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Invariant chiral differential operators and the $\mathcal{W}_3$ algebra

Andrew R. Linshaw

Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, United States

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ABSTRACT

Attached to a vector space V is a vertex algebra $\mathcal{S}(V)$ known as the $\beta\gamma$-system or algebra of chiral differential operators on V. It is analogous to the Weyl algebra $\mathcal{D}(V)$, and is related to $\mathcal{D}(V)$ via the Zhu functor. If G is a connected Lie group with Lie algebra $\mathfrak{g}$, and V is a linear G-representation, there is an action of the corresponding affine algebra on $\mathcal{S}(V)$. The invariant space $\mathcal{S}(V)^{\mathfrak{g}[t]}$ is a commutant subalgebra of $\mathcal{S}(V)$, and plays the role of the classical invariant ring $\mathcal{D}(V)^G$. When G is an abelian Lie group acting diagonally on V, we find a finite set of generators for $\mathcal{S}(V)^{\mathfrak{g}[t]}$, and show that $\mathcal{S}(V)^{\mathfrak{g}[t]}$ is a simple vertex algebra and a member of a Howe pair. The Zamolodchikov $\mathcal{W}_3$ algebra with $c = -2$ plays a fundamental role in the structure of $\mathcal{S}(V)^{\mathfrak{g}[t]}$.

1. Introduction

Let G be a connected, reductive Lie group acting algebraically on a smooth variety X. Throughout this paper, our base field will always be **C**. The ring $\mathcal{D}(X)^G$ of invariant differential operators on X has been much studied in recent years. In the case where X is the homogeneous space G/K, $\mathcal{D}(X)^G$ was originally studied by Harish-Chandra in order to understand the various function spaces attached to X [8,9]. In general, $\mathcal{D}(X)^G$ is not a homomorphic image of the universal enveloping algebra of a Lie algebra, but it is believed that $\mathcal{D}(X)^G$ shares many properties of enveloping algebras. For example, the center of $\mathcal{D}(X)^G$ is always a polynomial ring [12]. In the case where G is a torus, the structure and representation theory of the rings $\mathcal{D}(X)^G$ were studied extensively in [16], but much less is known about $\mathcal{D}(X)^G$ when G is nonabelian. The first step in this direction was taken by Schwarz in [17], in which he considered the special but nontrivial case where $G = SL(3)$ and X is the adjoint representation. In this case, he found generators for $\mathcal{D}(X)^G$, showed that $\mathcal{D}(X)^G$ is an FCR algebra, and classified its finite-dimensional modules.

1.1. *A vertex algebra analogue of $\mathcal{D}(X)^G$*

In [15], Malikov–Schechtman–Vaintrob introduced a sheaf of vertex algebras on any smooth variety X known as the chiral de Rham complex. For an affine open set $V \subset X$, the algebra of sections over V is just a copy of the $bc\beta\gamma$-system $\mathcal{S}(V) \otimes \mathcal{E}(V)$, localized over the function ring $\mathcal{O}(V)$. A natural question is whether there exists a subsheaf of "chiral differential operators" on X, whose space of sections over V is just the (localized) $\beta\gamma$-system $\mathcal{S}(V)$. For general X, there is a cohomological obstruction to the existence of such a sheaf, but it does exist in certain special cases such as affine spaces and certain homogeneous spaces [15,7].

In this paper, we focus on the case where X is the affine space $V = \mathbf{C}^n$, and we take $\mathcal{S}(V)$ to be our algebra of chiral differential operators on V. $\mathcal{S}(V)$ is related to $\mathcal{D}(V)$ via the *Zhu functor*, which attaches to every vertex algebra $\mathcal{V}$ an associative algebra $A(\mathcal{V})$ known as the *Zhu algebra* of $\mathcal{V}$, together with a surjective linear map $\pi_{Zh} : \mathcal{V} \to A(\mathcal{V})$.

E-mail address: alinshaw@math.ucsd.edu.

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If V carries a linear action of a group G with Lie algebra $\mathfrak{g}$, the corresponding representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ induces a vertex algebra homomorphism

$$\mathcal{O}(\mathfrak{g}, B) \to \mathcal{S}(V). \tag{1.1}$$

Here $\mathcal{O}(\mathfrak{g}, B)$ is the current algebra of $\mathfrak{g}$ associated to the bilinear form $B(\xi, \eta) = -Tr(\rho(\xi)\rho(\eta))$ on $\mathfrak{g}$. Letting Θ denote the image of $\mathcal{O}(\mathfrak{g}, B)$ inside $\mathcal{S}(V)$, the commutant $\mathrm{Com}(\Theta, \mathcal{S}(V))$, which we denote by $\mathcal{S}(V)^{\Theta+}$, is just the invariant space $\mathcal{S}(V)^{\mathfrak{g}[t]}$. Accordingly, we call $\mathcal{S}(V)^{\Theta+}$ the algebra of *invariant chiral differential operators* on V. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}(V)^{\Theta+} & \stackrel{\iota}{\hookrightarrow} & \mathcal{S}(V) \\ \pi \downarrow & & \pi_{Zh} \downarrow \\ \mathcal{D}(V)^{G} & \stackrel{\iota}{\hookrightarrow} & \mathcal{D}(V). \end{array} \tag{1.2}$$

Here the horizontal maps are inclusions, and the map π on the left is the restriction of the Zhu map on $\mathcal{S}(V)$ to the subalgebra $\mathcal{S}(V)^{\Theta+}$. In general, π is not surjective, and $\mathcal{D}(V)^G$ need not be the Zhu algebra of $\mathcal{S}(V)^{\Theta+}$.

For a general vertex algebra $\mathcal{V}$ and subalgebra $\mathcal{A}$, the commutant $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ was introduced by Frenkel–Zhu in [4], generalizing a previous construction in representation theory [10] and conformal field theory [6] known as the coset construction. We regard $\mathcal{V}$ as a module over $\mathcal{A}$ via the left regular action, and we regard $\mathrm{Com}(\mathcal{A}, \mathcal{V})$, which we often denote by $\mathcal{V}^{\mathcal{A}+}$, as the invariant subalgebra. Finding a set of generators for $\mathcal{V}^{\mathcal{A}+}$, or even determining when it is finitely generated as a vertex algebra, is generally a non-trivial problem. It is also natural to study the double commutant $\mathrm{Com}(\mathcal{V}^{\mathcal{A}+}, \mathcal{V})$, which always contains $\mathcal{A}$. If $\mathcal{A} = \mathrm{Com}(\mathcal{V}^{\mathcal{A}+}, \mathcal{V})$, we say that $\mathcal{A}$ and $\mathcal{V}^{\mathcal{A}+}$ form a *Howe pair* inside $\mathcal{V}$. Since

$$\mathrm{Com}(\mathrm{Com}(\mathcal{V}^{\mathcal{A}+}, \mathcal{V}), \mathcal{V}) = \mathcal{V}^{\mathcal{A}+},$$

a subalgebra $\mathcal{B}$ is a member of a Howe pair if and only if $\mathcal{B} = \mathcal{V}^{\mathcal{A}+}$ for some $\mathcal{A}$.

Here are some natural questions one can ask about $\mathcal{S}(V)^{\Theta+}$ and its relationship to $\mathcal{D}(V)^G$.

Question 1.1. *When is $\mathcal{S}(V)^{\Theta+}$ finitely generated as a vertex algebra? Can we find a set of generators?*

Question 1.2. *When do $\mathcal{S}(V)^{\Theta+}$ and Θ form a Howe pair inside $\mathcal{S}(V)$? In the case where $G = SL(2)$ and V is the adjoint module, this question was answered affirmatively in* [13].

Question 1.3. *What are the vertex algebra ideals in $\mathcal{S}(V)^{\Theta+}$, and when is $\mathcal{S}(V)^{\Theta+}$ a simple vertex algebra?*

Question 1.4. *When is $\mathcal{S}(V)^{\Theta+}$ a conformal vertex algebra?*

Question 1.5. *When is $\pi : \mathcal{S}(V)^{\Theta+} \to \mathcal{D}(V)^G$ surjective? More generally, describe* $\mathrm{Im}(\pi)$ *and* $\mathrm{Coker}(\pi)$.

These questions are somewhat outside the realm of classical invariant theory because the Lie algebra $\mathfrak{g}[t]$ is both infinite-dimensional and non-reductive. Moreover, when G is nonabelian, $\mathcal{S}(V)$ need not decompose into a sum of irreducible $\mathcal{O}(\mathfrak{g}, B)$-modules. The case where G is simple and V is the adjoint module is of particular interest to us, since in this case $\mathcal{S}(V)^{\Theta+}$ is a subalgebra of the complex $(\mathcal{W}(\mathfrak{g})_{bas}, d)$ which computes the chiral equivariant cohomology of a point [14].

In this paper, we focus on the case where G is an abelian group acting faithfully and diagonalizably on V. This is much easier than the general case because $\mathcal{O}(\mathfrak{g}, B)$ is then a tensor product of Heisenberg vertex algebras, which act completely reducibly on $\mathcal{S}(V)$. For any such action, we find a finite set of generators for $\mathcal{S}(V)^{\Theta+}$, and show that $\mathcal{S}(V)^{\Theta+}$ is a simple vertex algebra. Moreover, $\mathcal{S}(V)^{\Theta+}$ and Θ always form a Howe pair inside $\mathcal{S}(V)$. For generic actions, we show that $\mathcal{S}(V)^{\Theta+}$ admits a k-parameter family of conformal structures where $k = \dim V - \dim \mathfrak{g}$, and we find a finite set of generators for $\mathrm{Im}(\pi)$. Finally, we show that $\mathrm{Coker}(\pi)$ is always a finitely generated module over $\mathrm{Im}(\pi)$ with generators corresponding to central elements of $\mathcal{D}(V)^G$. The Zamolodchikov $\mathcal{W}_3$ algebra of central charge $c = -2$ plays an important role in the structure of $\mathcal{S}(V)^{\Theta+}$. Our description relies on the fundamental papers [18,19] of W. Wang, in which he classified the irreducible modules of $\mathcal{W}_{3,-2}$.

In the case where G is nonabelian, very little is known about the structure of $\mathcal{S}(V)^{\Theta+}$, and the representation-theoretic techniques used in the abelian case cannot be expected to work. In a separate paper, we will use tools from commutative algebra to describe $\mathcal{S}(V)^{\Theta+}$ in the special cases where G is one of the classical Lie groups $SL(n)$, $SO(n)$, or $Sp(2n)$, and V is a direct sum of copies of the standard representation.

One hopes that the vertex algebra point of view can also shed some light on the classical algebras $\mathcal{D}(V)^G$. For example, the vertex algebra products on $\mathcal{S}(V)$ induce a family of bilinear operations $*_k$, $k \geq -1$ on $\mathcal{D}(V)^G$, which coincide with classical operations known as transvectants. $\mathcal{D}(V)^G$ is generally not simple as an associative algebra, but in the case where G is an abelian group acting diagonalizably on V, $\mathcal{D}(V)^G$ is always simple as a $*$-algebra in the obvious sense.

2. Invariant differential operators

Fix a basis $\{x_1, \dots, x_n\}$ for V and a corresponding dual basis $\{x'_1, \dots, x'_n\}$ for V^*. The Weyl algebra $\mathcal{D}(V)$ is generated by the linear functions x'_i and the first-order differential operators $\frac{\partial}{\partial x'_i}$, which satisfy $[\frac{\partial}{\partial x'_i}, x'_j] = \delta_{i,j}$. Equip $\mathcal{D}(V)$ with the Bernstein filtration

$$\mathcal{D}(V)_{(0)} \subset \mathcal{D}(V)_{(1)} \subset \cdots, \tag{2.1}$$

defined by $(x'_1)^{k_1} \cdots (x'_n)^{k_n} (\frac{\partial}{\partial x'_1})^{l_1} \cdots (\frac{\partial}{\partial x'_n})^{l_n} \in \mathcal{D}(V)_{(r)}$ if $k_1 + \dots + k_n + l_1 + \dots + l_n \leq r$. Given $\omega \in \mathcal{D}(V)_{(r)}$ and $\nu \in \mathcal{D}(V)_{(s)}$, $[\omega, \nu] \in \mathcal{D}(V)_{(r+s-2)}$, so that

$$gr\,\mathcal{D}(V) = \bigoplus_{r>0} \mathcal{D}(V)_{(r)}/\mathcal{D}(V)_{(r-1)} \cong \mathrm{Sym}(V \oplus V^*). \tag{2.2}$$

We say that $\deg(\alpha) = d$ if $\alpha \in \mathcal{D}(V)_{(d)}$ and $\alpha \notin \mathcal{D}(V)_{(d-1)}$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let V be a linear representation of G via $\rho : G \to \mathrm{Aut}(V)$. Then G acts on $\mathcal{D}(V)$ by algebra automorphisms, and induces an action $\rho^* : \mathfrak{g} \to \mathrm{Der}(\mathcal{D}(V))$ by derivations of degree zero. Since G is connected, the invariant ring $\mathcal{D}(V)^G$ coincides with $\mathcal{D}(V)^{\mathfrak{g}}$, where

$$\mathcal{D}(V)^{\mathfrak{g}} = \{\omega \in \mathcal{D}(V) \mid \rho^*(\xi)(\omega) = 0,\ \forall \xi \in \mathfrak{g}\}.$$

We will usually work with the action of $\mathfrak{g}$ rather than G, and for greater flexibility, we do not assume that the $\mathfrak{g}$-action comes from an action of a *reductive* group G.

The action of $\mathfrak{g}$ on $\mathcal{D}(V)$ can be realized by *inner* derivations: there is a Lie algebra homomorphism

$$\tau : \mathfrak{g} \to \mathcal{D}(V), \qquad \xi \mapsto -\sum_{i=1}^{n} x'_i \rho^*(\xi) \left(\frac{\partial}{\partial x'_i}\right). \tag{2.3}$$

$\tau(\xi)$ is just the linear vector field on V generated by ξ, so $\xi \in \mathfrak{g}$ acts on $\mathcal{D}(V)$ by $[\tau(\xi), -]$. Clearly τ extends to a map $\mathfrak{U}\mathfrak{g} \to \mathcal{D}(V)$, and

$$\mathcal{D}(V)^{\mathfrak{g}} = \mathrm{Com}(\tau(\mathfrak{U}\mathfrak{g}), \mathcal{D}(V)).$$

Since $\mathfrak{g}$ acts on $\mathcal{D}(V)$ by derivations of degree zero, (2.1) restricts to a filtration $\mathcal{D}(V)^{\mathfrak{g}}_{(0)} \subset \mathcal{D}(V)^{\mathfrak{g}}_{(1)} \subset \cdots$ on $\mathcal{D}(V)^{\mathfrak{g}}$, and $gr(\mathcal{D}(V)^{\mathfrak{g}}) \cong gr(\mathcal{D}(V))^{\mathfrak{g}} \cong \mathrm{Sym}(V \oplus V^*)^{\mathfrak{g}}$.

2.1. *The case where $\mathfrak{g}$ is abelian*

Our main focus is on the case where $\mathfrak{g}$ is the abelian Lie algebra $\mathbf{C}^m = gl(1) \oplus \cdots \oplus gl(1)$, acting diagonally on V. Let $R(V)$ be the $\mathbf{C}$-vector space of all diagonal representations of $\mathfrak{g}$. Given $\rho \in R(V)$ and $\xi \in \mathfrak{g}$, $\rho(\xi)$ is a diagonal matrix with entries $a^{\xi}_1, \dots, a^{\xi}_n$, which we regard as a vector $a^{\xi} = (a^{\xi}_1, \dots, a^{\xi}_n) \in \mathbf{C}^n$. Let $A(\rho) \subset \mathbf{C}^n$ be the subspace spanned by $\{\rho(\xi) \mid \xi \in \mathfrak{g}\}$.

The action of $GL(m)$ on $\mathfrak{g}$ induces a natural action of $GL(m)$ on $R(V)$, defined by

$$(g \cdot \rho)(\xi) = \rho(g^{-1} \cdot \xi) \tag{2.4}$$

for all $g \in GL(m)$. Clearly $A(\rho) = A(g \cdot \rho)$ for all $g \in GL(m)$. Note that $\dim\ \mathrm{Ker}(\rho) = \dim\ \mathrm{Ker}(g \cdot \rho)$ for all $g \in GL(m)$, so in particular $GL(m)$ acts on the dense open set $R^0(V) = \{\rho \in R(V) \mid \mathrm{Ker}(\rho) = 0\}$. The correspondence $\rho \mapsto A(\rho)$ identifies $R^0(V)/GL(m)$ with the Grassmannian $Gr(m, n)$ of m-dimensional subspaces of $\mathbf{C}^n$.

Given $\rho \in R(V)$, $\mathcal{D}(V)^{\mathfrak{g}} = \mathcal{D}(V)^{\mathfrak{g}'}$ where $\mathfrak{g}' = \mathfrak{g}/\mathrm{Ker}(\rho)$, so we may assume without loss of generality that $\rho \in R^0(V)$. We denote $\mathcal{D}(V)^{\mathfrak{g}}$ by $\mathcal{D}(V)^{\mathfrak{g}}_{\rho}$ when we need to emphasize the dependence on ρ. Given $\omega \in \mathcal{D}(V)$, the condition $\rho^*(\xi)(\omega) = 0$ for all $\xi \in \mathfrak{g}$ is equivalent to the condition that $\rho^*(g \cdot \xi)(\omega) = 0$ for all $\xi \in \mathfrak{g}$, so it follows that $\mathcal{D}(V)^{\mathfrak{g}}_{\rho} = \mathcal{D}(V)^{\mathfrak{g}}_{g \cdot \rho}$ for all $g \in GL(m)$. Hence the family of algebras $\mathcal{D}(V)^{\mathfrak{g}}_{\rho}$ is parametrized by the points $A(\rho) \in Gr(m, n)$.

Fix $\rho \in R^0(V)$, and choose a basis $\{\xi^1, \dots, \xi^m\}$ for $\mathfrak{g}$. Let $a^i = (a^i_1, \dots, a^i_n) \in \mathbf{C}^n$ be the vectors corresponding to the diagonal matrices $\rho(\xi^i)$, and let $A = A(\rho)$ be the subspace spanned by these vectors. The map $\tau : \mathfrak{g} \to \mathcal{D}(V)$ is defined by

$$\tau(\xi^i) = -\sum_{j=1}^{n} a^i_j x'_j \frac{\partial}{\partial x'_j}. \tag{2.5}$$

The Euler operators $\{e_j = x'_j \frac{\partial}{\partial x'_j} \mid j = 1, \dots, n\}$ lie in $\mathcal{D}(V)^{\mathfrak{g}}$, and we denote the polynomial algebra $\mathbf{C}[e_1, \dots, e_n]$ by E.

For each $j = 1, \dots, n$ and $d \in \mathbf{Z}$, define $v^d_j \in \mathcal{D}(V)$ by

$$v^d_j = \begin{cases} \left(\dfrac{\partial}{\partial x'_j}\right)^{-d} & d < 0 \\ 1 & d = 0 \\ (x'_j)^d & d > 0. \end{cases} \tag{2.6}$$

Let $\mathbf{Z}^n \subset \mathbf{C}^n$ denote the lattice generated by the standard basis, and for each lattice point $l = (l_1, \dots, l_n) \in \mathbf{Z}^n$, define

$$\omega_l = \prod_{j=1}^{n} v_j^{l_j}. \tag{2.7}$$

As a module over E,

$$\mathcal{D}(V) = \bigoplus_{l \in \mathbf{Z}^n} M_l, \tag{2.8}$$

where M_l is the free E-module generated by ω_l. Moreover, we have

$$[e_j, \omega_l] = l_j \omega_l, \tag{2.9}$$

so the $\mathbf{Z}^n$-grading (2.8) is just the eigenspace decomposition of $\mathcal{D}(V)$ under the family of diagonalizable operators $[e_j, -]$. In particular, (2.9) shows that

$$\rho^*(\xi^i)(\omega_l) = [\tau(\xi^i), \omega_l] = -\langle l, a^i \rangle \omega_l, \tag{2.10}$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbf{C}^n$. Hence ω_l lies in $\mathcal{D}(V)^{\mathfrak{g}}$ precisely when $l \in A^{\perp}$, so

$$\mathcal{D}(V)^{\mathfrak{g}} = \bigoplus_{l \in A^{\perp} \cap \mathbf{Z}^n} M_l. \tag{2.11}$$

For generic actions, the lattice $A^{\perp} \cap \mathbf{Z}^n$ has rank zero, so $\mathcal{D}(V)^{\mathfrak{g}} = M_0 = E$.

Consider the double commutant $\mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V))$, which always contains $T = \tau(\mathfrak{U}\mathfrak{g}) = \mathbf{C}[\tau(\xi_1) \dots, \tau(\xi_m)]$. Since $\mathrm{Com}(E, \mathcal{D}(V)) = E$, we have $\mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V)) = E$ for generic actions.

Suppose next that $A^{\perp} \cap \mathbf{Z}^n$ has rank r for some $0 < r \leq n - m$. For $i = 1, \dots, r$ let $\{l^i = (l_1^i, \dots, l_n^i)\}$ be a basis for $A^{\perp} \cap \mathbf{Z}^n$, and let L be the $\mathbf{C}$-vector space spanned by $\{l^1, \dots, l^r\}$. If $r < n - m$, we can choose vectors $s^k = (s_1^k, \dots, s_n^k) \in L^{\perp} \cap A^{\perp}$, so that $\{l^1, \dots, l^r, s^{r+1}, \dots, s^{n-m}\}$ is a basis for $A^{\perp}$. For $i = 1, \dots, r$ and $k = r+1, \dots, n-m$, define differential operators

$$\phi^i = \sum_{j=1}^{n} l_j^i e_j, \qquad \psi^k = \sum_{j=1}^{n} s_j^k e_j.$$

Note that $\mathbf{C}[e_1, \dots, e_n] = T \otimes \Psi \otimes \Phi$, where $\Phi = \mathbf{C}[\phi^1, \dots \phi^r]$ and $\Psi = \mathbf{C}[\psi^{r+1}, \dots, \psi^{n-m}]$.

Theorem 2.1. $\mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V)) = T \otimes \Psi$. *Hence $\mathcal{D}(V)^{\mathfrak{g}}$ and T form a pair of mutual commutants inside $\mathcal{D}(V)$ precisely when $\Psi = \mathbf{C}$, which occurs when $A^{\perp} \cap \mathbf{Z}^n$ has rank $n - m$.*

Proof. By (2.9), for any lattice point $l \in A^{\perp} \cap \mathbf{Z}^n$, and for $k = r+1, \dots, n-m$ we have

$$[\psi^k, \omega_l] = \langle s^k, l \rangle \omega_l = 0$$

since $s^k \in L^{\perp}$. It follows that $\Psi \subset \mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V))$. Hence $T \otimes \Psi \subset \mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V))$. Moreover, since $[\phi^i, \omega_l] = \langle l^i, l \rangle \omega_l$ and $\{l^1, \dots, l^r\}$ form a basis for $A^{\perp} \cap \mathbf{Z}^n$, it follows that the variables ϕ^i cannot appear in any element $\omega \in \mathrm{Com}(\mathcal{D}(V)^{\mathfrak{g}}, \mathcal{D}(V))$. □

In the case $\Psi = \mathbf{C}$, we can recover the action ρ (up to $GL(m)$-equivalence) from the algebra $\mathcal{D}(V)^{\mathfrak{g}}$ by taking its commutant inside $\mathcal{D}(V)$, but otherwise $\mathcal{D}(V)^{\mathfrak{g}}$ does not determine the action.

3. Vertex algebras

We will assume that the reader is familiar with the basic notions in vertex algebra theory. For a list of references, see page 117 of [13]. We briefly describe the examples and constructions that we need, following the notation in [13].

Given a Lie algebra $\mathfrak{g}$ equipped with a symmetric $\mathfrak{g}$-invariant bilinear form B, the *current algebra* $\mathcal{O}(\mathfrak{g}, B)$ is the universal vertex algebra with generators $X^{\xi}(z)$, $\xi \in \mathfrak{g}$, which satisfy the OPE relations

$$X^{\xi}(z) X^{\eta}(w) \sim B(\xi, \eta)(z-w)^{-2} + X^{[\xi,\eta]}(w)(z-w)^{-1}.$$

Given a finite-dimensional vector space V, the $\beta\gamma$-system, or algebra of chiral differential operators $\mathcal{S}(V)$, was introduced in [5]. It is the unique vertex algebra with generators $\beta^x(z)$, $\gamma^{x'}(z)$ for $x \in V$, $x' \in V^*$, which satisfy

$$\beta^x(z)\gamma^{x'}(w) \sim \langle x', x \rangle (z-w)^{-1}, \qquad \gamma^{x'}(z)\beta^x(w) \sim -\langle x', x \rangle (z-w)^{-1},$$
$$\beta^x(z)\beta^y(w) \sim 0, \qquad \gamma^{x'}(z)\gamma^{y'}(w) \sim 0. \tag{3.1}$$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{C}^n$, $\mathcal{S}(V)$ has a Virasoro element

$$L^{\alpha}(z) = \sum_{i=1}^{n} (\alpha_i - 1) : \partial\beta^{x_i}(z)\gamma^{x_i'}(z) : + \alpha_i : \beta^{x_i}(z)\partial\gamma^{x_i'}(z) : \tag{3.2}$$

of central charge $\sum_{i=1}^{n} \left(12\alpha_i^2 - 12\alpha_i + 2\right)$. Here $\{x_1, \dots, x_n\}$ is any basis for V and $\{x_1', \dots, x_n'\}$ is the corresponding dual basis for V^*. An OPE calculation shows that $\beta^{x_i}(z)$, $\gamma^{x_i'}(z)$ are primary of conformal weights α_i, $1 - \alpha_i$, respectively.

$\mathcal{S}(V)$ has an additional $\mathbf{Z}$-grading which we call the $\beta\gamma$-charge. Define

$$v(z) = \sum_{i=1}^{n} : \beta^{x_i}(z)\gamma^{x_i'}(z) : . \tag{3.3}$$

The zeroth Fourier mode $v(0)$ acts diagonalizably on $\mathcal{S}(V)$; the $\beta\gamma$-charge grading is just the eigenspace decomposition of $\mathcal{S}(V)$ under $v(0)$. For $x \in V$ and $x' \in V^*$, $\beta^x(z)$ and $\gamma^{x'}(z)$ have $\beta\gamma$-charges -1 and 1, respectively.

There is also an odd vertex algebra $\mathcal{E}(V)$ known as a bc-system, or a semi-infinite exterior algebra, which is generated by $b^x(z)$, $c^{x'}(z)$ for $x \in V$ and $x' \in V^*$, which satisfy

$$b^x(z)c^{x'}(w) \sim \langle x', x\rangle (z-w)^{-1}, \qquad c^{x'}(z)b^x(w) \sim \langle x', x\rangle (z-w)^{-1},$$
$$b^x(z)b^y(w) \sim 0, \qquad c^{x'}(z)c^{y'}(w) \sim 0.$$

$\mathcal{E}(V)$ has an analogous conformal structure $L^{\alpha}(z)$ for any $\alpha \in \mathbf{C}^n$, and an analogous $\mathbf{Z}$-grading which we call the bc-charge. Define

$$q(z) = -\sum_{i=1}^{n} : b^{x_i}(z)c^{x_i'}(z) : . \tag{3.4}$$

The zeroth Fourier mode $q(0)$ acts diagonalizably on $\mathcal{S}(V)$, and the bc-charge grading is just the eigenspace decomposition of $\mathcal{E}(V)$ under $q(0)$. Clearly $b^x(z)$ and $c^{x'}(z)$ have bc-charges -1 and 1, respectively.

3.1. The commutant construction

Definition 3.1. Let $\mathcal{V}$ be a vertex algebra, and let $\mathcal{A}$ be a subalgebra. The commutant of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ or $\mathcal{V}^{\mathcal{A}+}$, is the subalgebra of vertex operators $v \in \mathcal{V}$ such that $[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. Equivalently, $a(z) \circ_n v(z) = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$.

We regard $\mathcal{V}$ as a module over $\mathcal{A}$, and we regard $\mathcal{V}^{\mathcal{A}+}$ as the invariant subalgebra. If $\mathcal{A}$ is a homomorphic image of a current algebra $\mathcal{O}(\mathfrak{g}, B)$, $\mathcal{V}^{\mathcal{A}+}$ is just the invariant space $\mathcal{V}^{\mathfrak{g}[t]}$. We will always assume that $\mathcal{V}$ is equipped with a weight grading, and that $\mathcal{A}$ is a graded subalgebra, so that $\mathcal{V}^{\mathcal{A}+}$ is also a graded subalgebra of $\mathcal{V}$.

Our main example of this construction comes from a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$. There is an induced vertex algebra homomorphism $\hat{\tau} : \mathcal{O}(\mathfrak{g}, B) \to \mathcal{S}(V)$, which is analogous to the map $\tau : \mathfrak{U}\mathfrak{g} \to \mathcal{D}(V)$ given by (2.3). Here B is the bilinear form $B(\xi, \eta) = -Tr(\rho(\xi)\rho(\eta))$ on $\mathfrak{g}$. In terms of a basis $\{x_1, \dots, x_n\}$ for V and dual basis $\{x_1', \dots x_n'\}$ for V^*, $\hat{\tau}$ is defined by

$$\hat{\tau}(X^{\xi}(z)) = \theta^{\xi}(z) = -\sum_{i=1}^{n} : \gamma^{x_i'}(z)\beta^{\rho(\xi)(x_i)}(z) : . \tag{3.5}$$

Definition 3.2. Let Θ denote the subalgebra $\hat{\tau}(\mathcal{O}(\mathfrak{g}, B)) \subset \mathcal{S}(V)$. The commutant algebra $\mathcal{S}(V)^{\Theta+}$ will be called the algebra of *invariant chiral differential operators* on V.

If $\mathcal{S}(V)$ is equipped with the conformal structure L^{α} given by (3.2), Θ is not a graded subalgebra of $\mathcal{S}(V)$ in general. For example, if $\mathfrak{g} = gl(n)$ and $V = \mathbf{C}^n$, Θ is graded by weight precisely when $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. However, when $\mathfrak{g}$ is abelian and its action on V is diagonal, $\theta^{\xi}(z)$ will be homogeneous of weight one for any α. Hence $\mathcal{S}(V)^{\Theta+}$ is also graded by weight, but this grading will depend on the choice of α.

3.2. The Zhu functor

Let $\mathcal{V}$ be a vertex algebra with weight grading $\mathcal{V} = \bigoplus_{n \in \mathbf{Z}} \mathcal{V}_n$. In [21], Zhu introduced a functor that attaches to $\mathcal{V}$ an associative algebra $A(\mathcal{V})$, together with a surjective linear map $\pi_{Zh} : \mathcal{V} \to A(\mathcal{V})$. For $a \in \mathcal{V}_m$ and $b \in \mathcal{V}$, we define

$$a * b = \mathrm{Res}_z \left(a(z) \frac{(z+1)^m}{z} b \right), \tag{3.6}$$

and extend $*$ by linearity to a bilinear operation $\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}$. Let $O(\mathcal{V})$ denote the subspace of $\mathcal{V}$ spanned by elements of the form

$$a\circ b=\mathrm{Res}_z\left(a(z)\frac{(z+1)^m}{z^2}b\right) \tag{3.7}$$

where $a\in\mathcal{V}_m$, and let $A(\mathcal{V})$ be the quotient $\mathcal{V}/O(\mathcal{V})$, with projection $\pi_{Zh}:\mathcal{V}\to A(\mathcal{V})$. For $a,b\in\mathcal{V}$, $a\sim b$ means $a-b\in O(\mathcal{V})$, and $[a]$ denotes the image of a in $A(\mathcal{V})$. A useful fact which is immediate from (3.6) and (3.7) is that for $a\in\mathcal{V}_m$,

$$\partial a\sim ma. \tag{3.8}$$

Theorem 3.3 (*Zhu*). *$O(\mathcal{V})$ is a two-sided ideal in V under the product $*$, and $(A(\mathcal{V}),*)$ is an associative algebra with unit* [1]*. The assignment $\mathcal{V}\mapsto A(\mathcal{V})$ is functorial. If $\mathcal{I}$ is a vertex algebra ideal of $\mathcal{V}$, we have*

$$A(\mathcal{V}/\mathcal{I})\cong A(\mathcal{V})/I,\quad I=\pi_{Zh}(\mathcal{I}). \tag{3.9}$$

The main application of the Zhu functor is to study the representation theory of $\mathcal{V}$, or at least reduce it to a more classical problem. Let $M=\bigoplus_{n\geq 0}M_n$ be a module over $\mathcal{V}$ such that for $a\in\mathcal{V}_m$, $a(n)M_k\subset M_{m+k-n-1}$ for all $n\in\mathbf{Z}$. Given $a\in\mathcal{V}_m$, the Fourier mode $a(m-1)$ acts on each M_k. The subspace M_0 is then a module over $A(\mathcal{V})$ with action $[a]\mapsto a(m-1)\in\mathrm{End}(M_0)$. In fact, $M\mapsto M_0$ provides a one-to-one correspondence between irreducible $\mathbf{Z}_{\geq 0}$-graded $\mathcal{V}$-modules and irreducible $A(\mathcal{V})$-modules.

A vertex algebra $\mathcal{V}$ is said to be *strongly generated* by a subset $\{v_i(z)\mid i\in I\}$ if $\mathcal{V}$ is spanned by collection of iterated Wick products

$$\{:\partial^{k_1}v_{i_1}(z)\cdots\partial^{k_m}v_{i_m}(z):\mid k_1,\dots,k_m\geq 0\}.$$

Lemma 3.4. *Suppose that $\mathcal{V}$ is strongly generated by $\{v_i(z)\mid i\in I\}$, which are homogeneous of weights $d_i\geq 0$. Then $A(\mathcal{V})$ is generated as an associative algebra by the collection $\{\pi_{Zh}(v_i)\mid i\in I\}$.*

Proof. Let $\mathcal{C}$ be the algebra generated by $\{\pi_{Zh}(v_i)|i\in I\}$. We need to show that for any vertex operator $\omega\in\mathcal{V}$, we have $\pi_{Zh}(\omega)\in\mathcal{C}$. By strong generation, it suffices to prove this when ω is a monomial of the form

$$:\partial^{k_1}v_{i_1}\cdots\partial^{k_r}v_{i_r}:\ .$$

We proceed by induction on weight. Suppose first that ω has weight zero, so that $k_1=\cdots=k_r=0$ and $v_{i_1},\dots,v_{i_r}$ all have weight zero. Note that $v_{i_1}\circ_n(:v_{i_2}\cdots v_{i_r}:)$ has weight $-n-1$, and hence vanishes for all $n\geq 0$. It follows from (3.6) that

$$[v_{i_1}]*[:v_{i_2}\cdots v_{i_r}:]=[\omega].$$

Continuing in this way, we see that $[\omega]=[v_{i_1}]*[v_{i_2}]*\cdots*[v_{i_r}]\in\mathcal{C}$. Next, assume that $\pi_{Zh}(\omega)\in\mathcal{C}$ whenever $wt(\omega)<n$, and suppose that $\omega=\ :\partial^{k_1}v_i\cdots\partial^{k_r}v_r:$ has weight n. We calculate

$$[\partial^{k_1}v_{i_1}]*[:\partial^{k_2}v_{i_2}\cdots\partial^{k_r}v_{i_r}:]=[\omega]+\dots,$$

where $\cdots$ is a linear combination of terms of the form $[\partial^{k_1}v_{i_1}\circ_k(:\ \partial^{k_2}v_{i_2}\cdots\partial^{k_r}v_{i_r}\ :)]$ for $k\geq 0$. The vertex operators $\partial^{k_1}v_{i_1}\circ_k(:\partial^{k_2}v_{i_2}\cdots\partial^{k_r}v_{i_r}:)$ all have weight $n-k-1$, so by our inductive assumption, $[\partial^{k_1}v_{i_1}\circ_k(:\partial^{k_2}v_{i_2}\cdots\partial^{k_r}v_{i_r}:)]\in\mathcal{C}$. Applying the same argument to the vertex operator $:\ \partial^{k_2}v_{i_2}\cdots\partial^{k_r}v_{i_r}\ :$ and proceeding by induction on r, we see that $[\omega]\equiv[\partial^{k_1}v_{i_1}]*\cdots*[\partial^{k_n}v_{i_n}]$ modulo $\mathcal{C}$. Finally, by applying (3.8) repeatedly, we see that $[\omega]\in\mathcal{C}$, as claimed. □

Example 3.5. $\mathcal{V}=\mathcal{O}(\mathfrak{g},B)$ where each generator X^ξ has weight 1. Then $A(\mathcal{O}(\mathfrak{g},B))$ is generated by $\{[X^\xi]\mid\xi\in\mathfrak{g}\}$, and is isomorphic to the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$ via $[X^\xi]\mapsto\xi$.

Example 3.6. Let $\mathcal{V}=\mathcal{S}(V)$ where $V=\mathbf{C}^n$, and $\mathcal{S}(V)$ is equipped with the conformal structure L^α given by (3.2). Then $A(\mathcal{S}(V))$ is generated by $\{[\gamma^{x'_i}],[\beta^{x_i}]\}$ and is isomorphic to the Weyl algebra $\mathcal{D}(V)$ with generators x'_i, $\frac{\partial}{\partial x'_i}$ via

$$[\gamma^{x'_i}]\mapsto x'_i,\qquad[\beta^{x_i}]\mapsto\frac{\partial}{\partial x'_i}.$$

Even though the structure of $A(\mathcal{S}(V))$ is independent of the choice of α, the Zhu map $\pi_{Zh}:\mathcal{S}(V)\to A(\mathcal{S}(V))$ does depend on α. For example, (3.6) shows that

$$\pi_{Zh}(:\gamma^{x'_i}\beta^{x_i}:)=x'_i\frac{\partial}{\partial x'_i}+1-\alpha_i. \tag{3.10}$$

We will be particularly concerned with the interaction between the commutant construction and the Zhu functor. If $a, b \in \mathcal{V}$ are (super)commuting vertex operators, $[a]$ and $[b]$ are (super)commuting elements of $A(\mathcal{V})$. Hence for any subalgebra $\mathcal{B} \subset \mathcal{V}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Com}(\mathcal{B}, \mathcal{V}) & \overset{\iota}{\hookrightarrow} & \mathcal{V} \\ \pi \downarrow & & \pi_{Zh} \downarrow \\ \mathrm{Com}(B, A(\mathcal{V})) & \overset{\iota}{\hookrightarrow} & A(\mathcal{V}). \end{array} \tag{3.11}$$

Here B denotes the subalgebra $\pi_{Zh}(\mathcal{B}) \subset A(\mathcal{V})$, and $\mathrm{Com}(B, A(\mathcal{V}))$ denotes the (super)commutant of B inside $A(\mathcal{V})$. The horizontal maps are inclusions, and π is the restriction of the Zhu map on $\mathcal{V}$ to $\mathrm{Com}(\mathcal{B}, \mathcal{V})$. Clearly $\mathrm{Im}(\pi)$ is a subalgebra of $\mathrm{Com}(B, A(\mathcal{V}))$. A natural problem is to describe $\mathrm{Im}(\pi)$ and $\mathrm{Coker}(\pi)$. In our main example $\mathcal{V} = \mathcal{S}(V)$ and $\mathcal{A} = \Theta$, we have $\pi_{Zh}(\Theta) = \tau(\mathfrak{U}\mathfrak{g}) \subset \mathcal{D}(V)$ and $\mathrm{Com}(\tau(\mathfrak{U}\mathfrak{g}), \mathcal{D}(V)) = \mathcal{D}(V)^{\mathfrak{g}}$, so (3.11) specializes to (1.2).

4. The Friedan–Martinec–Shenker bosonization

4.1. Bosonization of fermions

First we describe the bosonization of fermions and the well-known boson–fermion correspondence due to [3]. Let A be the Heisenberg algebra with generators $j(n), n \in \mathbf{Z}$, and κ, satisfying $[j(n), j(m)] = n\delta_{n+m,0}\kappa$. The field $j(z) = \sum_{n\in\mathbf{Z}} j(n)z^{-n-1}$ satisfies the OPE

$$j(z)j(w) \sim (z-w)^{-2},$$

and generates a Heisenberg vertex algebra $\mathcal{H}$ of central charge 1. Define the *free bosonic scalar field*

$$\phi(z) = q + j(0)\ln z - \sum_{n\neq 0} \frac{j(n)}{n} x^{-n},$$

where q satisfies $[q, j(n)] = \delta_{n,0}$. Clearly $\partial\phi(z) = j(z)$, and we have the OPE

$$\phi(z)\phi(w) \sim \ln(z-w).$$

Given $\alpha \in \mathbf{C}$, let $\mathcal{H}_\alpha$ denote the irreducible representation of A generated by the vacuum vector v_α satisfying

$$j(n)v_\alpha = \alpha\delta_{n,0}v_\alpha, \quad n \geq 0. \tag{4.1}$$

Given $\eta \in \mathbf{C}$, the operator $\mathrm{e}^{\eta q}(v_\alpha) = v_{\alpha+\eta}$, so $\mathrm{e}^{\eta q}$ maps $\mathcal{H}_\alpha \to \mathcal{H}_{\alpha+\eta}$. Define the vertex operator

$$X_\eta(z) = \mathrm{e}^{\eta\phi(z)} = \mathrm{e}^{\eta q} z^{\eta\alpha} \exp\left(\eta \sum_{n>0} j(-n)\frac{z^n}{n}\right) \exp\left(\eta \sum_{n<0} j(-n)\frac{z^n}{n}\right).$$

The X_η satisfy the OPEs

$$j(z)X_\eta(w) = \eta X_\eta(w)(z-w)^{-1} + \frac{1}{\eta}\partial X_\eta(w),$$
$$X_\eta(z)X_\nu(w) = (z-w)^{\eta\nu} : X_\eta(z)X_\nu(w) : .$$

If we take $\eta = \pm 1$, the pair of (fermionic) fields X_1, X_{-1} generate the lattice vertex algebra V_L associated to the one-dimensional lattice $L = \mathbf{Z}$. The state space of V_L is just $\sum_{n\in\mathbf{Z}} \mathcal{H}_n = \mathcal{H} \otimes_{\mathbf{C}} L$. It follows that

$$X_1(z)X_{-1}(w) \sim (z-w)^{-1}, \qquad X_{-1}(z)X_1(w) \sim (z-w)^{-1},$$
$$X_1(z)X_1(w) \sim 0, \qquad X_{-1}(z)X_{-1}(w) \sim 0,$$

so the map $\mathcal{E} \to V_L$ sending $b \mapsto X_{-1}, c \to X_1$ is a vertex algebra isomorphism. Here $\mathcal{E}$ denotes the bc-system $\mathcal{E}(V)$ in the case where V is one-dimensional.

4.2. Bosonization of bosons

Next, we describe the bosonization of bosons, following [2]. Recall that $\mathcal{E}$ has the grading $\mathcal{E} = \oplus_{l\in\mathbf{Z}} \mathcal{E}^l$ by bc-charge. As in [2], define $N(s) = \sum_{l\in\mathbf{Z}} \mathcal{E}^l \otimes \mathcal{H}_{i(s+l)}$, which is a module over the vertex algebra $\mathcal{E} \otimes V_{L'}$. Here L' is the one-dimensional lattice $i\mathbf{Z}$, and $V_{L'}$ is generated by $X_{\pm i}$. We define a map $\epsilon : \mathcal{S} \to \mathcal{E} \otimes V_{L'}$ by

$$\beta \mapsto \partial b \otimes X_{-i}, \qquad \gamma \mapsto c \otimes X_i. \tag{4.2}$$

It is straightforward to check that (4.2) is a vertex algebra homomorphism, which is injective since $\mathcal{S}$ is simple. Moreover Proposition 3 of [2] shows that the image of (4.2) coincides with the kernel of $c(0) : N(s) \to N(s-1)$. Let $\mathcal{E}'$ be the subalgebra of $\mathcal{E}$ generated by c and ∂b, which coincides with the kernel of $c(0) : \mathcal{E} \to \mathcal{E}$. It follows that

$$\epsilon(\mathcal{S}) \subset \mathcal{E}' \otimes V_{L'}. \tag{4.3}$$

5. $\mathcal{W}$ algebras

The $\mathcal{W}$ algebras are vertex algebras which arise as extended symmetry algebras of two-dimensional conformal field theories. For each integer $n \geq 2$ and $c \in \mathbf{C}$, the algebra $\mathcal{W}_{n,c}$ of central charge c is generated by fields of conformal weights $2, 3, \ldots, n$. In the case $n = 2$, $\mathcal{W}_{2,c}$ is just the Virasoro algebra of central charge c. In contrast to the Virasoro algebra, the generating fields for $\mathcal{W}_{n,c}$ for $n \geq 3$ have nonlinear terms in their OPEs, which makes the representation theory of these algebras highly nontrivial. One also considers various limits of $\mathcal{W}$ algebras denoted by $\mathcal{W}_{1+\infty,c}$ which may be defined as modules over the universal central extension $\hat{\mathcal{D}}$ of the Lie algebra $\mathcal{D}$ of differential operators on the circle [11].

We will be particularly concerned with the $\mathcal{W}_3$ algebra, which was introduced by Zamolodchikov in [20] and studied extensively in [1]. Our discussion is taken directly from [18,19]. First, let $\mathcal{F}(\mathcal{W}_3)$ denote the free associative algebra with generators $L_m, W_m, m \in \mathbf{Z}$. Let $\hat{\mathcal{F}}(\mathcal{W}_3)$ be the completion of $\mathcal{F}(\mathcal{W}_3)$ consisting of (possibly) infinite sums of monomials in $\mathcal{F}(\mathcal{W}_3)$ such that for each $N > 0$, only finitely many terms depend only on the variables L_n, W_n for $n \leq N$. For a fixed central charge $c \in \mathbf{C}$, let $\mathfrak{U}\mathcal{W}_{3,c}$ be the quotient of $\hat{\mathcal{F}}(\mathcal{W}_3)$ by the ideal generated by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \tag{5.1}$$

$$[L_m, W_n] = (2m-n)W_{m+n}, \tag{5.2}$$

$$\begin{aligned}[W_m, W_n] \;=\; &(m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right)L_{m+n} \\ &+ \frac{16}{22+5c}(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m,-n}.\end{aligned} \tag{5.3}$$

Here

$$\Lambda_m = \sum_{n\leq -2} L_n L_{m-n} + \sum_{n>-2} L_{m-n}L_n - \frac{3}{10}(m+2)(m+3)L_m.$$

Let

$$\mathcal{W}_{3,c,\pm} = \{L_n, W_n, \pm n > 0\}, \qquad \mathcal{W}_{3,c,0} = \{L_0, W_0\}.$$

The Verma module $\mathcal{M}_c(t, w)$ of highest weight (t, w) is the induced module

$$\mathfrak{U}\mathcal{W}_{3,c} \otimes_{\mathcal{W}_{3,c,+}\oplus\mathcal{W}_{3,c,0}} \mathbf{C}_{t,w},$$

where $\mathbf{C}_{t,w}$ is the one-dimensional $\mathcal{W}_{3,c,+} \oplus \mathcal{W}_{3,c,0}$-module generated by the vector $v_{t,w}$ such that

$$\mathcal{W}_{3,c,+}(v_{t,w}) = 0, \qquad L_0(v_{t,w}) = tv_{t,w}, \qquad W_0(v_{t,w}) = wv_{t,w}.$$

A vector $v \in \mathcal{M}_c(t, w)$ is called *singular* if $\mathcal{W}_{3,c,+}(v) = 0$. In the case $t = w = 0$, the vectors

$$L_{-1}(v_{0,0}), \qquad W_{-1}(v_{0,0}), \qquad W_{-2}(v_{0,0}) \tag{5.4}$$

are singular vectors in $\mathcal{M}_c(0, 0)$. The *vacuum module* $\mathcal{V}\mathcal{W}_{3,c}$ is defined to be the quotient of $\mathcal{M}_c(0, 0)$ by the $\mathfrak{U}\mathcal{W}_{3,c}$-submodule generated by the vectors (5.4). $\mathcal{V}\mathcal{W}_{3,c}$ has the structure of a vertex algebra which is freely generated by the vertex operators

$$L(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}, \qquad W(z) = \sum_{n\in\mathbf{Z}} W_n z^{-n-3}.$$

In particular, the vertex operators

$$\{\partial^{i_1}L(z)\cdots\partial^{i_m}L(z)\partial^{j_1}W(z)\cdots\partial^{j_n}W(z) \mid 0 \leq i_1 \leq \cdots \leq i_m, 0 \leq j_1 \leq \cdots \leq j_n\}$$

which correspond to $i_1!\cdots i_m!j_1!\cdots j_n!L_{-i_1-2}\cdots L_{-i_m-2}W_{-j_1-3}\cdots W_{-j_n-3}v_{0,0}$ under the state-operator correspondence, form a basis for $\mathcal{V}\mathcal{W}_{3,c}$. By Lemma 4.1 of [19], the Zhu algebra $A(\mathcal{V}\mathcal{W}_{3,c})$ is just the polynomial algebra $\mathbf{C}[l, w]$ where $l = \pi_{Zh}(L)$ and $w = \pi_{Zh}(W)$.

Let $\mathcal{I}_c$ denote the maximal proper $\mathfrak{U}\mathcal{W}_{3,c}$-submodule of $\mathcal{V}\mathcal{W}_{3,c}$, which is a vertex algebra ideal. The quotient $\mathcal{V}\mathcal{W}_{3,c}/\mathcal{I}_c$ is a simple vertex algebra which we denote by $\mathcal{W}_{3,c}$. Let $I_c = \pi_{Zh}(\mathcal{I}_c)$, which is an ideal of $\mathbf{C}[l, w]$. By (3.9), we have $A(\mathcal{W}_{3,c}) = \mathbf{C}[l, w]/I_c$. Generically, $\mathcal{I}_c = 0$, so that $\mathcal{V}\mathcal{W}_{3,c} = \mathcal{W}_{3,c}$. We will be primarily concerned with the non-generic case $c = -2$, in which $\mathcal{I}_{-2} \neq 0$. The generators $L(z), W(z) \in \mathcal{V}\mathcal{W}_{3,-2}$ satisfy the following OPEs:

$$L(z)L(w) \sim -(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}, \tag{5.5}$$

$$L(z)W(w) \sim 3W(w)(z-w)^{-2} + \partial W(w)(z-w)^{-1}, \tag{5.6}$$

$$\begin{aligned}W(z)W(w) \;\sim\; &-\frac{2}{3}(z-w)^{-6} + 2L(w)(z-w)^{-4} + \partial L(w)(z-w)^{-3} \\ &+ \left(\frac{8}{3}:L(w)L(w): - \frac{1}{2}\partial^2 L(w)\right)(z-w)^{-2} + \left(\frac{4}{3}\partial(:L(w)L(w):) - \frac{1}{3}\partial^3 L(w)\right)(z-w)^{-1}.\end{aligned} \tag{5.7}$$

The simple vertex algebra $\mathcal{W}_{3,-2}$ also has generators $L(z), W(z)$ satisfying (5.5)–(5.7), but $\mathcal{W}_{3,-2}$ is no longer freely generated.

In order to avoid introducing extra notation, we will *not* use the change of variables $\tilde{W}(z) = \frac{1}{2}\sqrt{6}W(z)$ given by Eq. 3.13 of [19]. By Lemma 4.3 of [19], the ideal $I_{-2} \subset \mathbf{C}[l, w]$ is generated (in our variables) by the polynomial

$$w^2 - \frac{2}{27}l^2(8l+1). \tag{5.8}$$

5.1. The representation theory of $\mathcal{W}_{3,-2}$

In [19], Wang gave a complete classification of the irreducible modules over the simple vertex algebra $\mathcal{W}_{3,-2}$. An important ingredient in his classification is the following realization of $\mathcal{W}_{3,-2}$ as a subalgebra of the Heisenberg algebra $\mathcal{H}$ with generator $j(z)$ satisfying $j(z)j(w) \sim (z-w)^{-2}$. Define

$$L_{\mathcal{H}} = \frac{1}{2}(:j^2:) + \partial j, \qquad W_{\mathcal{H}} = \frac{2}{3\sqrt{6}}(:j^3:) + \frac{1}{\sqrt{6}}(:j\partial j:) + \frac{1}{6\sqrt{6}}\partial^2 j. \tag{5.9}$$

The map $\mathcal{W}_{3,-2} \hookrightarrow \mathcal{H}$ sending $L \mapsto L_{\mathcal{H}}$ and $W \mapsto W_{\mathcal{H}}$ is a vertex algebra homomorphism, so we may regard any $\mathcal{H}$-module as a $\mathcal{W}_{3,-2}$-module. Given $\alpha \in \mathbf{C}$, consider the irreducible $\mathcal{H}$-module $\mathcal{H}_\alpha$ defined by (4.1), and let V_α denote the irreducible quotient of the $\mathcal{W}_{3,-2}$-submodule of $\mathcal{H}_\alpha$ generated by v_α. It is easily checked that the generator v_α is a highest weight vector of $\mathcal{W}_{3,-2}$ with highest weight

$$\left(\frac{1}{2}\alpha(\alpha-1), \frac{1}{3\sqrt{6}}\alpha(\alpha-1)(2\alpha-1)\right). \tag{5.10}$$

The main result of [19] is that the modules $\{V_\alpha \mid \alpha \in \mathbf{C}\}$ account for all the irreducible modules of $\mathcal{W}_{3,-2}$.

6. The commutant algebra $\mathcal{S}(V)^{\Theta_+}$ for $\mathfrak{g} = gl(1)$ and $V = \mathbf{C}$

In this section, we describe $\mathcal{S}(V)^{\Theta_+}$ in the case where $\mathfrak{g} = gl(1)$ and $V = \mathbf{C}$, where the action $\rho : \mathfrak{g} \to End\ V$ is by multiplication. Fix a basis ξ of $\mathfrak{g}$ and a basis x of V, such that $\rho(\xi)(x) = x$. Then $\mathcal{S} = \mathcal{S}(V)$ is generated by $\beta(z) = \beta^x(z)$ and $\gamma(z) = \gamma^{x'}(z)$, and the map (2.5) is given by

$$\mathfrak{g} \to \mathcal{D} = \mathcal{D}(V), \qquad \xi \mapsto -x'\frac{\mathrm{d}}{\mathrm{d}x'}.$$

In this case, $\mathcal{O}(\mathfrak{g}, B)$ is just the Heisenberg algebra $\mathcal{H}$ of central charge -1, and the action of $\mathcal{H}$ on $\mathcal{S}$ given by (3.5) is

$$\theta(z) = -:\gamma(z)\beta(z):, \tag{6.1}$$

which clearly satisfies

$$\theta(z)\theta(w) \sim -(z-w)^{-2}. \tag{6.2}$$

As usual, Θ will denote the subalgebra of $\mathcal{S}$ generated by $\theta(z)$. Since $-\theta(0)$ is the $\beta\gamma$-charge operator, $\mathcal{S}^{\Theta_+}$ must lie in the subalgebra $\mathcal{S}^0$ of $\beta\gamma$-charge zero.

Let $:\theta^n:$ denote the n-fold iterated Wick product of θ with itself. It is clear from (6.2) that each $:\theta^n:$ lies in $\mathcal{S}^0$ but not in $\mathcal{S}^{\Theta_+}$. A natural place to look for elements in $\mathcal{S}^{\Theta_+}$ is to begin with the operators $:\theta^n:$ and try to "quantum correct" them so that they lie in $\mathcal{S}^{\Theta_+}$. As a polynomial in $\beta, \partial\beta, \ldots, \gamma, \partial\gamma, \ldots$, note that

$$:\theta^n: \ = (-1)^n\beta^n\gamma^n + \nu_n,$$

where ν_n has degree at most $2n-2$. By a quantum correction, we mean an element $\omega_n \in \mathcal{S}$ of polynomial degree at most $2n-2$, so that $:\theta^n: + \omega_n \in \mathcal{S}^{\Theta_+}$.

Clearly θ has no such correction ω_1, because ω_1 would have to be a scalar, in which case $\theta \circ_1 (\theta + \omega_1) = \theta \circ_1 \theta = -1$. However, the next lemma shows that we can find such ω_n for all $n \geq 2$.

Lemma 6.1. *Let*

$$\omega_2 =: \beta(\partial\gamma): - :(\partial\beta)\gamma:,$$
$$\omega_3 = -\frac{9}{2}:\beta^2\gamma(\partial\gamma): + \frac{9}{2}:\beta(\partial\beta)\gamma^2: - \frac{3}{2}:\beta(\partial^2\gamma): - \frac{3}{2}:(\partial^2\beta)\gamma: + 6:(\partial\beta)(\partial\gamma):.$$

Then $:\theta^2: + \omega_2 \in \mathcal{S}^{\Theta_+}$ *and* $:\theta^3: + \omega_3 \in \mathcal{S}^{\Theta_+}$. *Since* $:(\theta^n):$ *and* $:(:\theta^i:)(:\theta^j:):$ *have the same leading term as polynomials in* $\beta, \partial\beta, \ldots, \gamma, \partial\gamma, \ldots$ *for* $i + j = n$, *it follows that for any* $n \geq 2$ *we can find* ω_n *such that* $:\theta^n: + \omega_n \in \mathcal{S}^{\Theta_+}$.

Proof. This is a straightforward OPE calculation. □

Next, define vertex operators $L_{\mathcal{S}}, W_{\mathcal{S}} \in \mathcal{S}^{\Theta+}$ as follows:

$$L_{\mathcal{S}} = \frac{1}{2}(:\theta^2: + \omega_2) = \frac{1}{2}(:\beta^2\gamma^2:) - :(\partial\beta)\gamma: + :\beta(\partial\gamma):, \tag{6.3}$$

$$\begin{aligned} W_{\mathcal{S}} &= -\sqrt{\frac{2}{27}}(:\theta^3: + \omega_3) \\ &= \sqrt{\frac{2}{27}}(:\beta^3\gamma^3:) - \sqrt{\frac{3}{2}}(:\beta(\partial\beta)\gamma^2:) + \sqrt{\frac{3}{2}}(:\beta^2\gamma(\partial\gamma):) \\ &\quad + \sqrt{\frac{1}{6}}(:(\partial^2\beta)\gamma:) - \sqrt{\frac{8}{3}}(:(\partial\beta)(\partial\gamma):) + \sqrt{\frac{1}{6}}(:\beta(\partial^2\gamma):). \end{aligned} \tag{6.4}$$

Let $\mathcal{W} \subset \mathcal{S}^{\Theta+}$ be the vertex algebra generated by $L_{\mathcal{S}}, W_{\mathcal{S}}$. An OPE calculation shows that the map

$$\mathcal{V}\mathcal{W}_{3,-2} \to \mathcal{S}^{\Theta+}, \qquad L \mapsto L_{\mathcal{S}}, \qquad W \mapsto W_{\mathcal{S}} \tag{6.5}$$

is a vertex algebra homomorphism. Moreover, the ideal $\mathcal{I}_{-2}$ is annihilated by (6.5), so this map descends to a map

$$f : \mathcal{W}_{3,-2} \hookrightarrow \mathcal{S}^{\Theta+}. \tag{6.6}$$

In fact, (6.6) is related to the realization of $\mathcal{W}_{3,-2}$ as a subalgebra of $\mathcal{H}$ defined earlier. First, under the boson–fermion correspondence,

$$L_{\mathcal{H}} \mapsto L_{\mathcal{E}} = \ :\partial bc:, \tag{6.7}$$

$$W_{\mathcal{H}} \mapsto W_{\mathcal{E}} = \frac{1}{\sqrt{6}}(:(\partial^2 b)c: - :(\partial b)(\partial c):). \tag{6.8}$$

Next, under the map $\epsilon : \mathcal{S} \to \mathcal{E} \otimes \mathcal{H}$ given by (4.2), we have

$$L_{\mathcal{S}} \mapsto L_{\mathcal{E}} \otimes 1, \qquad W_{\mathcal{S}} \mapsto W_{\mathcal{E}} \otimes 1. \tag{6.9}$$

The subalgebra $\mathcal{S}^0$ of $\beta\gamma$-charge zero has a natural set of generators

$$\{J^i =: \beta(\partial^i\gamma):, i \geq 0\},$$

and it is well known that $\mathcal{S}^0$ is isomorphic to $\mathcal{W}_{1+\infty,-1}$ [11]. One of the main results of [18] is that $\epsilon : \mathcal{S} \to \mathcal{E} \otimes \mathcal{H}$ restricts to an isomorphism

$$\mathcal{S}^0 \cong \mathcal{A} \otimes \mathcal{H}, \tag{6.10}$$

where $\mathcal{A} \cong \mathcal{W}_{3,-2}$ is the subalgebra of $\mathcal{E}$ generated by $L_{\mathcal{E}}$ and $W_{\mathcal{E}}$. By (6.9), ϵ maps $\mathcal{W}$ onto $\mathcal{A} \otimes 1$. Similarly, $\epsilon(\theta) = i(1 \otimes j)$, so ϵ maps Θ onto $1 \otimes \mathcal{H}$, and $\mathcal{S}^0 = \mathcal{W} \otimes \Theta$.

For each $d \in \mathbf{Z}$, the subspace $\mathcal{S}^d$ of $\beta\gamma$-charge d is a module over $\mathcal{S}^0$, which is in fact irreducible [11,19]. Define $v^d(z) \in \mathcal{S}^d$ by

$$v^d(z) = \begin{cases} \beta(z)^{-d} & d < 0 \\ 1 & d = 0 \\ \gamma(z)^d & d > 0. \end{cases} \tag{6.11}$$

Here $\beta(z)^{-d}$ and $\gamma(z)^d$ denote the d-fold iterated Wick products $:\beta(z)\cdots\beta(z):$ and $:\gamma(z)\cdots\gamma(z):$, respectively. Each $v^d(z)$ is a highest weight vector for the action of $\mathcal{W}_{3,-2}$, and the highest weight of $v^d(z)$ is given by (5.10) with

$$\begin{cases} \alpha = d & d \leq 0 \\ \alpha = d+1 & d > 0. \end{cases} \tag{6.12}$$

Moreover, $v^d(z)$ is also a highest weight vector for the action of $\mathcal{H}$, so $\mathcal{S}^d$ is generated by $v^d(z)$ as a module over $\mathcal{W}_{3,-2} \otimes \mathcal{H}$.

Theorem 6.2. *The map $f : \mathcal{W}_{3,-2} \hookrightarrow \mathcal{S}^{\Theta+}$ given by (6.6) is an isomorphism of vertex algebras. Moreover,* $\mathrm{Com}(\mathcal{S}^{\Theta+}, \mathcal{S}) = \Theta$. *Hence Θ and $\mathcal{S}^{\Theta+}$ form a Howe pair inside $\mathcal{S}$.*

Proof. Clearly $\mathcal{S}^{\Theta+} \subset \mathcal{S}^0$, and since $\mathcal{S}^0 = \mathcal{W} \otimes \Theta$, we have

$$\mathcal{S}^{\Theta+} = \mathrm{Com}(\Theta, \mathcal{W} \otimes \Theta) = \mathcal{W} \otimes \mathrm{Com}(\Theta, \Theta) = \mathcal{W}.$$

This proves the first statement. As for the second statement, it is clear from (5.10) and (6.12) that $\mathrm{Com}(\mathcal{S}^{\Theta+}, \mathcal{S}) \subset \mathcal{S}^0$. Hence

$$\mathrm{Com}(\mathcal{S}^{\Theta+}, \mathcal{S}) = \mathrm{Com}(\mathcal{W}, \mathcal{W} \otimes \Theta) = \Theta \otimes \mathrm{Com}(\mathcal{W}, \mathcal{W}) = \Theta. \quad \square$$

6.1. The map $\pi : \mathcal{S}^{\Theta+} \to \mathcal{D}^{\mathfrak{g}}$

Equip $\mathcal{S}$ with the conformal structure $L^{\alpha} = (\alpha - 1) : \partial\beta(z)\gamma(z) : + \alpha : \beta(z)\partial\gamma(z)$:, and consider the map $\pi : \mathcal{S}^{\Theta+} \to \mathcal{D}^{\mathfrak{g}}$ given by (1.2). In this case, $\mathcal{D}^{\mathfrak{g}}$ is just the polynomial algebra $\mathbf{C}[e]$, where e is the Euler operator $x' \frac{\mathrm{d}}{\mathrm{d}x'}$.

Lemma 6.3. *We have*

$$\pi(L_{\mathcal{S}}) = \frac{1}{2}(\mathrm{e}^2 + e), \qquad \pi(W_{\mathcal{S}}) = \frac{2}{3\sqrt{6}}\mathrm{e}^3 + \frac{1}{\sqrt{6}}\mathrm{e}^2 + \frac{1}{3\sqrt{6}}e. \tag{6.13}$$

In particular, $\pi(L_{\mathcal{S}})$ and $\pi(W_{\mathcal{S}})$ are independent of the choice of α.

Proof. This is a straightforward computation using (3.6) and the fact that $\pi_{Zh}(\gamma(z)) = x'$ and $\pi_{Zh}(\beta(z)) = \frac{\mathrm{d}}{\mathrm{d}x'}$. Note that $l = \pi(L_{\mathcal{S}})$ and $w = \pi(W_{\mathcal{S}})$ satisfy (5.8). □

Corollary 6.4. *For any conformal structure L^{α} on $\mathcal{S}$ as above,* $\mathrm{Im}(\pi)$ *is the subalgebra of* $\mathbf{C}[e]$ *generated by $\pi(L_{\mathcal{S}})$ and $\pi(W_{\mathcal{S}})$. Moreover,* $\mathrm{Coker}(\pi) = \mathbf{C}[e]/\mathrm{Im}(\pi)$ *has dimension one, and is spanned by the image of e in* $\mathrm{Coker}(\pi)$.

Proof. The first statement is immediate from Lemma 3.4, since $\mathcal{S}^{\Theta+}$ is strongly generated by $L_{\mathcal{S}}$ and $W_{\mathcal{S}}$ which have weights 2 and 3 respectively. The second statement follows from (3.10) and (6.13), because any polynomial in $\mathbf{C}[e]$ is equivalent to an element which is homogeneous of degree 1 modulo $\mathrm{Im}(\pi)$. □

7. $\mathcal{S}(V)^{\Theta+}$ for abelian Lie algebra actions

Fix a basis $\{x_1, \ldots, x_n\}$ for V and dual basis $\{x'_1, \ldots, x'_n\}$ for V^*. We regard $\mathcal{S}(V)$ as $\mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_n$, where $\mathcal{S}_j$ is the copy of $\mathcal{S}$ generated by $\beta^{x_j}(z), \gamma^{x'_j}(z)$. Let $f_j : \mathcal{S} \to \mathcal{S}(V)$ be the obvious map onto the jth factor. The subspace $\mathcal{S}_j^0$ of $\beta\gamma$-charge zero is isomorphic to $\mathcal{W}^j \otimes \mathcal{H}^j$, where $\mathcal{H}^j$ is generated by $\theta^j(z) = f_j(\theta(z))$, and $\mathcal{W}^j$ is generated by $L^j = f_j(L_{\mathcal{S}})$, $W^j = f_j(W_{\mathcal{S}})$. Moreover, as a module over $\mathcal{W}^j \otimes \mathcal{H}^j$, the space $\mathcal{S}_j^d$ of $\beta\gamma$-charge d is generated by the highest weight vector $v_j^d(z) = f_j(v^d(z))$, which is given by

$$v_j^d(z) = \begin{cases} \beta^{x_j}(z)^{-d} & d < 0 \\ 1 & d = 0 \\ \gamma^{x'_j}(z)^d & d > 0. \end{cases} \tag{7.1}$$

We denote by $\mathcal{S}'_j$ the linear span of the vectors $\{v_j^d(z) \mid d \in \mathbf{Z}\}$. Note that for any conformal structure L^{α} on $\mathcal{S}(V)$, the differential operators $v_j^d \in \mathcal{D}(V)$ defined by (2.6) correspond to $v_j^d(z)$ under the Zhu map. Let $\mathcal{B}$ denote the vertex algebra

$$\mathcal{S}_1^0 \otimes \cdots \otimes \mathcal{S}_n^0 \cong (\mathcal{W}^1 \otimes \mathcal{H}^1) \otimes \cdots \otimes (\mathcal{W}^n \otimes \mathcal{H}^n).$$

Clearly the space $\mathcal{S}(V)'$ consisting of highest-weight vectors for the action of $\mathcal{B}$ is just $\mathcal{S}'_1 \otimes \cdots \otimes \mathcal{S}'_n$. As usual, let $\mathbf{Z}^n \subset \mathbf{C}^n$ denote the standard lattice. For each lattice point $l = (l_1, \ldots, l_n) \in \mathbf{Z}^n$, define

$$\omega_l(z) =: v_1^{l_1}(z) \cdots v_n^{l_n}(z) :, \tag{7.2}$$

where $v_j^d(z)$ is given by (7.1). For example, in the case $n = 2$ and $l = (2, -3) \in \mathbf{Z}^2$, we have

$$\omega_l(z) =: v_1^2(z) v_2^{-3}(z) :=: \gamma^{x_1}(z)\gamma^{x_1}(z)\beta^{x_2}(z)\beta^{x_2}(z)\beta^{x_2}(z) : .$$

For any conformal structure L^{α} on $\mathcal{S}(V)$, $\omega_l(z)$ corresponds under the Zhu map to the element $\omega_l \in \mathcal{D}(V)$ given by (2.7).

Lemma 7.1. *For each $l \in \mathbf{Z}^n$, the $\mathcal{B}$-module $\mathcal{M}_l$ generated by $\omega_l(z)$ is irreducible. Moreover, as a module over $\mathcal{B}$,*

$$\mathcal{S}(V) = \bigoplus_{l \in \mathbf{Z}^n} \mathcal{M}_l. \tag{7.3}$$

Proof. This is immediate from the description of $\mathcal{S}^d$ as the irreducible $\mathcal{S}^0$-module generated by $v_d(z)$, and the fact that $\mathcal{S}(V)' = \mathcal{S}'_1 \otimes \cdots \otimes \mathcal{S}'_n$. □

Note that $\theta^j(z) \circ_0 \omega_l(z) = -l_j \omega_l(z)$, so the $\mathbf{Z}^n$-grading on $\mathcal{S}(V)$ above is just the eigenspace decomposition of $\mathcal{S}(V)$ under the family of diagonalizable operators $-\theta^j(z) \circ_0$.

For the remainder of this section, $\mathfrak{g}$ will denote the abelian Lie algebra

$$\mathbf{C}^m = gl(1) \oplus \cdots \oplus gl(1),$$

and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ will be a faithful, diagonal action. Let $A(\rho) \subset \mathbf{C}^n$ be the subspace spanned by $\{\rho(\xi) \mid \xi \in \mathfrak{g}\}$. As in the classical setting, we denote $\mathcal{S}(V)^{\Theta+}$ by $\mathcal{S}(V)^{\Theta+}_{\rho}$ when we need to emphasize the dependence on ρ. Clearly $\mathcal{S}(V)^{\Theta+}_{\rho} = \mathcal{S}(V)^{\Theta+}_{g\cdot\rho}$ for all $g \in GL(m)$, so the family of algebras $\mathcal{S}(V)^{\Theta+}_{\rho}$ is parametrized by the points $A(\rho) \in Gr(m, n)$.

Choose a basis $\{\xi^1, \ldots, \xi^m\}$ for $\mathfrak{g}$ such that the corresponding vectors

$$\rho(\xi^i) = a^i = (a^i_1, \ldots, a^i_n) \in \mathbf{C}^n$$

form an orthonormal basis for $A = A(\rho)$. Let $\theta^{\xi_i}(z)$ be the vertex operator corresponding to $\rho(\xi^i)$, and let Θ be the subalgebra of $\mathcal{B}$ generated by $\{\theta^{\xi_i}(z) \mid i = 1, \ldots, m\}$. By (3.5), we have

$$\theta^{\xi_i}(z) = \sum_{j=1}^{n} a_j \theta^j(z) = -\sum_{j=1}^{n} a_j : \gamma^{x'_j}(z)\beta^{x_j}(z) : .$$

Clearly $\theta^{\xi_i}(z)\theta^{\xi_j}(w) \sim -\langle a^i, a^j\rangle (z-w)^{-2} = \delta_{i,j}(z-w)^{-2}$.

If $m < n$, extend the set $\{a^1, \ldots, a^m\}$ to an orthonormal basis for $\mathbf{C}^n$ by adjoining vectors $b^i = (b^i_1, \ldots, b^i_n) \in \mathbf{C}^n$, for $i = m+1, \ldots, n$. Let

$$\phi^i(z) = \sum_{j=1}^{n} b^i_j \theta^j(z) = -\sum_{j=1}^{n} b^i_j : \gamma^{x'_j}(z)\beta^{x_j}(z) :$$

be the corresponding vertex operators, and let Φ be the subalgebra of $\mathcal{B}$ generated by $\{\phi^i(z) \mid i = m+1, \ldots, n\}$. The OPEs

$$\phi^i(z)\phi^j(w) \sim -\langle b^i, b^j\rangle (z-w)^{-2}, \qquad \theta^{\xi_i}(z)\phi^j(w) \sim -\langle a^i, b^j\rangle (z-w)^{-2}$$

show that the $\phi^i(z)$ pairwise commute and each generates a Heisenberg algebra of central charge -1, and that $\Phi \subset \mathcal{S}(V)^{\Theta+}$. In particular, we have the decomposition

$$\mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^n = \Theta \otimes \Phi.$$

Next, let $\mathcal{W}$ denote the subalgebra of $\mathcal{B}$ generated by $\{L^j(z), W^j(z) \mid j = 1, \ldots, n\}$. Theorem 6.2 shows that $\mathcal{W}$ commutes with both Θ and Φ, so we have the decomposition

$$\mathcal{B} = \mathcal{W} \otimes \Theta \otimes \Phi. \tag{7.4}$$

In particular, the subalgebra $\mathcal{B}' = \mathcal{W} \otimes \Phi$ lies in the commutant $\mathcal{S}(V)^{\Theta+}$. Let $\mathcal{M}'_l$ denote the $\mathcal{B}'$-submodule of $\mathcal{M}_l$ generated by $\omega_l(z)$, which is clearly irreducible as a $\mathcal{B}'$-module.

In order to describe $\mathcal{S}(V)^{\Theta+}$, we first describe the larger space $\mathcal{S}(V)^{\Theta>}$ which is annihilated by $\theta^{\xi_i}(k)$ for $i = 1, \ldots, m$ and $k > 0$. Then $\mathcal{S}(V)^{\Theta+}$ is just the subspace of $\mathcal{S}(V)^{\Theta>}$ which is annihilated by $\theta^{\xi_i}(0)$, for $i = 1, \ldots, m$. It is clear from (7.4) and the irreducibility of $\mathcal{M}_l$ as a $\mathcal{B}$-module that $\mathcal{S}(V)^{\Theta>} \cap \mathcal{M}_l = \mathcal{M}'_l$, so

$$\mathcal{S}(V)^{\Theta>} = \bigoplus_{l \in \mathbf{Z}^n} \mathcal{M}'_l. \tag{7.5}$$

Theorem 7.2. *As a module over $\mathcal{B}'$,*

$$\mathcal{S}(V)^{\Theta+} = \bigoplus_{l \in A^{\perp} \cap \mathbf{Z}^n} \mathcal{M}'_l. \tag{7.6}$$

Proof. Let $\omega(z) \in \mathcal{S}(V)^{\Theta+}$. Since ω lies in the larger space $\mathcal{S}(V)^{\Theta>}$ which is a direct sum of irreducible, cyclic $\mathcal{B}'$-modules $\mathcal{M}'_l$ with generators $\omega_l(z)$, we may assume without loss of generality that $\omega(z) = \omega_l(z)$ for some l. An OPE calculation shows that

$$\theta^{\xi_i}(z)\omega_l(w) \sim -\langle a^i, l\rangle \omega_l(w)(z-w)^{-1}. \tag{7.7}$$

Hence $\omega_l \in \mathcal{S}(V)^{\Theta+}$ if and only if l lies in the sublattice $A^{\perp} \cap \mathbf{Z}^n$. $\square$

Our next step is to find a *finite* generating set for $\mathcal{S}(V)^{\Theta+}$. Generically, $A^{\perp} \cap \mathbf{Z}^n$ has rank zero, so $\mathcal{S}(V)^{\Theta+} = \mathcal{B}'$, which is (strongly) generated by the set

$$\{\phi^i(z), L^j(z), W^j(z) | i = m+1, \ldots, n,\ j = 1, \ldots, n\}.$$

If $A^{\perp} \cap \mathbf{Z}^n$ has rank r for some $0 < r \le n - m$, choose a basis $\{l^1, \ldots, l^r\}$ for $A^{\perp} \cap \mathbf{Z}^n$. We claim that for any $l \in A^{\perp} \cap \mathbf{Z}^n$, $\omega_l(z)$ lies in the vertex subalgebra generated by

$$\{\omega_{l^1}(z), \ldots, \omega_{l^r}(z), \omega_{-l^1}(z), \ldots, \omega_{-l^r}(z)\}.$$

It suffices to prove that given lattice points $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$ in $\mathbf{Z}^n$, $\omega_{l+l'}(z) = k\omega_l(z) \circ_d \omega_{l'}(z)$ for some $k \neq 0$ and $d \in \mathbf{Z}$.

First, consider the special case where $l = (l_1, 0, \ldots, 0)$ and $l' = (l'_1, 0, \ldots, 0)$. If $l_1 l'_1 \geq 0$, we have $\omega_l(z) \circ_{-1} \omega_{l'}(z) = \omega_{l+l'}(z)$. Suppose next that $l_1 < 0$ and $l'_1 > 0$, so that $\omega_l(z) = \beta^{x_1}(z)^{-l_1}$ and $\omega_{l'}(z) = \gamma^{x'_1}(z)^{l'_1}$. Let

$$d_1 = \min\{-l_1, l'_1\}, \qquad e_1 = \max\{-l_1, l'_1\}, \qquad d = d_1 - 1.$$

An OPE calculation shows that

$$\omega_l(z) \circ_d \omega_{l'}(z) = \frac{e_1!}{(e_1 - d_1)!}\omega_{l+l'}(z), \tag{7.8}$$

where as usual $0! = 1$. Similarly, if $l_1 > 0$ and $l'_1 < 0$, we take $d_1 = \min\{l_1, -l'_1\}$, $e_1 = \max\{l_1, -l'_1\}$, and $d = d_1 - 1$. We have

$$\omega_l(z) \circ_d \omega_{l'}(z) = -\frac{e_1!}{(e_1 - d_1)!}\omega_{l+l'}(z). \tag{7.9}$$

Now consider the general case $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$. For $j = 1, \ldots, n$, define

$$d_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \min\{|l_j|, |l'_j|\}, & l_j l'_j < 0, \end{cases} \qquad e_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \max\{|l_j|, |l'_j|\}, & l_j l'_j < 0, \end{cases}$$

$$k_j = \begin{cases} 0 & l_j \leq 0 \\ d_j & l_j > 0, \end{cases} \qquad d = -1 + \sum_{j=1}^{n} d_j.$$

Using (7.8) and (7.9) repeatedly, we calculate

$$\omega_l(z) \circ_d \omega_{l'}(z) = \left(\prod_{j=1}^{n} (-1)^{k_j} \frac{e_j!}{(e_j - d_j)!}\right) \omega_{l+l'}(z),$$

which shows that $\omega_{l+l'}(z)$ lies in the vertex algebra generated by $\omega_l(z)$ and $\omega_{l'}(z)$. Thus we have proved

Theorem 7.3. *Let $\{l^1, \ldots, l^r\}$ be a basis for the lattice $A^\perp \cap \mathbf{Z}^n$, as above. Then $\mathcal{S}(V)^{\Theta+}$ is generated as a vertex algebra by $\mathcal{B}'$ together with the additional vertex operators*

$$\omega_{l^1}(z), \ldots, \omega_{l^r}(z), \qquad \omega_{-l^1}(z), \ldots, \omega_{-l^r}(z).$$

In particular, $\mathcal{S}(V)^{\Theta+}$ is finitely generated as a vertex algebra.

In the generic case where $A^\perp \cap \mathbf{Z}^n = 0$ and $\mathcal{S}(V)^{\Theta+} = \mathcal{B}'$, we claim that $\mathcal{S}(V)^{\Theta+}$ has a natural $(n - m)$-parameter family of conformal structures for which the generators $\phi^i(z)$, $L^j(z)$, $W^j(z)$ are primary of conformal weights 1, 2, 3, respectively. Note first that $\mathcal{W}$ has the conformal structure $L_{\mathcal{W}}(z) = \sum_{j=1}^{n} L^j(z)$ of central charge $-2n$.

It is well known that for $k \neq 0$ and $c \in \mathbf{C}$, the Heisenberg algebra $\mathcal{H}$ of central charge k admits a Virasoro element $L^c(z) = \frac{1}{2k} j(z)j(z) + c\partial j(z)$ of central charge $1 - 12c^2k$, under which the generator $j(z)$ is primary of weight one. Hence given $\lambda = (\lambda_{m+1}, \ldots, \lambda_n) \in \mathbf{C}^{n-m}$ the Heisenberg algebra generated by $\phi^i(z)$ has a conformal structure

$$L^{\lambda_i}(z) = -\frac{1}{2} : \phi^i(z)\phi^i(z) : + \lambda_i \partial \phi^i(z)$$

of central charge $1 + 12\lambda_i^2$. Since $\phi^i(z)$ and $\phi^j(z)$ commute for $i \neq j$, it follows that $L^\lambda_\Phi(z) = \sum_{i=m+1}^{n} L^{\lambda_i}(z)$ is a conformal structure on Φ of central charge $\sum_{i=m+1}^{n} 1 + 12\lambda_i^2$. Finally,

$$L_{\mathcal{B}'}(z) = L_{\mathcal{W}}(z) \otimes 1 + 1 \otimes L^\lambda_\Phi(z) \in \mathcal{W} \otimes \Phi = \mathcal{B}'$$

is a conformal structure on $\mathcal{B}'$ of central charge $-2n + \sum_{i=m+1}^{n} 1 + 12\lambda_i^2$ with the desired properties.

When the lattice $A^\perp \cap \mathbf{Z}^n$ has positive rank, the vertex algebras $\mathcal{S}(V)^{\Theta+}$ have a very rich structure which depends sensitively on $A^\perp \cap \mathbf{Z}^n$. In general, the set of generators for $\mathcal{S}(V)^{\Theta+}$ given by Theorem 7.3 will not be a set of *strong* generators, and the conformal structure $L_{\mathcal{B}'}$ on $\mathcal{B}'$ will not extend to a conformal structure on all of $\mathcal{S}(V)^{\Theta+}$.

Theorem 7.4. *For any action of $\mathfrak{g}$ on V, $\mathrm{Com}(\mathcal{S}(V)^{\Theta+}, \mathcal{S}(V)) = \Theta$. Hence $\mathcal{S}(V)^{\Theta+}$ and Θ form a Howe pair inside $\mathcal{S}(V)$.*

Proof. Since $\mathcal{B}' \subset \mathcal{S}(V)^{\Theta+}$, we have $\Theta \subset \mathrm{Com}(\mathcal{S}(V)^{\Theta+}, \mathcal{S}(V)) \subset \mathrm{Com}(\mathcal{B}', \mathcal{S}(V))$, so it suffices to show that $\mathrm{Com}(\mathcal{B}', \mathcal{S}(V)) = \Theta$. Recall that $\mathcal{B}' = \mathcal{W} \otimes \Phi$ and $\Theta \otimes \Phi = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^n$. Since $\mathrm{Com}(\mathcal{W}^i, \mathcal{S}_i) = \mathcal{H}^i$ by Theorem 6.2, it follows that $\mathrm{Com}(\mathcal{W}, \mathcal{S}(V)) = \Theta \otimes \Phi$. Then

$$\mathrm{Com}(\mathcal{B}', \mathcal{S}(V)) = \mathrm{Com}\,(\Phi, \mathrm{Com}(\mathcal{W}, \mathcal{S}(V))) = \mathrm{Com}(\Phi, \Theta \otimes \Phi) = \Theta \otimes \mathrm{Com}(\Phi, \Phi) = \Theta. \quad \square$$

This result shows that we can always recover the action of $\mathfrak{g}$ (up to $GL(m)$-equivalence) from $\mathcal{S}(V)^{\Theta+}$, by taking its commutant inside $\mathcal{S}(V)$. This stands in contrast to Theorem 2.1, which shows that we can reconstruct the action from $\mathcal{D}(V)^{\mathfrak{g}}$ only when $A^{\perp} \cap \mathbf{Z}^n$ has rank $n - m$.

Theorem 7.5. *For any action of $\mathfrak{g}$ on V, $\mathcal{S}(V)^{\Theta+}$ is a simple vertex algebra.*

Proof. Given a non-zero ideal $\mathcal{I} \subset \mathcal{S}(V)^{\Theta+}$, we need to show that $1 \in \mathcal{I}$. Let $\omega(z)$ be a non-zero element of $\mathcal{I}$. Since each $\mathcal{M}'_l$ is irreducible as a module over $\mathcal{B}'$, we may assume without loss of generality that

$$\omega(z) = \sum_{l \in \mathbf{Z}^n} c_l \omega_l(z) \tag{7.10}$$

for constants $c_l \in \mathbf{C}$, such that $c_l \neq 0$ for only finitely many values of l.

For each lattice point $l = (l_1, \ldots, l_n) \in \mathbf{Z}^n$, both $\omega_l(z)$ and $\omega_{-l}(z)$ have degree $d = \sum_{j=1}^n |l_j|$ as polynomials in the variables $\beta^{x_j}(z)$ and $\gamma^{x'_j}(z)$. Let d be the maximal degree of terms $\omega_l(z)$ appearing in (7.10) with non-zero coefficient c_l, and let l be such a lattice point for which $\omega_l(z)$ has degree d. An OPE calculation shows that

$$\omega_{-l}(z) \circ_{d-1} \omega_{l'}(z) = \begin{cases} 0 & l' \neq l \\ \left(\prod_{j=1}^n (-1)^{k_j} |l_j|! \right) 1 & l' = l \end{cases} \tag{7.11}$$

where $k_j = \min\{0, l_j\}$, for all lattice points l' appearing in (7.10) with non-zero coefficient. It follows from (7.11) that

$$\frac{1}{c_l \left(\prod_{j=1}^n (-1)^{k_j} |l_j|! \right)} \omega_{-l}(z) \circ_{d-1} \omega(z) = 1. \quad \square$$

7.1. The map $\pi : \mathcal{S}(V)^{\Theta+} \to \mathcal{D}(V)^{\mathfrak{g}}$

Equip $\mathcal{S}(V)$ with the conformal structure L^{α} given by (3.2), for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{C}^n$. Suppose first that $A^{\perp} \cap \mathbf{Z}^n$ has rank zero, so that $\mathcal{S}(V)^{\Theta+} = \mathcal{B}'$, and $\mathcal{D}(V)^{\mathfrak{g}} = \mathbf{C}[e_1, \ldots, e_n] = E$. Let $\pi : \mathcal{S}(V)^{\Theta+} \to \mathcal{D}(V)^{\mathfrak{g}}$ be the map given by (1.2). By Lemma 6.3, for $j = 1, \ldots, n$ we have

$$\pi(L^j(z)) = \frac{1}{2}(e_i^2 + e_j), \qquad \pi(W^j(z)) = \frac{2}{3\sqrt{6}} e_j^3 + \frac{1}{\sqrt{6}} e_j^2 + \frac{1}{3\sqrt{6}} e_j.$$

Moreover, (3.10) shows that $\pi(\phi^i(z)) = \langle b^i, \alpha \rangle - \sum_{j=1}^n b^i_j (e_j + 1)$. Since $\mathcal{B}'$ is strongly generated by $\{\phi^i(z), L^j(z), W^j(z) \mid i = m+1, \ldots, n, j = 1, \ldots, n\}$, it follows from Lemma 3.4 that $\mathrm{Im}(\pi)$ is generated by the collection

$$\{\pi(\phi^i(z)), \pi(L^j(z)), \pi(W^j(z)) \mid i = m+1, \ldots, n, \ j = 1, \ldots, n\}.$$

The map π is not surjective, but $\mathrm{Coker}(\pi)$ is generated as a module over $\mathrm{Im}(\pi)$ by the collection $\{t^{\xi_i} \mid i = 1, \ldots, m\}$, where t^{ξ_i} is the image of

$$\pi_{Zh}(\theta^{\xi_i}(z)) = \langle a^i, \alpha \rangle - \sum_{j=1}^n a^i_j (e_j + 1)$$

in $\mathrm{Coker}(\pi) = E/\pi(\mathcal{B}')$. Unlike the case where V is one-dimensional, π depends on the choice of α.

Suppose next that the lattice $A^{\perp} \cap \mathbf{Z}^n = 0$ has positive rank. Clearly $\pi_{Zh}(\mathcal{M}_l) = M_l$ for all l, so $\pi(\mathcal{M}'_l) \subset M_l$. This map need not be surjective, but since M_l is the free E-module generated by ω_l, and $E/\pi(\mathcal{B}')$ is generated as a $\pi(\mathcal{B}')$-module by $\{t^{\xi_i} \mid i = 1, \ldots, m\}$, it follows that each $M_l/\pi(\mathcal{M}'_l)$ is generated as a $\pi(\mathcal{B}')$-module by $\{t_l^{\xi_i} \mid i = 1, \ldots, m\}$, where $t_l^{\xi_i}$ is the image of $\pi_{Zh}(\theta^{\xi_i}(z))\omega_l$ in $M_l/\pi(\mathcal{M}'_l)$.

Theorem 7.6. *For any action of $\mathfrak{g}$ on V,* $\mathrm{Coker}(\pi)$ *is generated as a module over* $\mathrm{Im}(\pi)$ *by the collection $\{t^{\xi_i} \mid i = 1, \ldots, m\}$. In particular,* $\mathrm{Coker}(\pi)$ *is a finitely generated module over* $\mathrm{Im}(\pi)$ *with generators corresponding to central elements of $\mathcal{D}(V)^{\mathfrak{g}}$.*

Proof. First, since $\pi(\omega_l(z)) = \omega_l$ for all l, it is clear that the generators $t_l^{\xi_i}$ of $M_l/\pi(\mathcal{M}'_l)$ lie in the $\mathrm{Im}(\pi)$-module generated by $\{t^{\xi_i} \mid i = 1, \ldots, m\}$, which proves the first statement. Finally, the fact that the elements $\pi_{Zh}(\theta^{\xi_i}(z))$ corresponding to t^{ξ_i} each lie in the center of $\mathcal{D}(V)^{\mathfrak{g}}$ is immediate from (2.10). $\square$

7.2. A vertex algebra bundle over the Grassmannian $Gr(m, n)$

As ρ varies over the space $R^0(V)$ of effective actions, recall that $\mathcal{S}(V)^{\Theta+}_{\rho}$ is uniquely determined by the point $A(\rho) \in Gr(m, n)$. The algebras $\mathcal{S}(V)^{\Theta+}_{\rho}$ do not form a fiber bundle over $Gr(m, n)$. However, the subspace of $\mathcal{S}(V)^{\Theta+}_{\rho}$ of degree zero in the $A(\rho)^{\perp} \cap \mathbf{Z}^n$-grading (7.6) is just $\mathcal{B}'_{\rho} = \mathcal{B}'$, and the algebras $\mathcal{B}'_{\rho}$ form a bundle of vertex algebras $\mathcal{E}$ over $Gr(m, n)$. The classical analogue of $\mathcal{E}$ is not interesting; it is just the trivial bundle whose fiber over each point is the polynomial algebra E.

For each ρ, recall that $\mathcal{B}'_{\rho} = \mathcal{W}_{\rho} \otimes \Phi_{\rho}$, where $\mathcal{W}_{\rho}$ is generated by $\{L^j(z), W^j(z) \mid j = 1, \dots, n\}$, and Φ_{ρ} is generated by $\{\phi^i(z) \mid i = m+1, \dots, n\}$. Since $\mathcal{W}_{\rho}$ is independent of ρ, it gives rise to a trivial subbundle of $\mathcal{E}$. As a vector space, note that $\Phi_{\rho} = \text{Sym}\left(\bigoplus_{k\geq 1} A(\rho)^{\perp}_k\right)$, where $A(\rho)^{\perp}_k$ is the copy of $A(\rho)^{\perp}$ spanned by the vectors $\partial^k \phi^i(z)$ for $i = m+1, \dots, n$. It follows that the factor Φ_{ρ} in the fiber over $A(\rho)$ gives rise to the following subbundle of $\mathcal{E}$:

$$\text{Sym}\left(\bigoplus_{k\geq 1} \mathcal{F}_k\right), \tag{7.12}$$

where $\mathcal{F}_k$ is the quotient of the rank n trivial bundle over $Gr(m, n)$ by the tautological bundle. Since each $\mathcal{F}_k$ has weight k, the weighted components of the bundle (7.12) are all finite-dimensional. The non-triviality of this bundle is closely related to Theorem 7.4.

8. Vertex algebra operations and transvectants on $\mathcal{D}(V)^{\mathfrak{g}}$

If we fix a basis $\{x_1, \dots, x_n\}$ for V and a dual basis $\{x'_1, \dots, x'_n\}$ for V^*, $\mathcal{S}(V)$ has a basis consisting of iterated Wick products of the form

$$\mu(z) =: \partial^{k_1}\gamma^{x'_{i_1}}(z) \cdots \partial^{k_r}\gamma^{x'_{i_r}}(z)\partial^{l_1}\beta^{x_{j_1}}(z) \cdots \partial^{l_s}\beta^{x_{j_s}}(z) : .$$

Define gradings *degree* and *level* on $\mathcal{S}(V)$ as follows:

$$\deg(\mu) = r + s, \qquad \text{lev}(\mu) = \sum_{i=1}^{r} k_i + \sum_{j=1}^{s} l_j,$$

and let $\mathcal{S}(V)^{(n)}[d]$ denote the subspace of level n and degree d. The gradings

$$\mathcal{S}(V) = \bigoplus_{n\geq 0} \mathcal{S}(V)^{(n)} = \bigoplus_{n,d\geq 0} \mathcal{S}(V)^{(n)}[d] = \bigoplus_{d\geq 0} \mathcal{S}(V)[d] \tag{8.1}$$

are clearly independent of our choice of basis on V, since an automorphism of V has the effect of replacing β^{x_i} and $\gamma^{x'_i}$ with linear combinations of the β^{x_i}'s and $\gamma^{x'_i}$'s, respectively.

Let $\sigma : \mathcal{D}(V) \to gr\,\mathcal{D}(V) = \text{Sym}(V \oplus V^*)$ be the map

$$x'_{i_1} \cdots x'_{i_r} \frac{\partial}{\partial x'_{j_1}} \cdots \frac{\partial}{\partial x'_{j_s}} \mapsto x'_{i_1} \cdots x'_{i_r} x_{j_1} \cdots x_{j_s}, \tag{8.2}$$

which is a linear isomorphism. Any bilinear product $*$ on $\text{Sym}(V \oplus V^*)$ corresponds to a bilinear product on $\mathcal{D}(V)$, which we also denote by $*$, as follows:

$$\omega * \nu = \sigma^{-1}(\sigma(\omega) * \sigma(\omega)),$$

for $\omega, \nu \in \mathcal{D}(V)$, Moreover, $\omega_1, \dots, \omega_k$ generate $\mathcal{D}(V)$ as a ring if and only if $\sigma(\omega_1), \dots, \sigma(\omega_k)$ generate $\text{Sym}(V \oplus V^*)$ as a ring. The map $f : \text{Sym}(V \oplus V^*) \to \mathcal{S}(V)^{(0)}$ given by

$$x'_{i_1} \cdots x'_{i_r} x_{j_1} \cdots x_{j_s}, \mapsto : \gamma^{x'_{i_1}}(z) \cdots \gamma^{x'_{i_r}}(z)\beta^{x_{j_1}}(z) \cdots \beta^{x_{j_s}}(z) :, \tag{8.3}$$

is a linear isomorphism, so that $f \circ \sigma : \mathcal{D}(V) \to \mathcal{S}(V)^{(0)}$ is a linear isomorphism as well.

$\mathcal{S}(V)^{(0)}$ has a family of bilinear products $*_k$ which are induced by the circle products on $\mathcal{S}(V)$. Given $\omega(z), \nu(z) \in \mathcal{S}(V)^{(0)}$, define

$$\omega(z) *_k \nu(z) = p(\omega(z) \circ_k \nu(z)), \tag{8.4}$$

where $p : \mathcal{S}(V) \to \mathcal{S}(V)^{(0)}$ is the projection onto the subspace of level zero. Clearly $\omega(z) *_k \nu(z) = 0$ whenever $k < -1$ because $p \circ \partial$ acts by zero on $\mathcal{S}(V)^{(0)}$. For $k \geq -1$, $*_k$ is homogeneous of degree $-2k - 2$.

Via (8.3), we may pull back the products $*_k$, $k \geq -1$ to obtain a family of bilinear products on $\text{Sym}(V \oplus V^*)$, which we also denote by $*_k$. In fact, these products have a classical description. Let

$$\Gamma = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x'_i} - \frac{\partial}{\partial x'_i} \otimes \frac{\partial}{\partial x_i}, \tag{8.5}$$

and define the kth transvectant[1] on $\mathrm{Sym}(V \oplus V^*)$ by

$$[,]_k : \mathrm{Sym}(V \oplus V^*) \otimes \mathrm{Sym}(V \oplus V^*) \to \mathrm{Sym}(V \oplus V^*), \qquad [\omega, \nu]_k = m \circ \Gamma^k(\omega \otimes \nu).$$

Here m is the multiplication map sending $\omega \otimes \nu \mapsto \omega\nu$.

Theorem 8.1. *The product* $*_k$ *on* $\mathrm{Sym}(V \oplus V^*)$ *given by* (8.4) *coincides with the transvectant* $[,]_{k+1}$ *for* $k \geq -1$.

Proof. First consider the case $k = -1$. In this case $[,]_0$ is just ordinary multiplication. Recall the formula

$$:(:ab:)c: - :abc := \sum_{k\geq 0} \frac{1}{(k+1)!} \left(:(\partial^{k+1}a)(b \circ_k c): + (-1)^{|a||b|} :(\partial^{k+1}b)(a \circ_k c): \right),$$

which holds for any vertex operators a, b, c in a vertex algebra $\mathcal{A}$. It follows that the associator ideal in $\mathcal{S}(V)$ under the Wick product is annihilated by the projection p. Similarly, the commutator ideal in $\mathcal{S}(V)$ under the Wick product is annihilated by p, so $\mathcal{S}(V)^{(0)}$ is a polynomial algebra with product $*_{-1}$, and $f : \mathrm{Sym}(V \oplus V^*) \to \mathcal{S}(V)^{(0)}$ is an isomorphism of polynomial algebras. Hence given $\omega, \nu \in \mathrm{Sym}(V \oplus V^*)$, we have $[\omega, \nu]_0 = \omega\nu = \omega *_{-1} \nu$.

Next, if $k \geq 0$, it is clear from the definition of the vertex algebra products $\circ_k$ that given $\omega(z), \nu(z) \in \mathcal{S}(V)^{(0)}$, $\omega(z) *_k \nu(z)$ is just the sum of all possible contractions of $k+1$ factors of the form $\beta^{x_i}(z)$ or $\gamma^{x'_i}(z)$ appearing in $\omega(z)$ with $k+1$ factors of the form $\beta^{x_i}(z)$ or $\gamma^{x'_i}(z)$ appearing in $\nu(z)$. Here the contraction of $\beta^{x_i}(z)$ with $\gamma^{x_j}(z)$ is $\delta_{i,j}$, and the contraction of $\gamma^{x_i}(z)$ with $\beta^{x_j}(z)$ is $-\delta_{i,j}$. Similarly, it follows from (8.5) that given $\omega, \nu \in \mathrm{Sym}(V \oplus V^*)$, $[\omega, \nu]_{k+1}$ is the sum of all possible contractions of $k+1$ factors of the form x_i or x'_i appearing in ω with $k+1$ factors of the form x_i or x'_i appearing in ν. The contraction of x_i with x'_j is $\delta_{i,j}$ and the contraction of x'_i with x_j is $-\delta_{i,j}$. Since $f : \mathrm{Sym}(V \oplus V^*) \to \mathcal{S}(V)^{(0)}$ is the algebra isomorphism sending $x_i \mapsto \beta^{x_i}(z)$ and $x'_i \mapsto \gamma^{x'_i}(z)$, the claim follows. $\square$

Via $\sigma : \mathcal{D}(V) \to \mathrm{Sym}(V \oplus V^*)$ the products $*_k$ on $\mathrm{Sym}(V \oplus V^*)$ pull back to bilinear products on $\mathcal{D}(V)$, which we also denote by $*_k$. These products satisfy $\omega *_k \nu \in \mathcal{D}(V)_{(r+s-2k-2)}$ for $\omega \in \mathcal{D}(V)_{(r)}$ and $s \in \mathcal{D}(V)_{(s)}$. It is immediate from Theorem 8.1 that $*_{-1}$ and $*_0$ correspond to the ordinary associative product and bracket on $\mathcal{D}(V)$, respectively. Since the circle product $\circ_0$ is a derivation of every $\circ_k$, it follows that $\omega *_0$ is a derivation of $*_k$ for all $\omega \in \mathcal{D}(V)$ and $k \geq -1$.

We call $\mathcal{D}(V)$ equipped with the products $\{*_k \mid k \geq -1\}$ a $*$-algebra. A similar construction goes through in other settings as well. For example, given a Lie algebra $\mathfrak{g}$ equipped with a symmetric, invariant bilinear form B, $\mathfrak{U}\mathfrak{g}$ has a $*$-algebra structure (which depends on B). Given a $*$-algebra $\mathcal{A}$, we can define $*$-subalgebras, $*$-ideals, quotients, and homomorphisms in the obvious way. If V is a module over a Lie algebra $\mathfrak{g}$, $\mathcal{D}(V)^{\mathfrak{g}}$ is a $*$-subalgebra of $\mathcal{D}(V)$ because the action of $\xi \in \mathfrak{g}$ is given by $[\tau(\xi), -] = \tau(\xi) *_0$ which is a derivation of all the other products.

Given elements $\omega_1, \ldots, \omega_k \in \mathcal{D}(V)^{\mathfrak{g}}$, examples are known where $\omega_1, \ldots, \omega_k$ do not generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a ring, but do generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a $*$-algebra.[2] This phenomenon occurs in our main example, in which $\mathfrak{g}$ is the abelian Lie algebra $\mathbf{C}^m$ acting diagonally on $V = \mathbf{C}^n$. Recall that $\mathcal{D}(V)^{\mathfrak{g}} = \bigoplus_{l \in A^\perp \cap \mathbf{Z}^n} M_l$, where M_l is the free E-module generated by ω_l. Suppose that $A^\perp \cap \mathbf{Z}^n$ has rank r, and let $\{l^i = (l^i_1, \ldots, l^i_n) \mid i = 1, \ldots, r\}$ be a basis for $A^\perp \cap \mathbf{Z}^n$. In general, the collection

$$e_1, \ldots, e_n, \qquad \omega_{l^1}, \ldots, \omega_{l^r}, \qquad \omega_{-l^1}, \ldots, \omega_{-l^r} \tag{8.6}$$

is too small to generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a ring.

Theorem 8.2. *$\mathcal{D}(V)^{\mathfrak{g}}$ is generated as a $*$-algebra by the collection (8.6). Moreover, $\mathcal{D}(V)^{\mathfrak{g}}$ is simple as a $*$-algebra.*

Proof. To prove the first statement, it suffices to show that given lattice points $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$, $\omega_{l+l'}$ lies in the $*$-algebra generated by ω_l and $\omega_{l'}$. For $j = 1, \ldots, n$, define

$$d_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \min\{|l_j|, |l'_j|\}, & l_j l'_j < 0, \end{cases} \qquad e_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \max\{|l_j|, |l'_j|\}, & l_j l'_j < 0, \end{cases}$$

$$k_j = \begin{cases} 0 & l_j \leq 0 \\ d_j & l_j > 0, \end{cases} \qquad d = -1 + \sum_{j=1}^n d_j.$$

The same calculation as in the proof of Theorem 7.3 shows that

$$\omega_l *_d \omega_{l'} = \left(\prod_{j=1}^n (-1)^{k_j} \frac{e_j!}{(e_j - d_j)!} \right) \omega_{l+l'},$$

which shows that $\omega_{l+l'}$ lies in the $*$-algebra generated by ω_l and $\omega_{l'}$.

[1] I thank N. Wallach for explaining this construction to me.
[2] I thank N. Wallach for pointing this out to me.

As for the second statement, the argument is analogous to the proof of Theorem 7.5. Given a non-zero $*$-ideal $I \subset \mathcal{D}(V)^{\mathfrak{g}}$, we need to show that $1 \in I$. Let ω be a non-zero element of I. It is easy to check that for $i, j = 1, \dots, n$, and $l \in A^{\perp} \cap \mathbf{Z}^n$, we have

$$e_i *_1 e_j = -\delta_{i,j}, \qquad e_i *_1 \omega_l = 0.$$

By applying the operators $e_i *_1$ for $i = 1, \dots, n$, we can reduce ω to the form

$$\sum_{l \in \mathbf{Z}^n} c_l \omega_l \tag{8.7}$$

for constants $c_l \in \mathbf{C}$, such that $c_l \neq 0$ for only finitely many values of l. We may assume without loss of generality that ω is already of this form. Let d be the maximal degree (in the Bernstein filtration) of terms ω_l appearing in (8.7) with non-zero coefficient c_l, and let l be such a lattice point for which ω_l has degree d. We have

$$\omega_{-l} *_{d-1} \omega_{l'} = \begin{cases} 0 & l' \neq l \\ \left(\prod_{j=1}^{n} (-1)^{k_j} |l_j|! \right) 1 & l' = l \end{cases}$$

where $k_j = \min\{0, l_j\}$, for all l' appearing in (8.7). Hence

$$\frac{1}{c_l \left(\prod_{j=1}^{n} (-1)^{k_j} |l_j|! \right)} \omega_{-l} *_{d-1} \omega = 1. \quad \square$$

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