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Methods of judging shape of solitary wave and solution formulae for some evolution equations with nonlinear terms of high order

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Abstract

In this paper, we present several methods of judging shape of the solitary wave and solution formulae for some nonlinear evolution equations by means of Lienard equations. Then, using the judgement methods and solution formulae, we obtain solutions of the solitary wave for some of important nonlinear evolution equations, which include generalized modified Boussinesq, generalized nonlinear wave, generalized Fisher, generalized Klein–Gordon and generalized Zakharov equations. Some new solitary-wave solutions are found for the equations.

Keywords: Solitary wave; Exact solution; Generalized modified Boussinesq equation; Generalized nonlinear wave equation; Generalized Fisher equation; Generalized Klein–Gordon equation; Generalized Zakharov equation

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1. Introduction

The solitary waves of many nonlinear equations are important in theoretical physicists and applied mathematicians. In this paper, we consider some of nonlinear evolution equations with the solitary waves.

Soliton-producing nonlinear equations, which arise in a variety of physics, mechanics and biology, can have solitary waves with different shape. For example, compound KdV equation

$$u_t + auu_x + bu^2u_x + u_{xxx} = 0, \quad b \neq 0, \tag{1.1}$$

has the kink and bell profile solitary-wave solutions (see [1–3]). The compound KdV–Burgers equation

$$u_t + auu_x + bu^2u_x + ru_{xx} + u_{xxx} = 0, \quad r \neq 0. \tag{1.2}$$

has the kink profile solitary-wave solutions (see [3,4]). The improved Boussinesq equation

$$u_{tt} - u_{xx} + \frac{a}{2}\left(u^2\right)_{xx} - \beta u_{xxtt} = 0 \tag{1.3}$$

has the bell profile solitary-wave solutions (see [5]). The generalized Fisher equation

$$u_t - Du_{xx} = su\left(1 - u^\alpha\right)\left(u^\alpha + \beta\right) \tag{1.4}$$

has the kink profile solitary-wave solutions (see [6]). In [7–9], Zhang and Ma consider the exact solitory wave solutions for some evolution equations. Therefore, methods of judging shape of the solitary wave for some nonlinear evolution equations are valuable and important. We will present several methods of judging shape of the solitary wave in this paper.

Moreover, we find that the problem of looking for the solitary-wave solutions for many nonlinear evolution equations can be lead to solve the following Lienard equations:

$$a''(\xi) + ra'(\xi) + la(\xi) + ma^q(\xi) + na^{2q-1}(\xi) = 0 \tag{I}$$

or

$$a''(\xi) + la(\xi) + ma^q(\xi) + na^{2q-1}(\xi) = 0, \tag{II}$$

where $q > 0$. Kong gave the exact solution of a simple Lienard equation and its application in [10]. Now, we consider generalized modified Boussinesq equation

$$u_{tt} - \delta u_{xxtt} + ru_{xxt} - \left(b_1u + b_2u^{p+1} + b_3u^{2p+1}\right)_{xx} = 0, \quad p > 0. \tag{1.5}$$

Let $u(x,t) = u(x - vt) \equiv u(\xi)$ be a traveling-wave solutions for Eq. (1.5). Then, we have

$$v^2u''(\xi) - \delta v^2u^{(4)}(\xi) - rvu'''(\xi) - \left(b_1u + b_2u^{p+1} + b_3u^{2p+1}\right)_{\xi\xi} = 0.$$

Assume that the solutions of Eq. (1.5) satisfy the condition

$$u'(\xi), u''(\xi), u'''(\xi) \to 0, \quad \text{as } |\xi| \to \infty, \tag{1.6}$$

and asymptotic values of the solutions

$$C_\pm = \lim_{\xi \to \pm\infty} u(\xi) \tag{1.7}$$

satisfy

$$\left(b_1 - v^2\right)x + b_2 x^{p+1} + b_3 x^{2p+1} = 0. \tag{1.8}$$

Then, the solitary-wave solutions of Eq. (1.5) satisfy

$$u''(\xi) + \frac{r}{\delta v} u'(\xi) + \frac{b_1 - v^2}{\delta v^2} u(\xi) + \frac{b_2}{\delta v^2} u^{p+1}(\xi) + \frac{b_3}{\delta v^2} u^{2p+1}(\xi) = 0. \tag{1.9}$$

For the nonlinear wave equation,

$$\delta u_{tt} - k u_{xx} + r u_t + b_1 u + b_2 u^{p+1} + b_3 u^{2p+1} = 0, \quad p > 0, \ r \geqslant 0, \tag{1.10}$$

its solitary-wave solutions satisfy

$$\begin{aligned} & u''(\xi) - \frac{rv}{\delta v^2 - k} u'(\xi) + \frac{b_1}{\delta v^2 - k} u(\xi) \\ & \quad + \frac{b_2}{\delta v^2 - k} u^{p+1}(\xi) + \frac{b_3}{\delta v^2 - k} u^{2p+1}(\xi) = 0. \end{aligned} \tag{1.11}$$

Eqs. (1.9) and (1.11) are Lienard-type. They may be reduced to the Lienard equation (II) if $r = 0$.

It is easy to know that the generalized Zakharov equations

$$\begin{cases} H_{tt} - H_{xx} = (|u|^{2p})_{xx}, \\ i u_t + u_{xx} = H u + b_1 |u|^{2p} u + b_2 |u|^{4p} u, \end{cases} \tag{1.12}$$

can be reduced to the Lienard equation (II) by means of proper transform (see Section 6).

In this paper, we will consider shapes of the solitary waves and the exact solutions for the nonlinear evolution equations, which can be reduced to the Lienard equation (I) or (II).

This paper is organized as follows. In Section 2, we will discuss some properties and several methods of judging shape of the solitary wave for many nonlinear evolution equations by means of the Lienard equations (I) and (II). The methods of judging shape of the solitary wave are new and valuable, since solution functions can be assumed and solutions of the solitary wave are obtained more easily according their shape. Using assumption of solution functions, the kink and bell profile solitary-wave solutions for the Lienard equation (I) and (II) are found in Section 3. The explicit exact solutions of Eqs. (I) and (II) for any real parameter $q > 0$ are new results. Applying the results obtained in Sections 2 and 3, the kink and bell profile solitary-wave solutions for generalized modified Boussinesq equation (1.5), generalized nonlinear wave equation (1.10), and generalized Zakharov equation (1.12) are found in Sections 4, 5 and 6. In [5], the exact solutions of Eq. (1.5) were obtained only in cases $p = 1$, $\delta = \beta^2$, $b_1 = 1$, $b_2 = -\frac{a}{2}$, $b_3 = 0$ and $p = 1$, $\delta = \beta^2$, $b_1 = 1$, $b_2 = 0$, $b_3 = -\frac{a}{3}$. The exact solutions of Eq. (1.12) only for $p = 1$, $b_1 = 0$, $b_2 = 0$ were found in [11,12].

In this paper, we obtain the explicit exact solutions of these equations for any real number $p > 0$, which include previous results as special cases. Moreover, many new exact solitary-wave solutions are found for modified improved Boussinesq equation, generalized Fisher equation and Zakharov equation.

2. Properties of the solution functions and methods of judging shape of the solitary wave

We consider some important properties of the solution for the Lienard equations (I) and (II), which satisfy the condition

$$a'(\xi), a''(\xi) \to 0, \quad \text{as } |\xi| \to \infty. \tag{2.1}$$

Let $C_\pm = \lim_{\xi\to\pm\infty} a(\xi)$. Multiplying Eq. (I) by $a'(\xi)$ and integrating once, we have

$$\frac{1}{2}\big(a'(\xi)\big)^2 + r\int_{-\infty}^{\xi}\big(a'(\xi)\big)^2\,d\xi + \frac{l}{2}a^2(\xi) + \frac{m}{(q+1)}a^{q+1}(\xi) + \frac{n}{2q}a^{2q}(\xi) = c_1, \tag{2.2}$$

where c_1 is a integrating constant. Let $\xi \to -\infty$ in (2.2), then we obtain from the condition (2.1) that

$$\frac{l}{2}C_-^2 + \frac{m}{q+1}C_-^{q+1} + \frac{n}{2q}C_-^{2q} = c_1. \tag{2.3}$$

As $\xi \to +\infty$, substituting (2.3) into (2.2) yields

$$r\int_{-\infty}^{+\infty}\big(a'(\xi)\big)^2\,d\xi = \frac{l}{2}\big(C_-^2 - C_+^2\big) + \frac{m}{q+1}\big(C_-^{q+1} - C_+^{q+1}\big) + \frac{n}{2q}\big(C_-^{2q} - C_+^{2q}\big). \tag{2.4}$$

It follows from Eq. (I) that

$$lC_+ + mC_+^q + nC_+^{2q-1} = 0 \tag{2.5}$$

and

$$lC_- + mC_-^q + nC_-^{2q-1} = 0 \tag{2.6}$$

respectively. Combining the (2.5) and (2.6), we have

$$m\big(C_-^{q+1} - C_+^{q+1}\big) = -l\big(C_-^2 - C_+^2\big) - n\big(C_-^{2q} - C_+^{2q}\big). \tag{2.7}$$

By substituting (2.7) into (2.4), the following useful formulae for the solutions of the Lienard equation (I) with the condition (2.1) are obtained

$$\int_{-\infty}^{+\infty}\big(a'(\xi)\big)^2\,d\xi = \frac{q-1}{2r(q+1)}\left[l\big(C_-^2 - C_+^2\big) - \frac{n}{q}\big(C_-^{2q} - C_+^{2q}\big)\right] \tag{2.8}$$

and

$$\int_{-\infty}^{+\infty}\big(a'(\xi)\big)^2\,d\xi = \frac{q-1}{2rq}\left[l\big(C_-^2 - C_+^2\big) + \frac{m}{q+1}\big(C_-^{q+1} - C_+^{q+1}\big)\right]. \tag{2.9}$$

In view of (2.8) and (2.9), we have the following properties of solitary-wave solutions for some nonlinear evolution equations, solutions of which can be found by means the Lienard equations (I) or (II).

Property 1. Let $a(\xi)$ be a solution of Eq. (I) with the condition (2.1). Then, $a'(\xi)$ is square integrable in $(-\infty, +\infty)$.

Property 2. Let $a(\xi)$ be a solution of Eq. (I) with the condition (2.1). Then

$$r \quad \text{and} \quad (q-1)\left[l\left(C_-^2 - C_+^2\right) - \frac{n}{q}\left(C_-^{2q} - C_+^{2q}\right)\right]$$

have the same sign.

Property 3. Let $a(\xi)$ be a solution of Eq. (I) with the condition (2.1). Then for fixed C_- and C_+, the smaller $|r|$ is, the smaller dissipative effect is and the steeper wave shape of the $a(\xi)$.

Property 4. Let $a(\xi)$ be a solution of Eq. (II) with the condition (2.1). Then, the asymptotic values of $a(\xi)$, C_+ and C_- satisfy

$$l\left(C_-^2 - C_+^2\right) - \frac{n}{q}\left(C_-^{2q} - C_+^{2q}\right) = 0 \tag{2.10}$$

and

$$l\left(C_-^2 - C_+^2\right) + \frac{m}{q+1}\left(C_-^{q+1} - C_+^{q+1}\right) = 0. \tag{2.11}$$

The property is deduced from (2.4) and (2.7).

Property 5. Let q be a natural number. If the nonlinear evolution equations, which can be reduced to Eq. (II), have a solitary-wave solution of kink profile with asymptotic values $C_+ = -C_- \neq 0$, then the following formula holds:

$$m\left(1 - (-1)^{q+1}\right) = 0. \tag{2.12}$$

The property is obtained from (2.11) directly.

Using (2.8), (2.9) and above properties, the following judgement methods can be proved:

Judgement method 1. *The nonlinear evolution equations, which can be reduced to Eq.* (I), *only possibly have a solution of the kink profile with the asymptotic values* $|C_+| \neq |C_-|$, *and do not possess the solution of the bell profile with the same asymptotic values and the solution of the kink profile with the asymptotic values* $C_+ = -C_-$.

It is obtained from (2.8).

Judgement method 2. *The nonlinear evolution equations, which can be reduced to Eq.* (II)*, possibly have a solution of the bell profile with the same asymptotic values, or a solution of the kink profile with the asymptotic values $C_+ \neq C_-$, which satisfy* (2.10) *and* (2.11).

Property 4 implies Judgement method 2.

Judgement method 3. *A necessary condition of existing the solution of kink profile with the asymptotic values $|C_+| \neq |C_-|$ in the nonlinear evolution equations, which can be reduced to Eq.* (II)*, is $nl > 0$ and $m \neq 0$.*

Proof. The (2.10) implies $nl > 0$, since sign of $(C_-^2 - C_+^2)$ is the same as one of $(C_-^{2q} - C_+^{2q})$. It follows from the (2.11) and $|C_+| \neq |C_-|$ that $m \neq 0$. □

Judgement method 4. *In the case of $nl \leqslant 0$, or $n = 0$, or $m = 0$, the nonlinear evolution equations, which can be reduced to Eq.* (II)*, possibly have a solution of the bell profile, or a solution of the kink profile with the asymptotic values $C_+ = -C_- \neq 0$; and do not possess the solution of the kink profile with $|C_+| \neq |C_-|$.*

The judgement method is given by Judgement method 3.

Judgement method 5. *Assume that the nonlinear evolution equations can be reduced to the following equation*:

$$a''(\xi) + la(\xi) + ma^q(\xi) = 0, \quad m \neq 0, \tag{2.13}$$

where q is even. Then, its solutions only possibly have a solution of the bell profile, and do not possess the solution of the kink profile with the asymptotic values $|C_+| \neq |C_-|$, or $C_+ = -C_- \neq 0$.

Proof. The (2.13) is a special case of Eq. (II) for $n = 0$ and even q. Judgement method 4 shows that Eq. (2.13) only possible has solitary-wave solution of the bell profile or the kink profile with $C_+ = -C_- \neq 0$. Now, assume that there exists a kink profile soliton with $C_+ = -C_- \neq 0$ in (2.13), then (2.11) implies that

$$m\bigl(1 - (-1)^{q+1}\bigr)C_-^{q+1} = 0.$$

Using $C_- \neq 0$, we have $m = 0$, which is contradictory with assumption $m \neq 0$. The method 5 is obtained. □

These judgement methods inform us shapes and formulae of the possible solutions of the solitary wave for many nonlinear evolution equations, when the solutions are unknown. At the same time, the methods also are an guidance to find exact or approximation solutions.

Now, we consider two examples. The first one is KdV equation,

$$u_t + auu_x + u_{xxx} = 0 \quad (a \neq 0), \tag{2.14}$$

with conditions

$$u'(\xi), u''(\xi) \to 0, \quad |\xi| \to +\infty, \tag{2.15}$$

and the asymptotic values $C_\pm$ which satisfy

$$-vx + \frac{a}{2}x^2 = 0. \tag{2.16}$$

It is easy to know that the solution of the problem (2.14)–(2.16) satisfies

$$u''(\xi) - vu(\xi) + \frac{a}{2}u^2(\xi) = 0, \quad \xi = x - vt. \tag{2.17}$$

It follows from Judgement method 5 that the problem (2.14)–(2.16) only possibly has a solution of the bell profile, which has been found in [13]. Therefore, we do not need to consider other solutions of the solitary wave for the problem.

The second example is the modified improvement Boussinesq equation

$$u_{tt} - u_{xx} + \frac{a}{3}\left(u^3\right)_{xx} - \beta^2 u_{xxtt} = 0, \tag{2.18}$$

with condition (2.15) and the asymptotic values $C_\pm$ which satisfy

$$\left(v^2 - 1\right)x + \frac{a}{3}x^3 = 0. \tag{2.19}$$

The solution of above problem satisfies

$$u''(\xi) - \frac{v^2 - 1}{v^2\beta^2}u(\xi) - \frac{a}{3v^2\beta^2}u^3(\xi) = 0, \quad \xi = x - vt. \tag{2.20}$$

In view of Judgement method 4, the problem (2.15), (2.18) and (2.19) only possibly has a solution of the bell profile, which has been found in [5], or a solution of the kink profile with $C_+ = -C_- \neq 0$. Thus, We will try to find new such solitary wave in Section 4.

3. Exact solutions of the Lienard equations (I) and (II)

We first consider the kink profile solutions of the Lienard equation (I). Let

$$a(\xi) = \phi(\xi)^{1/(q-1)}, \tag{3.1}$$

then it follows from Eq. (I) that

$$\begin{aligned} &\frac{1}{q-1}\phi(\xi)\phi''(\xi) + \frac{2-q}{(q-1)^2}\phi'^2(\xi) + \frac{r}{q-1}\phi(\xi)\phi'(\xi) \\ &\quad + l\phi^2(\xi) + m\phi^3(\xi) + n\phi^4(\xi) = 0. \end{aligned} \tag{3.2}$$

Now, we assume that solution of Eq. (3.2) has the following form:

$$\phi(\xi) = \frac{Ae^{\alpha(\xi+\xi_0)}}{(1 + e^{\alpha(\xi+\xi_0)})} = \frac{A}{2}\left[1 + \tanh\frac{\alpha}{2}(\xi + \xi_0)\right]. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{cases} \alpha^2 + r(q-1)\alpha + (q-1)^2 l = 0, \\ -\alpha^2 + r\alpha + 2(q-1)l + (q-1)mA = 0, \\ nA^2 + mA + l = 0. \end{cases} \tag{3.4}$$

Solve (3.4) lead to the following two groups of solutions:

$$\begin{cases} A_{1,2} = -\frac{q}{(q-1)(1+q)n}\Big[(q-1)m \mp r\sqrt{-\frac{(q-1)^2 n}{q}}\,\Big], \\ \alpha_{1,2} = \pm\sqrt{-\frac{(q-1)^2 n}{q}}A_{1,2}, \\ l_{1,2} = -nA_{1,2}^2 - mA_{1,2}. \end{cases} \tag{3.5}$$

Substituting (3.5) into (3.4) implies the following

Theorem 1. *Suppose that $n < 0$, $q \neq 1$.*

(1) *If q makes $A_1^{1/(q-1)}$ meaningful and $l = l_1 = -nA_1^2 - mA_1$, then Lienard equation* (I) *has a kink profile solution*

$$a_1(\xi) = \left(\frac{A_1}{2}\left[1 + \tanh\frac{1}{2}\sqrt{-\frac{(q-1)^2 n}{q}}A_1(\xi + \xi_0)\right]\right)^{1/(q-1)}. \tag{3.6}$$

(2) *If q makes $(A_2)^{1/(q-1)}$ meaningful and $l = l_2 = -nA_2^2 - mA_2$, then Lienard equation* (I) *has another kink profile solution*

$$a_2(\xi) = \left(\frac{A_2}{2}\left[1 - \tanh\frac{1}{2}\sqrt{-\frac{(q-1)^2 n}{q}}A_2(\xi + \xi_0)\right]\right)^{1/(q-1)}, \tag{3.7}$$

where A_1 and A_2 is given by (3.5).

It is easy to verify that the solutions (3.6) and (3.7) for the Lienard equation (I) satisfy the condition (2.1) and their asymptotic values satisfy the algebraic equation

$$lx + mx^q + nx^{2q-1} = 0. \tag{3.8}$$

Direct calculation implies that $a_1(\xi)$ and $a_2(\xi)$ satisfy formulae (2.8) and (2.9).

When $r = 0$, we have that

$$A_1 = A_2 = -\frac{qm}{(1+q)n}, \qquad l_1 = l_2 = \frac{qm^2}{(1+q)^2 n},$$
$$\alpha_{1,2} = \pm|q-1|\sqrt{-l}. \tag{3.9}$$

Hence, the following theorem is obtained from Theorem 1.

Theorem 2. *Suppose that q makes $(m)^{1/(q-1)}$ meaningful and $q \neq 1$, $m \neq 0$, $n < 0$. Then Lienard equation* (II) *has a kink profile solution with $|C_+| \neq |C_-|$,*

$$a(\xi) = \left(-\frac{mq}{2n(1+q)}\left[1 \pm \tanh\frac{1}{2}|q-1|\sqrt{-l}\,(\xi + \xi_0)\right]\right)^{1/(q-1)}, \tag{3.10}$$

where $l = \frac{qm^2}{(1+q)^2 n}$.

It is clear that the solution (3.10) satisfies the conditions (2.1) and its asymptotic values satisfy (2.10) and (2.11).

Now, we are to find the bell profile solution for Eq. (II). Substituting Eq. (3.1) into (II) implies that

$$\frac{1}{q-1}\phi(\xi)\phi''(\xi)+\frac{2-q}{(q-1)^2}\bigl(\phi'(\xi)\bigr)^2+l\phi^2(\xi)+m\phi^3(\xi)+n\phi^4(\xi)=0. \tag{3.11}$$

We assume that solution of Eq. (3.11) has the following form:

$$\phi(\xi)=\frac{Ae^{\alpha(\xi+\xi_0)}}{(1+e^{\alpha(\xi+\xi_0)})^2+Be^{\alpha(\xi+\xi_0)}}=\frac{A\operatorname{sech}^2\frac{\alpha}{2}(\xi+\xi_0)}{4+B\operatorname{sech}^2\frac{\alpha}{2}(\xi+\xi_0)}, \tag{3.12}$$

where A, B and α are constant to be given, ξ_0 is an any parameter. Substituting (3.12) into (3.11), we obtain

$$\begin{cases} \frac{1}{(q-1)^2}\alpha^2+l=0, \\ -\frac{1}{q-1}\alpha^2(2+B)+2l(2+B)+mA=0, \\ \frac{2(1-2q)}{(q-1)^2}\alpha^2+[2+(2+B)^2]l+mA(2+B)+nA^2=0. \end{cases} \tag{3.13}$$

Solve (3.13) lead to the following two groups of solutions:

$$\begin{cases} \alpha=|q-1|\sqrt{-l} \quad (l<0), \\ A_{1,2}=\pm\frac{2|l|(1+q)\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}, \\ B_{1,2}=2\Bigl(-1\pm\frac{m\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}\Bigr). \end{cases} \tag{3.14}$$

Substituting (3.14) into (3.12), we get two solutions of Eq. (3.11)

$$\phi_1(\xi)=\frac{\frac{|l|(1+q)\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}\operatorname{sech}^2\frac{|q-1|}{2}\sqrt{-l}\,(\xi+\xi_0)}{2+\bigl(-1+\frac{m\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}\bigr)\operatorname{sech}^2\frac{|q-1|}{2}\sqrt{-l}\,(\xi+\xi_0)} \tag{3.15}$$

and

$$\phi_2(\xi)=\frac{-\frac{|l|(1+q)\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}\operatorname{sech}^2\frac{|q-1|}{2}\sqrt{-l}\,(\xi+\xi_0)}{2-\bigl(1+\frac{m\sqrt{q}}{\sqrt{qm^2-nl(1+q)^2}}\bigr)\operatorname{sech}^2\frac{|q-1|}{2}\sqrt{-l}\,(\xi+\xi_0)}. \tag{3.16}$$

It is easy to verify that

$qm^2-nl(1+q)^2>0$, under the condition of $l<0$, $q\neq 1$ and $n\geqslant 0$;

$\phi_1(\xi)>0$, $\forall\xi\in R$, if $n>0$, or $n=0$ and $m>0$;

$\phi_2(\xi)<0$, $\forall\xi\in R$, if $n>0$, or $n=0$ and $m<0$.

From the above discussion, the following theorem is obtained.

Theorem 3. *Suppose that $l<0$ and $q\neq 1$.*

(1) *If $n>0$, or $n=0$, $m>0$, then Lienard equation* (II) *has a bell profile solution*

$$a(\xi)=\bigl[\phi_1(\xi)\bigr]^{1/(q-1)}. \tag{3.17}$$

(2) *If q makes $(e)^{1/(q-1)}$ meaningful for any negative number e, and if the condition $n > 0$, or $n = 0$, $m < 0$ holds, then Eq.* (II) *has another bell profile solution*

$$a(\xi) = \left[\phi_2(\xi)\right]^{1/(q-1)}, \tag{3.18}$$

where $\phi_1(\xi)$ and $\phi_2(\xi)$ is given by (3.15) *and* (3.16), *respectively.*

It is easy to verify that the solutions (3.17) and (3.18) of the Lienard equation (II) satisfy the condition (2.1) and $a(\xi) \to 0$, as $|\xi| \to \infty$. Hence, the solutions have bell profile.

Using the similar deduction, the following theorem is gotten

Theorem 4. *If $l > 0$, $n < 0$, Eq.* (II) *with $m = 0$, $q = 2$, has the following kink solution with $C_+ = -C_- \neq 0$:*

$$a(\xi) = \pm\sqrt{-\frac{l}{n}}\tanh\frac{1}{2}\sqrt{2l}(\xi + \xi_0). \tag{3.19}$$

In Theorems 1–4, the kink solutions for Eq. (I), and the bell and kink solutions for Eq. (II) are found. But, it is possible to have other solutions of the bell or kink profile for Eqs. (I) and (II). For example, the equation

$$a''(\xi) + la(\xi) + na^{2k+1}(\xi) = 0, \quad k = 1, 2, \ldots, \tag{3.20}$$

is possible to have the kink profile solitons with $C_+ = -C_- \neq 0$ in view of the judgments 4. We only find the soliton solution of the kink profile for Eq. (3.21) in Theorem 4 when $k = 1$. Find the kink profile soliton of (3.21) for $k \neq 1$ is a valuable problem.

In the Sections 4–6, we will apply the results given in Sections 2 and 3 to find the solitary-wave solutions of several nonlinear evolution equations.

4. The exact solitary-wave solutions for generalized modified Boussinesq equation

Consider solutions of the solitary wave for the generalized modified Boussinesq equation

$$u_{tt} - \delta u_{xxtt} + ru_{xxt} - \left(b_1 u + b_2 u^{p+1} + b_3 u^{2p+1}\right)_{xx} = 0, \quad p > 0. \tag{4.1}$$

Let $p = 1$, $\delta = \beta^2$, $r = 0$, $b_1 = 1$, $b_2 = -\frac{a}{2}$, $b_3 = 0$; and $p = 1$, $\delta = \beta^2$, $r = 0$, $b_1 = 1$, $b_2 = 0$, $b_3 = -\frac{a}{3}$; in (4.1), respectively, we have the improved Boussinesq equation

$$u_{tt} - u_{xx} + \frac{a}{2}\left(u^2\right)_{xx} - \beta^2 u_{xxtt} = 0, \tag{4.2}$$

and modified improved Boussinesq equation

$$u_{tt} - u_{xx} + \frac{a}{3}\left(u^3\right)_{xx} - \beta^2 u_{xxtt} = 0. \tag{4.3}$$

Let $u(x,t) = u(x - vt) \equiv u(\xi)$ be a traveling-wave solutions for Eq. (4.1) in the conditions (1.6), (1.7). Then, the solitary waves satisfy the equation

$$u''(\xi) + \frac{r}{\delta v}u'(\xi) + \frac{b_1 - v^2}{\delta v^2}u(\xi) + \frac{b_2}{\delta v^2}u^{p+1}(\xi) + \frac{b_3}{\delta v^2}u^{2p+1}(\xi) = 0. \tag{4.4}$$

Now, we apply the judgments to (4.4). The judgment 1 and Theorem 1 imply that Eq. (4.1) has solutions of the kink profile for $r \neq 0$, $b_3\delta < 0$. When $r = 0$, $\delta(b_1 - v^2) < 0$, $b_3 \geqslant 0$, Eq. (4.1) has solutions of the bell profile in view of the judgment 4 and Theorem 3. It follows from the judgment 3 and Theorem 2 that Eq. (4.1) has solutions of the kink profile with $|C_+| \neq |C_-|$ for $r = 0$, $b_3\delta < 0$, $b_2 \neq 0$.

Comparing (4.4) with Eq. (I), Theorem 1 implies the following

Theorem 5. *Suppose that* $b_3\delta < 0$,

$$A_{1,2} = -\frac{p+1}{b_3(p+2)}\left[b_2 \mp r\sqrt{-\frac{b_3}{\delta(p+1)}}\right],$$
$$v_{1,2} = \sqrt{b_1 + b_2 A_{1,2} + b_3 A_{1,2}^2}.$$

(1) *If* p *makes* $A_1^{1/p}$ *meaningful and* $b_1 + b_2 A_1 + b_3 A_1^2 > 0$, *then Eq.* (4.1) *has a solution of the kink profile with* $|C_+| \neq |C_-|$,

$$u(x,t) = \left(\frac{A_1}{2}\left[1 + \tanh\frac{p}{2v_1}\sqrt{-\frac{b_3}{\delta(p+1)}}A_1(x \pm v_1 t + \xi_0)\right]\right)^{1/p}. \tag{4.5}$$

(2) *If* p *makes* $A_2^{1/p}$ *meaningful and* $b_1 + b_2 A_2 + b_3 A_2^2 > 0$, *then Eq.* (4.1) *has a solution of the kink profile with* $|C_+| \neq |C_-|$,

$$u(x,t) = \left(\frac{A_2}{2}\left[1 - \tanh\frac{p}{2v_2}\sqrt{-\frac{b_3}{\delta(p+1)}}A_2(x \pm v_2 t + \xi_0)\right]\right)^{1/p}. \tag{4.6}$$

For the generalized modified Boussinesq equation without dissipative term ($r = 0$),

$$u_{tt} - \delta u_{xxtt} - \left(b_1 u + b_2 u^{p+1} + b_3 u^{2p+1}\right)_{xx} = 0, \quad p > 0, \tag{4.7}$$

and the conditions (1.6), (1.7), its solitary-wave solutions satisfy

$$u''(\xi) + \frac{b_1 - v^2}{\delta v^2}u(\xi) + \frac{b_2}{\delta v^2}u^{p+1}(\xi) + \frac{b_3}{\delta v^2}u^{2p+1}(\xi) = 0. \tag{4.8}$$

Comparing (4.8) with Eq. (II), Theorems 2–4 imply that

Theorem 6. *Suppose that* v *is traveling wave velocity.*

(1) *Assume that* $\delta(b_1 - v^2) < 0$. *If* $\delta b_3 > 0$; *or* $b_3 = 0$, $\delta b_2 > 0$, *then Eq.* (4.7) *has a bell profile solitary-wave solution*

$$u(x,t) = \left(\frac{\frac{|b_1 - v^2|(p+2)\sqrt{p+1}}{\sqrt{(p+1)b_2^2 - b_3(b_1 - v^2)(p+2)^2}}\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1 - v^2}{\delta v^2}}(x - vt + \xi_0)}{2 + \left(-1 + \frac{b_2|\delta|\sqrt{p+1}}{\delta\sqrt{(p+1)b_2^2 - b_3(b_1 - v^2)(p+2)^2}}\right)\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1 - v^2}{\delta v^2}}(x - vt + \xi_0)}\right)^{1/p}. \tag{4.9}$$

(2) *Assume that* $\delta(b_1 - v^2) < 0$, *and* p *makes* $e^{1/p}$ *meaningful for any negative number* e. *If* $b_3\delta > 0$; *or* $b_3 = 0$, $b_2\delta < 0$, *then Eq.* (4.7) *has a bell profile solitary-wave solution*

$$u(x,t)=\left(\frac{-\frac{|b_1-v^2|(p+2)\sqrt{p+1}}{\sqrt{(p+1)b_2^2-b_3(b_1-v^2)(p+2)^2}}\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1-v^2}{\delta v^2}}(x-vt+\xi_0)}{2-\left(1+\frac{b_2|\delta|\sqrt{p+1}}{\delta\sqrt{(p+1)b_2^2-b_3(b_1-v^2)(p+2)^2}}\right)\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1-v^2}{\delta v^2}}(x-vt+\xi_0)}\right)^{1/p}. \tag{4.10}$$

(3) *Assume that* $\delta b_3 < 0$ *and* p *makes* $(-b_2b_3)^{1/p}$ *meaningful. Then Eq.* (4.7) *has a kink profile solitary-wave solution with* $|C_+| \neq |C_-|$,

$$u(x,t)=\left(-\frac{b_2(p+1)}{2b_3(p+2)}\left[1\pm\tanh\frac{p}{2}\sqrt{-\frac{b_1-v^2}{\delta v^2}}(x-vt+\xi_0)\right]\right)^{1/p}, \tag{4.11}$$

where wave velocity

$$v^2=b_1-\frac{b_2^2(p+1)}{b_3(p+2)^2}.$$

(4) *Assume that* $p=1$ *and* $b_2=0$*. If* $b_3\delta<0$ *and* $\delta(b_1-v^2)>0$*, then Eq.* (4.7) *has a kink profile solitary-wave solution with* $|C_+|=|C_-|$,

$$u(x,t)=\pm\sqrt{-\frac{b_1-v^2}{b_3}}\tanh\frac{1}{2}\sqrt{\frac{2(b_1-v^2)}{\delta v^2}}(x-vt+\xi_0). \tag{4.12}$$

Let $p=1$, $\delta=\beta^2$, $b_1=1$, $b_2=-\frac{a}{2}$ and $b_3=0$, then Theorem 6 implies a bell profile solution of the improved Boussinesq equation (4.2),

$$u(x,t)=\frac{3(1-v^2)}{a}\operatorname{sech}^2\frac{\sqrt{v^2-1}}{2v\beta}(x-vt+\xi_0). \tag{4.13}$$

Let $p=1$, $\delta=\beta^2$, $b_1=1$, $b_2=0$ and $b_3=-\frac{a}{3}$ in (4.9), (4.10) and (4.12), we get a bell profile solution of the modified improved Boussinesq equation (4.3),

$$u(x,t)=\pm\sqrt{\frac{6(1-v^2)}{a}}\operatorname{sech}^2\frac{\sqrt{v^2-1}}{v\beta}(x-vt+\xi_0) \tag{4.14}$$

and a kink profile solution of Eq. (4.3) with $C_+=-C_-$,

$$u(x,t)=\pm\sqrt{\frac{3(1-v^2)}{a}}\tanh\frac{1}{2}\sqrt{\frac{2(1-v^2)}{v^2\beta^2}}(x-vt+\xi_0). \tag{4.15}$$

The solution (4.15) is a new soliton for the modified improved Boussinesq equation (4.3).

5. The solitary-wave solutions for generalized nonlinear wave equation

Consider generalized nonlinear wave equation

$$\delta u_{tt}-ku_{xx}+ru_t+b_1u+b_2u^{p+1}+b_3u^{2p+1}=0,\quad p>0,\ r\geqslant 0. \tag{5.1}$$

Equation (5.1) includes many important model problems, for example, generalized Fisher equation, approximation equation for the Sinh–Gordon equation, ϕ^4 equation, one-dimensional Klein–Gordon equation, Landau–Ginzburg–Higgs equation, Duffing equation and nonlinear telegram equation. Therefore, find solutions of the generalized nonlinear wave equation (5.1) is valuable. Let $\delta = 0$, $r = 1$, $k = D$, $b_1 = -s\beta$, $b_2 = -s(1-\beta)$, $b_3 = s$ and $\alpha = p$ in (5.1), we have the generalized Fisher equation,

$$u_t - Du_{xx} - s\beta u - s(1-\beta)u^{\alpha+1} + su^{2\alpha+1} = 0. \tag{5.2}$$

Take $r = 0$, $\delta = 1$ in (5.1) implies generalized one-dimensional Klein–Gordon equation [14]

$$u_{tt} - ku_{xx} + b_1 u + b_2 u^{p+1} + b_3 u^{2p+1} = 0. \tag{5.3}$$

It is easy to get that the traveling-wave solutions $u(x,t) = u(x - vt) \equiv u(\xi)$ for Eq. (5.1) satisfy

$$u''(\xi) - \frac{rv}{\delta v^2 - k}u'(\xi) + \frac{b_1}{\delta v^2 - k}u(\xi) + \frac{b_2}{\delta v^2 - k}u^{p+1}(\xi) + \frac{b_3}{\delta v^2 - k}u^{2p+1}(\xi) = 0, \tag{5.4}$$

which is a Lienard-type equation. Thus, if $r > 0$, $b_3(\delta v^2 - k) < 0$, Eq. (5.1) has solution of the kink profile in view of the judgments 1 and Theorem 1. Eq. (5.1) has solution of the bell profile in view of the judgments 4 and Theorem 3 when $r = 0$, $b_1 b_3 < 0$. In the case of $r = 0$, $b_3(\delta v^2 - k) < 0$, $b_2 \neq 0$, Eq. (5.1) has solution of the kink profile with $|C_+| \neq |C_-|$ in view of the judgments 3 and Theorem 2.

Comparing (5.4) with Eq. (I), Theorem 1 implies the following

Theorem 7. *Suppose that* $b_2^2 - 4b_1 b_3 \geqslant 0$,

$$A_{1,2} = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}}{2b_3},$$
$$v_{1,2} = \frac{|k|((p+1)b_1 - b_3 A_{1,2}^2)}{\sqrt{k[\delta((p+1)b_1 - b_3 A_{1,2}^2)^2 + (p+1)r^2 b_3 A_{1,2}^2]}}.$$

(1) *If* p *makes* $A_1^{1/p}$ *meaningful and* $k[\delta((p+1)b_1 - b_3 A_1^2)^2 + (p+1)r^2 b_3 A_1^2] > 0$, *then Eq.* (5.1) *has solutions of the kink profile*

$$u(x,t) = \left(\frac{A_1}{2}\left[1 \pm \tanh \frac{p}{2}\sqrt{-\frac{b_3 A_1^2}{(\delta v^2 - k)(p+1)}}\,(x \mp v_1 t + \xi_0)\right]\right)^{1/p}. \tag{5.5}$$

(2) *If* p *makes* $A_2^{1/p}$ *meaningful and* $k[\delta((p+1)b_1 - b_3 A_2^2)^2 + (p+1)r^2 b_3 A_2^2] > 0$, *then Eq.* (5.1) *has solutions of the kink profile*

$$u(x,t) = \left(\frac{A_2}{2}\left[1 \pm \tanh \frac{p}{2}\sqrt{-\frac{b_3 A_2^2}{(\delta v^2 - k)(p+1)}}\,(x \mp v_2 t + \xi_0)\right]\right)^{1/p}. \tag{5.6}$$

Let $\delta = 0$, $r = 1$, $k = D$, $b_1 = -s\beta$, $b_2 = -s(1-\beta)$, $b_3 = s$, $\alpha = p$, then above theorem implies the solitary-wave solutions for the generalized Fisher equation (5.2),

$$u(x,t) = \left(\frac{1}{2}\left[1 \pm \tanh \frac{\alpha}{2}\sqrt{\frac{s}{D(\alpha+1)}} \times \left(x \pm \sqrt{\frac{Ds}{\alpha+1}}\big((\alpha+1)\beta+1\big)t + \xi_0\right)\right]\right)^{1/\alpha} \tag{5.7}$$

and

$$u(x,t) = \left(-\frac{\beta}{2}\left[1 \pm \tanh \frac{\alpha}{2}\sqrt{\frac{s}{D(\alpha+1)}}\beta \times \left(x \pm \sqrt{\frac{Ds}{\alpha+1}}\big((\alpha+1)+\beta\big)t + \xi_0\right)\right]\right)^{1/\alpha}, \tag{5.8}$$

where the solution (5.7) was given in [6] and the solution (5.8) is new solitary waves for the Fisher equation (5.2).

Let $\delta = 0$, $r = 1$, $k = 1$, $b_1 = a$, $b_2 = -(a+1)$, $b_3 = 1$, $p = 1$ in Theorem 7, then we have Diffusion equation

$$u_t - u_{xx} = u(u-a)(1-u) \quad (-1 \leqslant a < 1), \tag{5.9}$$

and its solitary-wave solutions

$$u(x,t) = \frac{1}{2}\left[1 \pm \tanh \frac{\sqrt{2}}{4}\left(x \pm \frac{1-2a}{\sqrt{2}}t + \xi_0\right)\right] \tag{5.10}$$

and

$$u(x,t) = \frac{a}{2}\left[1 \pm \tanh \frac{\sqrt{2}a}{4}\left(x \pm \frac{a-2}{\sqrt{2}}t + \xi_0\right)\right], \tag{5.11}$$

which are the same as the solutions given in [15].

Let $\delta = 0$, $r = 1$, $k = 1$, $b_1 = -1$, $b_2 = 0$, $b_3 = 1$, $p = \frac{\alpha}{2}$ in Theorem 7, then we have another generalized Fisher equation

$$u_t - u_{xx} = u - u^{\alpha+1}, \quad \alpha \in (0, +\infty), \tag{5.12}$$

and its solitary-wave solutions

$$u(x,t) = \left\{\frac{1}{2}\left[1 - \tanh \frac{\alpha}{2\sqrt{2\alpha+4}}\left(x - \frac{\alpha+4}{\sqrt{2\alpha+4}}t + \xi_0\right)\right]\right\}^{2/\alpha} \tag{5.13}$$

and

$$u(x,t) = \left\{\frac{1}{2}\left[1 + \tanh \frac{\alpha}{2\sqrt{2\alpha+4}}\left(x + \frac{\alpha+4}{\sqrt{2\alpha+4}}t + \xi_0\right)\right]\right\}^{2/\alpha}, \tag{5.14}$$

where (5.13) has been given in [16] and the solution (5.14) is our new result.

When $r = 0$, we can assume $\delta = 1$ without generality. Thus, Eq. (5.1) is reduced to the one-dimensional Klein–Gordon equation (5.3). Its solitary-wave solutions satisfy

$$u''(\xi) + \frac{b_1}{v^2-k}u(\xi) + \frac{b_2}{v^2-k}u^{p+1}(\xi) + \frac{b_3}{v^2-k}u^{2p+1}(\xi) = 0. \tag{5.15}$$

Comparing (5.15) with Eq. (II) and taking $q = p+1$, $l = \frac{b_1}{v^2-k}$, $m = \frac{b_2}{v^2-k}$, $n = \frac{b_3}{v^2-k}$ Theorems 2–4 imply that

Theorem 8. *Suppose that v is traveling wave velocity.*

(1) *Assume that $b_1(v^2-k) < 0$. If $b_3(v^2-k) > 0$; or $b_3 = 0$, $b_2(v^2-k) > 0$, then Eq.* (5.3) *has a bell profile solitary-wave solution*

$$u(x,t) = \left(\frac{\frac{|b_1|(p+2)\sqrt{p+1}}{\sqrt{(p+1)b_2^2 - b_3 b_1 (p+2)^2}}\operatorname{sech}^2 \frac{p}{2}\sqrt{-\frac{b_1}{v^2-k}}\,(x-vt+\xi_0)}{2+\left(-1+\frac{b_2|v^2-k|\sqrt{p+1}}{(v^2-k)\sqrt{(p+1)b_2^2-b_3b_1(p+2)^2}}\right)\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1}{v^2-k}}\,(x-vt+\xi_0)}\right)^{1/p}. \tag{5.16}$$

(2) *Assume that $b_1(v^2-k) < 0$, and p makes $e^{1/p}$ meaningful for any negative number e. If $b_3(v^2-k) > 0$; or $b_3 = 0$, $b_2(v^2-k) < 0$, then Eq.* (5.3) *has another bell profile solitary-wave solution*

$$u(x,t) = \left(\frac{-\frac{|b_1|(p+2)\sqrt{p+1}}{\sqrt{(p+1)b_2^2 - b_3 b_1 (p+2)^2}}\operatorname{sech}^2 \frac{p}{2}\sqrt{-\frac{b_1}{v^2-k}}\,(x-vt+\xi_0)}{2-\left(1+\frac{b_2|v^2-k|\sqrt{p+1}}{(v^2-k)\sqrt{(p+1)b_2^2-b_3b_1(p+2)^2}}\right)\operatorname{sech}^2\frac{p}{2}\sqrt{-\frac{b_1}{v^2-k}}\,(x-vt+\xi_0)}\right)^{1/p}. \tag{5.17}$$

(3) *Assume that $b_3(v^2-k) < 0, b_2 \neq 0$, p makes $(-b_2b_3)^{1/p}$ meaningful and $b_1 = \frac{(p+1)b_2^2}{(p+2)^2 b_3}$ holds. Then Eq.* (5.3) *has a kink profile solitary-wave solution with $|C_+| \neq |C_-|$,*

$$u(x,t) = \left(-\frac{b_2(p+1)}{2b_3(p+2)}\left[1 \pm \tanh\frac{p}{2}\sqrt{-\frac{b_1}{v^2-k}}\,(x-vt+\xi_0)\right]\right)^{1/p}. \tag{5.18}$$

(4) *Assume that $p = 1$ and $b_2 = 0$. If $b_1(v^2-k) > 0$ and $b_3(v^2-k) < 0$, then Eq.* (5.3) *has a kink profile solitary-wave solution with $|C_+| = |C_-|$,*

$$u(x,t) = \pm\sqrt{-\frac{b_1}{b_3}}\tanh\frac{1}{2}\sqrt{\frac{2b_1}{v^2-k}}\,(x-vt+\xi_0). \tag{5.19}$$

6. The exact solitary-wave solutions for generalized Zakharov equations

Consider the generalized Zakharov equations

$$\begin{cases} H_{tt} - H_{xx} = (|u|^{2p})_{xx}, \\ iu_t + u_{xx} = Hu + b_1|u|^{2p}u + b_2|u|^{4p}u, \end{cases} \tag{6.1}$$

which are reduced to the famous Zakharov equations (see [11,12])

$$\begin{cases} H_{tt} - H_{xx} = (|u|^2)_{xx}, \\ iu_t + u_{xx} = Hu, \end{cases} \tag{6.1$'$}$$

if $p = 1$, $b_1 = 0$, $b_2 = 0$. We will find solutions of the following form for (6.1):

$$\begin{cases} u(x,t) = e^{i[kx-(kv+\omega)t]}a(x-vt), \\ H(x,t) = H(x-vt). \end{cases} \tag{6.2}$$

Assume that

$$a'(\xi), H'(\xi) \to 0, \quad |\xi| \to \infty, \tag{6.3}$$

and

$$\left(v^2-1\right)C_-^H = C_-^{2p}, \qquad \left(v^2-1\right)C_+^H = C_+^{2p}, \tag{6.4}$$

where $\xi = x - vt$, $C_\pm^H = \lim_{|\xi|\to\pm\infty} H(\xi)$, $C_\pm = \lim_{|\xi|\to\pm\infty} a(\xi)$. Let $k = \frac{v}{2}$. Substitute (6.2) into (6.1) implies

$$H(\xi) = \frac{1}{v^2-1}a^{2p}(\xi), \tag{6.5}$$

and $a(\xi)$ satisfies

$$a''(\xi) + \frac{v^2+4\omega}{4}a(\xi) - \frac{b_1(v^2-1)+1}{v^2-1}a^{2p+1}(\xi) - b_2a^{4p+1}(\xi) = 0. \tag{6.6}$$

Equation (6.6) is Lienard (II)-type. Thus, if $v^2 + 4\omega < 0$, $b_2 \leqslant 0$, Eq. (6.1) has solutions of the bell profile in view of the judgments 4 and Theorem 3. Equation (6.1) has solutions of the kink profile with $|C_+| \neq |C_-|$ in view of the judgments 3 and Theorem 2 when $b_2(v^2 + 4\omega) < 0$, $b_1(v^2 - 1) + 1 \neq 0$.

Comparing (6.6) with Eq. (II), Theorems 2–4 imply:

Theorem 9. *Suppose that v is traveling wave velocity and*

$$G = \frac{\sqrt{2p+1}}{2\sqrt{(2p+1)(b_1(v^2-1)+1)^2 + b_2(p+1)^2(v^2-1)^2(v^2+4\omega)}}.$$

(1) *Assume that $v^2 + 4\omega < 0$. If $b_2 < 0$; or $b_2 = 0$, $(b_1(v^2-1)+1)(v^2-1) < 0$, then Eq.* (6.1) *has bell profile solitary-wave solutions*

$$\begin{cases} u(x,t) = e^{i[\frac{v}{2}x-(\frac{v^2}{2}+\omega)t]} \\ \qquad \times \left(\dfrac{(p+1)G|(v^2-1)(v^2+4\omega)|\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}{2-\left(1+\frac{2|v^2-1|(b_1(v^2-1)+1)G}{(v^2-1)}\right)\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}\right)^{1/(2p)}, \\ H(x,t) = \frac{1}{v^2-1}\left(\dfrac{(p+1)G|(v^2-1)(v^2+4\omega)|\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}{2-\left(1+\frac{2|v^2-1|(b_1(v^2-1)+1)G}{(v^2-1)}\right)\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}\right). \end{cases} \tag{6.7}$$

(2) *Assume that $v^2 + 4\omega < 0$, and p makes $e^{1/2p}$ meaningful for any negative number e. If $b_2 < 0$; or $b_2 = 0$, $(b_1(v^2 - 1) + 1)(v^2 - 1) > 0$, then Eq.* (6.1) *has bell profile solitary-wave solutions*

$$\begin{cases} u(x,t) = e^{i[\frac{v}{2}x - (\frac{v^2}{2}+\omega)t]} \\ \qquad \times \left(\dfrac{-(p+1)G|(v^2-1)(v^2+4\omega)| \operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}{2-\left(1-\frac{2|v^2-1|(b_1(v^2-1)+1)G}{(v^2-1)}\right)\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)} \right)^{1/(2p)}, \\ H(x,t) = \frac{1}{v^2-1}\left(\dfrac{-(p+1)G|(v^2-1)(v^2+4\omega)| \operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)}{2-\left(1-\frac{2|v^2-1|(b_1(v^2-1)+1)G}{(v^2-1)}\right)\operatorname{sech}^2 p\sqrt{-\frac{v^2+4\omega}{4}}\,(x-vt+\xi_0)} \right). \end{cases} \tag{6.8}$$

(3) *Assume that $b_2 > 0$, p makes $(-(b_1(v^2-1)+1)(v^2-1))^{1/(2p)}$ meaningful. Let the wave velocity v satisfy $b_2(p+1)^2(v^2+4\omega)(v^2-1)^2 = -(2p+1)(b_1(v^2-1)+1)^2$. Then Eq.* (6.1) *has a kink profile solitary-wave solution with $|C_+| \neq |C_-|$,*

$$\begin{cases} u(x,t) = e^{i[\frac{v}{2}x - (\frac{v^2}{2}+\omega)t]} \\ \quad \times \left(\frac{-(2p+1)(b_1(v^2-1)+1)}{4b_2(p+1)(v^2-1)} \left[1 \pm \tanh p\sqrt{-\frac{v^2+4\omega}{4}}(x - vt + \xi_0) \right] \right)^{1/(2p)}, \\ H(x,t) = -\frac{(2p+1)(b_1(v^2-1)+1)}{4b_2(p+1)(v^2-1)^2}\left(1 \pm \tanh p\sqrt{-\frac{v^2+4\omega}{4}}(x - vt + \xi_0)\right). \end{cases} \tag{6.9}$$

(4) *Assume that $p = 1$ and $b_2 = 0$. If $v^2 + 4\omega > 0$, $b_1 + \frac{1}{v^2-1} > 0$, then Eq.* (6.1) *has a kink profile solitary-wave solution with $|C_+| = |C_-|$,*

$$\begin{cases} u(x,t) = \pm e^{i[\frac{v}{2}x - (\frac{v^2}{2}+\omega)t]} \sqrt{\frac{(v^2+4\omega)(v^2-1)}{4(b_1(v^2-1)+1)}} \tanh \frac{1}{2}\sqrt{\frac{v^2+4\omega}{2}}\,(x - vt + \xi_0), \\ H(x,t) = \frac{1}{v^2-1}\left(\pm \sqrt{\frac{(v^2+4\omega)(v^2-1)}{4(b_1(v^2-1)+1)}} \tanh \frac{1}{2}\sqrt{\frac{v^2+4\omega}{2}}\,(x - vt + \xi_0)\right)^2. \end{cases} \tag{6.10}$$

Let $p = 1$, $b_1 = 0$, $b_2 = 0$ in the (6.7)*, we have solutions of the bell profile for Zakharov equation* (6.1′)*,*

$$\begin{cases} u(x,t) = e^{i[\frac{v}{2}x - (\frac{v^2}{2}+\omega)t]} \sqrt{\frac{(v^2+4\omega)(v^2-1)}{2}} \operatorname{sech} \sqrt{-\frac{v^2+4\omega}{4}}\,(x - vt + \xi_0), \\ H(x,t) = \frac{v^2+4\omega}{2} \operatorname{sech}^2 \sqrt{-\frac{v^2+4\omega}{4}}\,(x - vt + \xi_0), \end{cases}$$

which are the same as solutions given in [11] *and* [12].

Taking $b_1 = 0$ in (6.10), we have that

Theorem 10. *Assume that $v^2 + 4\omega > 0$, $v^2 - 1 > 0$. Then the Zakharov equation* (6.1′) *has solitary-wave solutions*

$$\begin{cases} u(x,t) = \pm e^{i[\frac{v}{2}x - (\frac{v^2}{2}+\omega)t]} \sqrt{\frac{(v^2+4\omega)(v^2-1)}{4}} \tanh \frac{1}{2}\sqrt{\frac{v^2+4\omega}{2}}\,(x - vt + \xi_0), \\ H(x,t) = \frac{v^2+4\omega}{4}\left(1 - \operatorname{sech}^2 \frac{1}{2}\sqrt{\frac{v^2+4\omega}{2}}\,(x - vt + \xi_0)\right), \end{cases} \tag{6.11}$$

which are a new solitary-wave solutions.

7. Remark

By means of proper transform, some of other nonlinear evolution equations can be led to the Lienard equation (I) or (II), for instance, generalized BBM equation

$$u_t + au^p u_x + bu^{2p} u_x + \delta u_{xxt} = 0, \quad \delta \neq 0,\ p > 0;$$

and generalized kP equation

$$\frac{\partial}{\partial x}\left\{u_t + au^p u_x + bu^{2p} u_x + ru_{xx} + \delta U_{xxx}\right\} + 3k^2 u_{yy} = 0, \quad \delta \neq 0,\ p > 0.$$

Hence, we can apply the approaches and formulae given in this paper to look for the explicit exact solitary-wave solutions of these equations.

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