{C}$. Now, if $g^{(k)}(\zeta_0) = 1$ for some $\zeta_0 \in \mathbb{C}$, then by (14), $g^{(k)}(\zeta) \equiv 1$, and so g is a polynomial of degree k, but this contradicts the fact that the zeros of g are of multiplicity at least $k+1$. Thus we have $g^{(k)}(\zeta) \neq 1$, and by Lemma 3, $g(\zeta) = \frac{1}{k!}\frac{(\zeta-a)^{k+1}}{\zeta-b}$, where $a \neq b$ are two complex numbers. But this contradicts the fact that all poles of g are multiple. This completes the proof of Theorem 1.

4. Auxiliary results for the proof of Theorem 2

Lemma 5. *Let $R(z) \not\equiv 0$ be a rational function and let $Q(z) = -z^n + C_{n-1} + \cdots + C_0$ be a polynomial ($n \geqslant 1$). Then for every $0 < \varepsilon < \frac{\pi}{2n}$, the function $h_z(t) = |R(tz)\exp(Q(tz))|$ is decreasing in $\{t \geqslant 1\}$ for every $|z| > L = L(\varepsilon)$ in the domain $S = V(0, 0, \frac{\pi}{2n} - \varepsilon)$.*

Proof. Denote $z = re^{i\theta}$. Let $R(z) = \frac{z^\ell + a_{\ell-1}z^{\ell-1} + \cdots + a_0}{b_m z^m + \cdots + b_0}$, $b_m \neq 0$. Then

$$\begin{aligned} h_z(t) &= |R(tz)| \cdot \exp(\operatorname{Re} Q(tz)) \\ &= |R(tz)| \exp\{\operatorname{Re}[-r^n t^n(\cos(n\theta) + i\sin(n\theta)) + C_{n-1}r^{n-1}t^{n-1}(\cos((n-1)\theta) + i\sin((n-1)\theta)) + \cdots + C_0]\}. \end{aligned}$$

It is enough to prove that for sufficiently large z in S, $\frac{h_z(t+\Delta t)}{h_z(t)} < 1$ for small enough positive Δt. There are $d_1, \ldots, d_{n-1} \in \mathbb{R}$ such that

$$\frac{h_z(t+\Delta t)}{h_z(t)} = \left|\frac{R((t+\Delta t)z)}{R(tz)}\right| \cdot \exp\bigl[-r^n\bigl((t+\Delta t)^n - t^n\bigr)\cos(n\theta) + d_{n-1}r^{n-1}\bigl((t+\Delta t)^{n-1} - t^{n-1}\bigr) + \cdots$$
$$+ d_1 r\bigl((t+\Delta t) - t\bigr)\bigr]$$
$$= \left|\frac{R((t+\Delta t)z)}{R(tz)}\right| \cdot \exp\left[-n\cos(n\theta) r^n t^{n-1}\Delta t + r^n \sum_{k=2}^{n} e_{k,n} t^{n-k}\Delta t^k + r^{n-1}\sum_{k=1}^{n-1} e_{k,n-1}(\Delta t)^k t^{n-1+k} + \cdots\right.$$
$$\left. + e_{1,1} r\Delta t\right],$$

where $e_{k,n} = C_n^k$, $2 \leqslant k \leqslant n$, $e_{1,1} = d_1$ and $\{e_{j,\ell}\colon\ 2 \leqslant \ell \leqslant n-1,\ 1 \leqslant j \leqslant \ell\}$ are real numbers. Set $A := A(\varepsilon) = \cos(\frac{\pi}{2} - n\varepsilon) > 0$, then the last expression is

$$\frac{R((t+\Delta t)z)}{R(tz)} \exp\left[-Ar^n t^{n-1}\Delta t\left(1 + O(\Delta t) + O\left(\frac{1}{r}\right)\right)\right] \quad (\Delta t \to 0,\ r \to \infty)$$
$$< \left|\frac{R((t+\Delta t)z)}{R(tz)}\right| \exp\left(-\frac{A}{2} r^n t^{n-1}\Delta t\right).$$

Claim. *There exists $k > 0$ such that for $t \geqslant 1$ and large enough z, $|\frac{R((t+\Delta t)z)}{R(tz)}| < 1 + k\Delta t$, for small enough Δt.*

Proof. Obviously, it is enough to consider the case when $R(z)$ is a polynomial. So assume $R(z) = a_n z^n + \cdots + a_0$. We have

$$\frac{R((t+\Delta t)z)}{R(tz)} = \frac{a_n((t+\Delta t)z)^n[1 + \frac{a_{n-1}}{(t+\Delta t)z} + \cdots + \frac{a_0}{((t+\Delta t)z)^n}]}{a_n t^n z^n[1 + \frac{a_{n-1}}{tz} + \cdots + \frac{a_0}{(tz)^n}]} = \left(1 + \frac{\Delta t}{t}\right)^n \cdot \frac{1 + \sum_{k=0}^{n-1} \frac{a_k}{((t+\Delta t)z)^{n-k}}}{1 + \sum_{k=0}^{n-1} \frac{a_k}{(tz)^{n-k}}}.$$

For each $0 \leqslant k \leqslant n-1$, when $\Delta t \to 0$ (since $t \geqslant 1$ and $|z|$ is big), we have

$$\frac{a_k}{((t+\Delta t)z)^{n-k}} = \frac{a_k}{(tz)^{n-k}(1 + \frac{\Delta t}{t})^{n-k}} = \frac{a_k}{(tz)^{n-k}}\bigl(1 + O(\Delta t)\bigr).$$

Thus,

$$\frac{R((t+\Delta t)z)}{R(tz)} = \left(1 + \frac{\Delta t}{t}\right)^n \bigl(1 + O(\Delta t)\bigr),$$

and the claim is proved. □

Thus, if r is such that $\frac{A}{2}r^n > 2k$, then for small enough Δt,

$$\left|\frac{R((t+\Delta t)z)}{R(tz)}\right| \exp\left(-\frac{A}{2} r^n t^{n-1}\Delta t\right) < (1 + k\Delta t)\exp(-2k\Delta t) < (1 + k\Delta t)(1 - k\Delta t) < 1,$$

and Lemma 5 is proved. □

Lemma 6. *If $f(z)$ is a meromorphic function in the finite plane, then*

$$T(r, f) < O\bigl\{T(2r, f') + \log r\bigr\}, \quad r \to \infty.$$

This lemma is a corollary to Chuang Chi-Tai's inequality [10, pp. 95–96].

Lemma 7. *Let $h(z)$ be analytic in $S = V(z_0, \theta_0, A)$ for large enough $|z|$. Suppose that $h(z) \overset{\forall}{\Rightarrow} k \in \mathbb{C}$ in S_ε, for every $0 < \varepsilon < A$, where $S_\varepsilon := V(z_0, \theta_0, A - \varepsilon)$. Then $zh'(z) \overset{\forall}{\Rightarrow} 0$ in S_ε for every $0 < \varepsilon < A$.*

Proof. Without loss of generality, assume that $k = 0$. Let $0 < \varepsilon < A$. Then $h(z) \overset{\forall}{\Rightarrow} 0$ in $S_{\varepsilon/2}$. Let $z \in S_{\varepsilon/2}$ and denote $c = c(\zeta_1, |z| tg\frac{\varepsilon}{2})$. Then

$$\bigl|h'(z)\bigr| = \left|\frac{1}{2\pi i}\int_C \frac{h(\zeta)}{(\zeta - z)^2}\,d\zeta\right| \leqslant \frac{|z| tg\frac{\varepsilon}{2}\max_{\zeta\in C}|h(\zeta)|}{|z|^2 tg^2\frac{\varepsilon}{2}}.$$

So $|zh'(z)| \leqslant \frac{\max_{\zeta\in\mathbb{C}}|h(\zeta)|}{tg\frac{\varepsilon}{2}}$, and the lemma is proved. □

We also need the following lemma.

Lemma 8. *(See [10, p. 25].) If $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$, then $\underline{\lim}_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$.*

The following lemma is due to H. King-lai.

Lemma 9. *(See [10, p. 99].) Let $f(z)$ be a meromorphic function in $\{|z| < R\}$, $R \leqslant \infty$. If $f(0) \neq 0, \infty$, then for every $k \in \mathbb{N}$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) < C_k\left\{1 + \log^+\log^+\frac{1}{|f(0)|} + \log+\frac{1}{r} + \log+\frac{1}{\rho - r} + \log^+\rho + \log^+ T(\rho, f)\right\},$$

where $0 < r < \rho < R$ and C_k is a constant depending only on k.

We shall also use the following result.

Theorem L. *(See J.K. Langley [3].) Let f be a meromorphic function of finite order in $\mathbb{C}$ and let $k \geqslant 2$ be an integer, such that the kth derivative $f^{(k)}$ has finitely many zeros. Then f has finitely many poles.*

The Phragmen–Lindelöf Principle, presented in the following two theorems, will play a central role in our proof.

Theorem PL1. *(See [8, p. 177].) Let f be analytic in $D = V(0, 0, \frac{\pi}{2\lambda})$. Suppose that $\log\mu(r, f) \overset{as}{<} r^{\rho}$ for some $\rho < \lambda$. If for every $\zeta \in \partial D$, $\overline{\lim}_{z\to\zeta,\, z\in D} |f(z)| \leqslant M$, then $|f(z)| \leqslant M$ in D.*

Here $\mu(r, f) = \sup_{-\frac{\pi}{2\lambda}<\theta<\frac{\pi}{2\lambda}} |f(re^{i\theta})|$.

Theorem PL2. *(See [8, p. 179].) If $f(z) \to a$ along two rays and f is bounded and analytic in the angle between them, then $f(z) \underset{z\to\infty}{\to} a$ uniformly in the whole angle.*

5. Proof of Theorem 2

We divide into two cases.

Case (A). *f has infinitely many poles.* There exists a holomorphic function $T(z)$ such that $T^{(k)}(z) = a(z)$ and since the poles of f are exactly the poles of $f - T$, we have by Theorem L that the equation $(f(z) - T(z))^{(k)} = 0$ has infinitely many roots, so $f^{(k)}(z) - a(z)$ takes the value 0 infinitely many times.

Case (B). *f has finitely many poles.* If, to the contrary, $f^{(k)}(z) - a(z)$ has only finitely many zeros, then we have

$$f^{(k)}(z) = P_1(z)\exp\bigl(Q_1(z)\bigr) + P_2(z)\exp\bigl(Q_2(z)\bigr), \tag{19}$$

where $P_1 = P$, $Q_1 = Q$, P_2 is a rational function and Q_2 is a polynomial.

Case (B) is now divided into two subcases.

Case (BI). *Suppose that $\rho(f)$ is a fraction.* Since $\rho(a)$ is an integer, $\rho(f) \neq \rho(a)$. If $\rho(f) < \rho(a)$, then if $|Q_1| \neq |Q_2|$, we have a contradiction to (19). If $|Q_1| = |Q_2|$, then they must be positive integers. In this case, also the leading coefficients in Q_1 and in Q_2 must be equal, because otherwise, the order of the right-hand side of (19) is $|Q_1|$, a contradiction. So assume that the leading coefficient in Q_1 and in Q_2 is a_1. Then by multiplying (19) in $\exp(-a_1 z^{|Q_1|})$, we get a contradiction by comparing the order of both sides of the resulting identity. If $\rho(f) > \rho(a)$, then we get by (19) that $\rho(f) = \rho(P_2\exp(Q_2))$, and this is impossible since $\rho(P_2\exp(Q_2))$ is an integer.

Case (BII). *Suppose now that $\rho(f)$ is an integer.* Separate into cases.

(i) $|Q_1| > |Q_2,|$. Then

$$T(r,a) \sim M_1 r^{|Q_1|} \quad \text{for some } M_1 > 0, \tag{20}$$

and by (19) also

$$T\bigl(r, f^{(k)}\bigr) \sim M_1 r^{|Q_1|} \quad \text{as } r \to \infty. \tag{21}$$

Now, by Lemma 6, for all $r > 0$, we have

$$T(r, f) < C_k T\bigl(2^k r, f^{(k)}\bigr) + D_k \log r + E_k \quad \text{for some positive constants } C_k, D_k, E_k. \tag{22}$$

By (21), we have $T(2^k r, f^{(k)}) = O(r^{|Q_1|})$ and then by (20) and (22), we get

$$T(r, f) = O\bigl(T(r,a)\bigr). \tag{23}$$

Also by Lemmas 8 and 9,

$$T\big(r, f^{(k)}\big) = O\big(T(r, f)\big). \tag{24}$$

So from (20), (21) and (24), we have

$$T(r, a) = O\big(T(r, f)\big). \tag{25}$$

By (23) and (25), we get a contradiction to condition (C_2) of Theorem 2.

(ii) $|Q_1| = |Q_2|$. If $|Q_1| = |Q_2| = 0$, then $f^{(k)}(z)$ is a rational function and so is $f(z)$. (The theorem holds then if and only if $f(z) \equiv C$, $|C| > 1$ and $a(z) \equiv 0$.) If $|Q_1| = |Q_2| > 0$, then if $\rho(f) = \rho(f^{(k)}) < |Q_1|$, then the leading coefficients of $Q_1(z)$ and $Q_2(z)$ must be equal, say a_1, and we get a contradiction by multiplying (19) in $\exp(-a_1 z^{|Q_1|})$. The case $\rho(f) > |Q_1|$ is impossible by (19). Suppose $\rho(f) = |Q_1|$, then if the leading coefficients of $Q_1(z)$ and $Q_2(z)$ were not equal, we would deduce that $r^{|Q_1|} = O(T(r, f))$.

Hence (25) holds (and also (23)), and we have again a contradiction to condition (C_2). So the leading coefficients of $Q_1(z)$ and $Q_2(z)$ must be equal. In this case we have again that (23) and (25) hold and we get a contradiction. (The possibility of $f^{(k)}(z) = 0$ is of course excluded.)

Observe that running over Case (BI) and on the case $|Q_1| = |Q_2| = 0$ in (ii) of Case (BII), show that in the case $\rho(f) = 0$, the theorem holds under condition (C_2) alone.

So we are left with the case

(iii) $|Q_2| > |Q_1|$. Let $m_1 = |Q_1|$, $m_2 = |Q_2|$.

Without loss of generality, we may assume that $Q_2(z) = -z^{m_2} + \cdots$. Suppose first that f has finitely many zeros. Then $f(z) = R(z)\exp(\tilde{Q}(z))$, where $R(z)$ is a rational function and $\tilde{Q}(z)$ is a polynomial, with $|\tilde{Q}| = m_2$. Then $f^{(k)}(z) = \tilde{R}(z)\exp(\tilde{Q}(z))$, where $\tilde{R}(z)$ is a rational function. If $f^{(k)}(z) - a(z)$ has only finitely many zeros in $\mathbb{C}$, then

$$\tilde{R}(z)\exp\big(\tilde{Q}(z)\big) - P_1(z)\exp\big(Q_1(z)\big) = P_2(z)\exp\big(Q_2(z)\big). \tag{26}$$

We must have that $|\tilde{Q}| = m_2$ and that the leading coefficient in $\tilde{Q}$ must be -1. Multiply now (26) in $\exp(z^{m_2})$ and by comparing the order of both sides of the resulting equation, we get a contradiction.

Thus we can assume that f has infinitely many zeros $\{z_n\}$, and since all of them are of multiplicity at least $k+1$, we get

$$f(z_n) = f'(z_n) = \cdots = f^{(k)}(z_n) = 0. \tag{27}$$

Let S be a subsequence of $\{z_n\}$ (denote it also by $\{z_n\}$), such that $\arg(z_n)$ converges to α. By (19) and (27), we have

$$\alpha = \frac{\pi}{2m_2} + \frac{\pi}{m_2}\ell, \quad 0 \leqslant \ell \leqslant 2m_2 - 1.$$

Without loss of generality, assume that $\alpha = \frac{\pi}{2m_2}$. Denote $f(z) = f_1(z) + f_2(z)$, where

$$f_i^{(k)}(z) = P_i(z)\exp\big(Q_i(z)\big) \quad (i = 1, 2). \tag{28}$$

Take r_0 sufficiently large such that there are no zeros or poles of $P_2(z)$ in $\{|z| \geqslant r_0\}$ and also no zeros of $P_1(z)$ there. For all $m \in \mathbb{Z}$ and for every $0 < \varepsilon < \frac{\pi}{2m_2}$, we have $z^m \exp(Q_2(z)) \overset{\forall}{\Rightarrow} 0$ in S_ε, where $S_\varepsilon := V(0, 0, \frac{\pi}{2m_2} - \varepsilon)$.

There exists $a_2 \in \mathbb{C}$ such that

$$\int_{r_0}^{z} P_2(u)\exp\big(Q_2(u)\big)\,du \overset{\forall}{\Rightarrow} a_2 \quad \text{in } S_\varepsilon. \tag{29}$$

The integral path can be taken to be the segment from r_0 to $|z|$ and then the arc γ_z on $C(0, |z|)$ from $|z|$ to z counterclockwise. This limit exists uniformly in S_ε. To justify (29), first note that the limit exists when z is positive and then observe that $\int_{\gamma_z} P_2(u)\exp(Q_2(u))\,du \overset{\forall}{\Rightarrow} 0$ in S_ε. Thus we have

$$\int_{r_0}^{z} P_2(u)\exp\big(Q_2(u)\big)\,du = a_2 + o(1)$$

uniformly in S_ε.

Next we estimate the $o(1)$. We write

$$a_2 - \int_{r_0}^{z} P_2(u)\exp\big(Q_2(u)\big)\,du = \int_{z}^{\infty} P_2(u)\exp\big(Q_2(u)\big)\,du.$$

For the right-hand side of this equation, we can take the path as the ray from z to ∞, in the direction of $\arg(z)$. Integrating by parts, we have

$$\int_z^\infty P_2(u)\exp\bigl(Q_2(u)\bigr)\,du = \int_z^\infty \frac{P_2(u)}{Q_2'(u)}Q_2'(u)\exp\bigl(Q_2(u)\bigr)\,du = -\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr) - \int_z^\infty \exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{P_2(u)}{Q_2'(u)}\right)du$$
$$= -\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr) - \int_z^\infty \frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'^2(u)}\exp\bigl(Q_2(u)\bigr)\,du.$$

We shall prove now that

$$\int_z^\infty \left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_i'(u)^2}\right)\exp\bigl(Q_2(u)\bigr)\,du = o\left(\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr)\right)$$

as $z\to\infty$ uniformly in S_ε. Again we integrate by parts and obtain

$$\int_z^\infty \left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^2}\right)\exp\bigl(Q_2(u)\bigr)\,du$$
$$= -\frac{Q_2'(z)P_2'(z)-P_2(z)Q_2''(z)}{Q_2'(z)^3}\exp\bigl(Q_2(z)\bigr) - \int_z^\infty \exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^3}\right)du.$$

Applying Lemma 4 twice, there exists $k>0$, such that for sufficiently large u in S_ε,

$$\left|u^2\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'^3(u)}\right)\exp\bigl(Q_2(u)\bigr)\right| \leqslant \left|\frac{kP_2(u)}{Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\right|.$$

Thus, for large enough z in S_ε,

$$\left|\int_z^\infty \exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^3}\right)du\right| \leqslant k\int_z^\infty \left|\frac{P_2(u)}{u^2Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\right|du$$
$$= \frac{k}{|z|}\int_1^\infty \frac{1}{t^2}\left|\frac{P_2(tz)}{Q_2'(tz)^2}\exp\bigl(Q_2(tz)\bigr)\right|dt. \tag{30}$$

By Lemma 5, there is $L_\varepsilon>0$, such that for every $z\in S_\varepsilon$, $|z|>L_\varepsilon$, the function $h_z(t) := |\frac{P_2(tz)}{Q_2'(tz)^2}\exp(Q_2(tz))|$ is decreasing in $\{t\geqslant 1\}$. Thus we have by (30) that for z in S_ε, $|z|>L_\varepsilon$,

$$\left|\int_z^\infty \exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^3}\right)du\right| \leqslant \frac{k}{|z|}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right|\left|\int_1^\infty \frac{dt}{t^2}\right|$$
$$= \frac{k}{|z|}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right|. \tag{31}$$

By Lemma 4, we also have that for large enough z in S_ε,

$$\left|\frac{Q_2'(z)P_2'(z)-P_2(z)Q_2''(z)}{Q_2'(z)^3}\exp\bigl(Q_2(z)\bigr)\right| \leqslant \frac{k'}{|z|}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right| \tag{32}$$

for some $k'>0$.

From (31) and (32), we have

$$\left|\int_z^\infty \left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^2}\right)\exp\bigl(Q_2(u)\bigr)\,du\right| \leqslant \frac{k+k'}{|z|}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right|;$$

and thus

$$\int_z^\infty \left(\frac{Q_2'(u)P_2'(u)-P_2(u)Q_2''(u)}{Q_2'(u)^2}\right)\exp\bigl(Q_2(u)\bigr)\,du = o\left(\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr)\right) \quad \text{as } z\to\infty \text{ uniformly in } S_\varepsilon.$$

So we can write

$$a_2 - \int_{r_0}^{z} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du = \int_{z}^{\infty} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du \sim -\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr),$$

and have

$$\frac{Q_2'(z)}{P_2(z)}\exp\bigl(-Q_2(z)\bigr)\left(a_2 - \int_{r_0}^{z} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du\right) \overset{\forall}{\Rightarrow} -1 \quad \text{in } S_\varepsilon.$$

Consider now the domain

$$S_\varepsilon^+ := V\left(0, \frac{\pi}{m_2}, \frac{\pi}{2m_2} - \varepsilon\right) \quad \text{for } 0 < \varepsilon < \frac{\pi}{2m_2}.$$

Integrating the $o(1)$ function gives

$$\begin{aligned} a_2 &- \int_{r_0}^{z} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du \\ &= a_2 - \frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr) + \frac{P_2(r_0)}{Q_2'(r_0)}\exp\bigl(Q_2(r_0)\bigr) + \int_{\Gamma_z} \frac{Q_2'(u)P_2'(u) - P_2(u)Q_2''(u)}{Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\,du, \end{aligned} \tag{33}$$

where Γ_z is the curve from r_0 to $r_0\frac{z}{|z|}$, counterclockwise on the arc$\{|u| = r_0\}$ and then on the segment from $r_0\frac{z}{|z|}$ to z in S_ε^+.

Integrating by parts, we obtain

$$\begin{aligned} &\int_{r_0}^{z}\left(\frac{Q_2'(u)P_2'(u) - P_2(u)Q_2''(u)}{Q_2'(u)^2}\right)\exp\bigl(Q_2(u)\bigr)\,du \\ &= \frac{Q_2'(z)P_2'(z) - P_2(z)Q_2''(z)}{Q_2'(z)^3}\exp\bigl(Q_2(z)\bigr) - \frac{Q_2'(r_0)P_2'(r_0) - P_2(r_0)Q_2''(r_0)}{Q_2'(r_0)^3}\exp\bigl(Q_2(r_0)\bigr) \\ &\quad - \int_{r_0}^{z}\exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u) - P_2(u)Q_2''(u)}{Q_2'(u)^3}\right)du. \end{aligned} \tag{34}$$

We have by Lemma 4, for $z \in S_\varepsilon^+$,

$$\left|\int_{\Gamma_z}\exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left(\frac{Q_2'(u)P_2'(u) - P_2(u)Q_2''(u)}{Q_2'(u)^3}\right)du\right| \leqslant k\int_{\Gamma_z}\left|\frac{P_2(u)}{u^2Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\right|du \tag{35}$$

for some $k > 0$.

Fix $0 < \delta < 1$, and apply Lemma 5 to $\frac{1}{h_z(t)}$ in S_ε^+. We then have that there exists $\tilde{k} > 0$ such that for large enough z, there is

$$\begin{aligned} k\int_{\Gamma_z}\left|\frac{P_2(u)}{u^2Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\right|du &\leqslant \frac{\tilde{k}}{|z|^{1-\delta}}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right|\cdot\int_{r_0\frac{z}{|z|}}^{z}\frac{|du|}{|u|^{1+\delta}} = \frac{\tilde{k}}{|z|^{1-\delta}}\left|\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)\right|\int_{r_0}^{|z|}\frac{dt}{t^{1+\delta}} \\ &= o(1)\left|\frac{P_2(z)}{Q_2'(z)}\exp\bigl(Q_2(z)\bigr)\right|. \end{aligned} \tag{36}$$

By (33)–(36), we have

$$\frac{Q_2'(z)}{P_2(z)}\exp\bigl(-Q_2(z)\bigr)\left(a_2 - \int_{r_0}^{z} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du\right) \overset{\forall}{\Rightarrow} -1 \tag{37}$$

in S_ε^+.

In the same fashion we have that (37) holds also in

$$S_\varepsilon^- := V\left(0, \frac{-\pi}{m_2}, \frac{\pi}{2m_2} - \varepsilon\right)$$

for every $0 < \varepsilon < \frac{\pi}{2m_2}$. (In fact, (37) holds for both S_ε^+ and S_ε^- with any constant from $\mathbb{C}$ instead of a_2.)

Now, for a given $0 < \varepsilon < \frac{\pi}{2m_2}$, applying Theorems PL1 and PL2 for the two angular domains, emanating from r_0, $S^+_{\varepsilon',r_0} := V(r_0, \frac{\pi}{m_2} - \varepsilon', \frac{\pi}{2m_2})$ and $S^+_{\varepsilon',r_0} := V(r_0, \frac{-\pi}{m_2} + \varepsilon', \frac{\pi}{2m_2})$, where $0 < \varepsilon' < \varepsilon$. Consider that (37) is true for *every* $0 < \varepsilon < \frac{\pi}{2m_2}$. We get by geometrical considerations, that when $m_2 \geqslant 2$, then for every $0 < \delta < \frac{3\pi}{2m_2}$, if r_0 is sufficiently large, then

$$\frac{Q_2'(z)}{P_2(z)} \exp\bigl(-Q_2(z)\bigr)\left(a_2 - \int_{r_0}^{z} P_2(u)\exp\bigl(Q_2(u)\bigr)\,du\right) \overset{\forall}{\Rightarrow} -1 \quad \text{in } \hat{S}_\delta, \tag{38}$$

where

$$\hat{S}_\delta := V\left(0, 0, \frac{3\pi}{2m_2} - \delta\right).$$

When $m_2 = 1$, then (38) occurs in

$$\hat{S}_\delta := V(0, 0, \pi - \delta), \tag{39}$$

where $\delta > 0$ can be arbitrary small if r_0 is large enough.

The reason for making the domains S^+_{ε',r_0} and S^-_{ε,r_0} emanating from r_0 is to avoid the poles of the function in the left-hand side of (38), in order to use Theorems PL1 and PL2. Note that in (38), if r_0 is large enough, then it is good for every $0 < \delta < \frac{3\pi}{2m_2}$, while in (39) $r_0 \to \infty$ as $\delta \to 0^+$.

Now, if $Q_1(z) \not\equiv \text{const}$, then we can similarly show that there exists $a_1 \in \mathbb{C}$, such that for every $0 < \delta < \frac{\pi}{2m_1}$, $\int_{r_0}^{z} P_1(u)\exp(Q_1(u))\,du \overset{\forall}{\Rightarrow} a_1$ in $T_\delta := V(0, \theta_0, \frac{\pi}{2m_1} - \delta)$. Here θ_0 depends on the argument of the coefficient of z^{m_1} in $Q_1(z)$. Estimating $a_1 - \int_{r_0}^{z} P_1(u)\exp(Q_1(u))\,du$ gives as in (38) that when $m_1 \geqslant 2$ and r_0 is sufficiently large, then

$$\frac{Q_1'(z)}{P_1(z)} \exp\bigl(-Q_1(z)\bigr)\left(a_1 - \int_{r_0}^{z} P_1(u)\exp\bigl(Q_1(u)\bigr)\,du\right) \overset{\forall}{\Rightarrow} -1 \quad \text{in } \hat{T}_\delta := V\left(0, \theta_0, \frac{3\pi}{2m_1} - \delta\right) \tag{40}$$

for every $0 < \delta < \frac{3\pi}{2m_1} - \delta$.

When $m_1 = 1$, then (40) occurs in

$$\hat{T}_\delta := V(0, \theta_0, \pi - \delta), \tag{41}$$

when δ can be arbitrarily small if r_0 is sufficiently large. Now, since $m_1 < m_2$, we can in any case choose θ_0 and δ, such that $\hat{T}_\delta$ contains $S^* := V(0, 0, \frac{\pi}{2m_2} + \varepsilon_0)$ for small ε_0 $(0 < \varepsilon_0 < \frac{\pi}{2m_1} - \frac{\pi}{2m_2})$. Thus, we have for $i = 1, 2$,

$$\frac{Q_i'(z)}{P_i(z)} \exp\bigl(-Q_i(z)\bigr)\left(a_i - \int_{r_0}^{z} P_i(u)\exp\bigl(Q_i(u)\bigr)\,du\right) \overset{\forall}{\Rightarrow} -1 \quad \text{in } S^*. \tag{42}$$

Integrating $f^{(k)}(u)$ from r_0 to z in S^* and considering (28) and (42), we have

$$f^{(k-1)}(z) - f^{(k-1)}(r_0) = a_1 + \bigl(1 + r_1(z)\bigr)\frac{P_1(z)}{Q_1'(z)}\exp\bigl(Q_1(z)\bigr) + a_2 + \bigl(1 + r_2(z)\bigr)\frac{P_2(z)}{Q_2''(z)}\exp\bigl(Q_2(z)\bigr), \tag{43}$$

where $r_2(z)$ is analytic in $\hat{S}_\delta$ and converges there uniformly to 0 as $z \to \infty$, and $r_1(z)$ has the same properties in $\hat{T}_\delta$.

Integrating (43) from r_0 to z gives

$$\begin{aligned} f^{(k-2)}(z) = {} & \bigl(a_1 + a_2 + f^{(k-1)}(r_0)\bigr)z + b_0 + \int_{r_0}^{z} \bigl(1 + r_1(u)\bigr)\frac{P_1(u)}{Q_1'(u)}\exp\bigl(Q_1(u)\bigr)\,du \\ & + \int_{r_0}^{z} \bigl(1 + r_2(u)\bigr)\frac{P_2(u)}{Q_2'(u)}\exp\bigl(Q_2(u)\bigr)\,du, \end{aligned} \tag{44}$$

where $b_0 \in \mathbb{C}$.

We shall now estimate the integrals in (44). We have

$$\int_{r_0}^{z} \bigl(1 + r_2(u)\bigr)\frac{P_2(u)}{Q_2'(u)}\exp\bigl(Q_2(u)\bigr)\,du \overset{\forall}{\Rightarrow} b_2$$

in S_ε, where $b_2 \in \mathbb{C}$. Now we use integration by parts to estimate the difference

$$\begin{aligned} b_2 - \int_{r_0}^{z} \big(1 + r_2(u)\big) \frac{P_2(u)}{Q_2'(u)} \exp\big(Q_2(u)\big)\, du &= \int_{z}^{\infty} \frac{P_2(u)}{Q_2'(u)} \exp\big(Q_2(u)\big)\big(1 + r_2(u)\big)\, du \\ &= -\exp\big(Q_2(z)\big)\big(1 + r_2(z)\big) \frac{P_2(z)}{Q_2'(z)^2} - \int_{z}^{\infty} \exp\big(Q_2(u)\big) \frac{d}{du}\left[\frac{P_2(u)}{Q_2'(u)^2}\big(1 + r_2(u)\big)\right] du \\ &= -\exp\big(Q_2(z)\big)\big(1 + r_2(z)\big) \frac{P_2(z)}{Q_2'(z)^2} + T(z), \end{aligned} \tag{45}$$

where

$$T(z) = -\int_{z}^{\infty} \exp\big(Q_2(u)\big)\left[\frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^4}\big(1 + r_2(u)\big) + r_2'(u)\frac{P_2(u)}{Q_2'(u)^2}\right] du.$$

We will show that

$$T(z) = o(1)\exp\big(Q_2(z)\big)\frac{P_2(z)}{Q_2'(z)^2} \quad \text{as } z \to \infty \text{ uniformly in } S_\varepsilon. \tag{46}$$

We have

$$\begin{aligned} T(z) &= \exp\big(Q_2(u)\big)\left[\frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^5}\big(1 + r_2(u)\big) + r_2'(u)\frac{P_2(u)}{Q_2'(u)^3}\right]\Bigg|_{z}^{\infty} \\ &\quad - \int_{z}^{\infty} \exp\big(Q_2(u)\big)\frac{d}{du}\left[\frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^5}\big(1 + r_2(u)\big) + r_2'(u)\frac{P_2(u)}{Q_2'(u)^3}\right] du. \end{aligned} \tag{47}$$

The left term in the right-hand side of (47) is obviously $o(1)\frac{\exp(Q_2(z))P_2(z)}{Q_2'(z)^2}$. By Lemmas 4 and 7, and similarly to (31) and (32), the right term in the right-hand side of (47) is $O(\frac{1}{|z|})\frac{\exp(Q_2(z))P_2(z)}{Q_2'(z)^2}$, so (46) is proved. Thus we conclude by (45) that

$$\left(b_2 - \int_{r_0}^{z} \big(1 + r_2(u)\big)\frac{P_2(u)}{Q_2'(u)}\exp\big(Q_2(u)\big)\, du\right)\frac{\exp(-Q_2(z))Q_2'(z)^2}{P_2(z)} \overset{\forall}{\Rightarrow} -1 \quad \text{in } S_\varepsilon. \tag{48}$$

Now, in S_ε^+,

$$\begin{aligned} b_2 - \int_{r_0}^{z} \big(1 + r_2(u)\big)\frac{P_2(u)}{Q_2'(u)}\exp\big(Q_2(u)\big)\, du &= b_2 - \frac{\exp(Q_2(z))}{Q_2'(z)^2}P_2(z)\big(1 + r_2(z)\big) + \frac{\exp(Q_2(r_0))P_2(r_0)(1 + r_2(r_0))}{Q_2'(r_0)^2} \\ &\quad + \int_{r_0}^{z} \frac{d}{du}\left[\big(1 + r_2(u)\big)\frac{P_2(u)}{Q_2'(u)^2}\right]\exp\big(Q_2(u)\big)\, du. \end{aligned} \tag{49}$$

We wish to show that (48) holds also in S_ε^+. Since $\frac{\exp(Q_2(z))}{Q_2'(z)^2}P_2(z) \overset{\forall}{\Rightarrow} 0$ in S_ε^+, we need to show that the integral on the right-hand side of (49) is $o(1)\frac{\exp(Q_2(z))}{Q_2'(z)^2}P_2(z)$ as $z \to \infty$, uniformly in S_ε^+.

Indeed,

$$\begin{aligned} &\int_{r_0}^{z} \frac{d}{du}\left[\big(1 + r_2(u)\big)\frac{P_2(u)}{Q_2'(u)^2}\right]\exp\big(Q_2(u)\big)\, du \\ &\quad = \int_{r_0}^{z} \frac{r_2'(u)P_2(u)}{Q_2'(u)^2}\exp\big(Q_2(u)\big)\, du + \int_{r_0}^{z} \frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^4}\exp\big(Q_2(u)\big)\, du. \end{aligned}$$

By Lemma 7,

$$\int_{r_0}^{z} \frac{r_2'(u)P_2(u)}{Q_2'(u)^2}\exp\bigl(Q_2(u)\bigr)\,du = o(1)\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr)$$

and

$$\begin{aligned}
&\int_{r_0}^{z} \frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^4}\exp\bigl(Q_2(u)\bigr)\,du \\
&\quad = \frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^5}\exp\bigl(Q_2(u)\bigr)\Big|_{r_0}^{z} \\
&\qquad - \int_{r_0}^{z} \exp\bigl(Q_2(u)\bigr)\frac{d}{du}\left[\frac{P_2'(u)Q_2'(u)^2 - 2Q_2'(u)Q_2''(u)P_2(u)}{Q_2'(u)^5}\right]du.
\end{aligned} \tag{50}$$

The left term in the right-hand side of (50) is $o(1)\frac{P_2(z)}{Q_2'(z)^2}\exp(Q_2(z))$ as $z \to \infty$, uniformly in S_ε^+. The right term is shown to be so, similarly to the discussion after (33). Thus, (48) holds also in S_ε^+ and similarly it holds in S_ε^-.

Again, by applying Theorems PL1 and PL2, (48) holds in $\hat{S}_\delta$ (see (38), (39)). In the same way, it can be shown that there exists $b_1 \in \mathbb{C}$, such that

$$\left(b_1 - \int_{r_0}^{z}\bigl(1+r_1(u)\bigr)\frac{P_1(u)}{Q_1'(u)}\exp\bigl(Q_1(u)\bigr)\,du\right)\frac{\exp(-Q_1(z))Q_1'(z)^2}{P_1(z)} \underset{z\to\infty}{\to} -1 \quad \text{uniformly in } \hat{T}_\delta \tag{51}$$

(see (40), (41)). By (48) and (51),

$$f^{(k-2)}(z) = Az + B + \bigl(1+S_1(z)\bigr)\frac{P_1(z)}{Q_1'(z)^2}\exp\bigl(Q_1(z)\bigr) + \bigl(1+S_2(z)\bigr)\frac{P_2(z)}{Q_2'(z)^2}\exp\bigl(Q_2(z)\bigr), \tag{52}$$

where $A = a_1 + a_2 + f^{(k-1)}(r_0)$, $B \in \mathbb{C}$ and $S_i(z) \overset{\forall}{\Rightarrow} 0$ in S^*, for $i = 1, 2$.

Now, for $n \geqslant N_0$, all the zeros z_n are in S^*. From (27), (28), (43) and (52), we have the following relations:

$$\begin{aligned}
&P_1(z_n)\exp\bigl(Q_1(z_n)\bigr) + P_2(z_n)\exp\bigl(Q_2(z_n)\bigr) = 0, \\
&\bigl(1+r_1(z_n)\bigr)\frac{P_1(z_n)}{Q_1'(z_n)}\exp\bigl(Q_1(z_n)\bigr) + \bigl(1+r_2(z_n)\bigr)\frac{P_2(z_n)}{Q_2'(z_n)}\exp\bigl(Q_2(z_n)\bigr) + A = 0, \\
&\bigl(1+S_1(z_n)\bigr)\frac{P_1(z_n)}{Q_1'(z_n)^2}\exp\bigl(Q_1(z_n)\bigr) + \bigl(1+S_2(z_n)\bigr)\frac{P_2(z_n)\exp(Q_2(z_n))}{Q_2'(z_n)^2} + Az_n + B_0 = 0.
\end{aligned} \tag{53}$$

From (53), we get

$$A\left[\frac{(1+o(1))}{Q_1'(z_n)^2} - \frac{(1+o(1))}{Q_2'(z_n)^2}\right] + (Az_n + B)\left[\frac{(1+o(1))}{Q_2'(z_n)} - \frac{(1+o(1))}{Q_1'(z_n)}\right] = 0,$$

and this implies

$$\begin{aligned}
&-A\left[\frac{1}{Q_1'(z_n)^2} - \frac{1}{Q_2'(z_n)^2}\right] - (Az_n + B)\left[\frac{1}{Q_2'(z_n)} - \frac{1}{Q_1'(z_n)}\right] \\
&\quad = A\left[\frac{o(1)}{Q_1'(z_n)^2} - \frac{o(1)}{Q_2'(z_n)^2}\right] + (Az_n + B)\left[\frac{o(1)}{Q_2'(z_n)} - \frac{o(1)}{Q_1'(z_n)}\right].
\end{aligned} \tag{54}$$

We claim that

$$A\left(\frac{1}{Q_1'(z)^2} - \frac{1}{Q_2'(z)^2}\right) + (Az + B)\left[\frac{1}{Q_2'(z)} - \frac{1}{Q_1'(z)}\right] \equiv 0. \tag{55}$$

If not, then $Az + B \not\equiv 0$, so we multiply (54) in $\frac{Q_1'(z_n)}{Az_n + B}$ and get

$$\frac{-A}{Az_n + B}\left(\frac{1}{Q_1'(z_n)} - \frac{Q_1'(z_n)}{Q_2'(z_n)^2}\right) - \left(\frac{Q_1'(z_n)}{Q_2'(z_n)} - 1\right) = \frac{A}{Az_n + B}\left(\frac{o(1)}{Q_1'(z_n)} - \frac{o(1)Q_1'(z_n)}{Q_2'(z_n)^2}\right) + \left(o(1)\frac{Q_1'(z_n)}{Q_2'(z_n)} + o(1)\right).$$

Let now $n \to \infty$ and we get that $1 = 0$, a contradiction.

Now multiply (55) in ${Q'_1}^2(z){Q'_2}^2(z)$ and get

$$A\big(Q'_2(z)^2 - Q'_1(z)^2\big) + (Az+B)\big(Q'_1(z)^2 Q'_2(z) - Q'_2(z)^2 Q'_1(z)\big) = 0.$$

Since $Az+B \not\equiv 0$ and $m_2 > m_1 \geqslant 1$, we have $Q'_1 = Q'_2$, a contradiction.

Now consider the case where $Q_1(z) \equiv \text{const}$, i.e., $a(z)$ is a polynomial. In the case where $a(z)$ is a nonzero constant, the theorem follows from [9, Theorem 3] or [4, p. 18]. If $a(z)$ is a general polynomial, then we integrate (19) in S^* (only once!) and get similarly to (43)

$$f^{(k-1)}(z) - a_1(z) = \frac{P_2(z)}{Q'_2(z)}\big(1 + r_2(z)\big)\exp\big(Q_2(z)\big), \tag{56}$$

where $a_1(z)$ is a polynomial such that $a'_1(z) = a(z)$, and $r_2(z) \underset{z\to\infty}{\overset{\forall}{\Longrightarrow}} 0$ in S^*. We divide (56) by (19), and get $\frac{a_1(z_n)}{a(z_n)} = \frac{1+r_2(z_n)}{Q_2(z_n)}$. Letting $n \to \infty$, we get $\infty = 0$, a contradiction.

This completes treating the case (iii) of Case (BII) which completes the proof of Theorem 2.

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