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A local Paley–Wiener theorem for compact symmetric spaces

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Abstract

The Fourier coefficients of a smooth K -invariant function on a compact symmetric space $M = U/K$ are given by integration of the function against the spherical functions. For functions with support in a neighborhood of the origin, we describe the size of the support by means of the exponential type of a holomorphic extension of the Fourier coefficients.

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1. Introduction

The classical Paley–Wiener theorem (also called the Paley–Wiener–Schwartz theorem) describes the image by the Fourier transform of the space of compactly supported smooth functions on \mathbb{R}^n . The theorem was generalized to Riemannian symmetric spaces of the non-compact type by Helgason and Gangolli (see [12, Theorem IV,7.1], [10]), to semisimple Lie groups by Arthur (see [1]), and to pseudo-Riemannian reductive symmetric spaces by van den Ban and

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Schlichtkrull (see [2]). More precisely, these theorems describe the Fourier image of the space of functions supported in a (generalized) ball of a given size. The image space consists of holomorphic (in the pseudo-Riemannian case, meromorphic) functions with exponential growth, and the size of the ball is reflected in the exponent of the exponential growth estimate.

In this paper we present an analogue of these theorems for Riemannian symmetric spaces of the compact type. Obviously the compact support is trivial in this case, and the important issue is the determination of the *size* of the support of a smooth function from the growth property of its Fourier transform. Let us illustrate this by recalling the corresponding result for Fourier series. Consider a smooth 2π -periodic function $f: \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}$, and suppose that f has support in $[-r, r] + 2\pi\mathbb{Z}$, where $0 < r < \pi$. We denote the space of such functions by $C_r^\infty(\mathbb{T})$. The Fourier transform of f is the Fourier coefficient map $n \mapsto \hat{f}(n)$ on \mathbb{Z} , where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt,$$

and it extends to a holomorphic function on \mathbb{C} , defined by the same formula with n replaced by $\lambda \in \mathbb{C}$. By the classical Paley–Wiener theorem for \mathbb{R} this holomorphic extension has at most exponential growth of type r , and every holomorphic function on \mathbb{C} of this type arises in this fashion from a unique function $f \in C_r^\infty(\mathbb{T})$. It is this ‘local’ Paley–Wiener theorem for \mathbb{T} that we generalize to an arbitrary Riemannian symmetric space M of the compact type. We consider spherical functions on M , and the relevant transform is the spherical Fourier transform.

The theorem presented here is known in some particular cases. In particular, it is known in the case of a compact Lie group U , viewed as a symmetric space for the product group $U \times U$ with the left \times right action. In this case, the theorem was obtained by Gonzalez (see [11]) by a simple reduction to the Euclidean case by means of the Weyl character formula. This result of Gonzalez plays a crucial role in our proof, and it is recalled in Section 8 below. Other cases in which the theorem is known, are as follows.

If the symmetric space has rank one, the spherical Fourier transform can be expressed in terms of a Jacobi transform, for which the Paley–Wiener theorem has been obtained by Koornwinder (see [14, p. 158]). As an example, we treat the special case $S^2 = \text{SU}(2)/\text{SO}(2)$ in the final section of this paper. In this case, the theorem of Koornwinder is due to Beurling (unpublished, see [14]).

If the symmetric space is of even multiplicity type, the local Paley–Wiener theorem has been achieved by Branson, Ólafsson and Pasquale (see [4]) by application of a holomorphic version of Opdam’s differential shift operators (developed in [16,17]). The method is strongly dependent on the assumption that the multiplicities are even. The theorem of Gonzalez is a particular case.

Finally, the theorem was obtained recently by Camporesi for the complex Grassmann manifolds by reduction to the rank one case, see [6].

We shall now give a brief outline of the paper. In Sections 2 and 3 we introduce the basic notations. In Section 4 we define the relevant Paley–Wiener space and state the main theorem, that the Fourier transform is bijective onto this space. The proof, that it maps into the space is given in Section 6. Here we rely on work of Opdam [17]. The theorem of Gonzalez, mentioned above, is recalled in Section 8, and the central argument of the present paper, establishing surjectivity, is given in the following Sections 9–11. An important ingredient is a result of Rais from [18], which has previously been applied in similar situations by Clozel and Delorme [7, Lemma 7], and by Flensted-Jensen [9, p. 30]. Finally, in Section 12 we treat S^2 as an example.

The result of [4] has been generalized to the Jacobi transform associated to a root system with a multiplicity function which is even, but not necessarily related to a symmetric space (see [5]). For the method of the present paper the geometry of the symmetric space is crucial, especially in Lemma 9.3, and we do not see how to generalize in this direction.

2. Basic notation

Let M be a Riemannian symmetric space of the compact type. We can write M as a homogeneous space $M = U/K$, where U is a connected compact semisimple Lie group which acts isometrically on M , and K a closed subgroup with the property that $U_0^\theta \subset K \subset U^\theta$ for an involution θ of U . Here U^θ denotes the subgroup of θ -fixed points, and U_0^θ its identity component. It should be emphasized that the pair (U, K) is in general not uniquely determined by M (see [13, Chapter VII]).

Let \mathfrak{u} denote the Lie algebra of U . We denote the involution of \mathfrak{u} corresponding to θ by the same symbol. Let $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ be the corresponding Cartan decomposition, then \mathfrak{k} is the Lie algebra of K , and \mathfrak{q} can be identified with the tangent space T_oM , where $o = eK \in M$ is the origin.

Recalling that the Killing form $B(X, Y)$ on \mathfrak{u} is negative definite, let $\langle \cdot, \cdot \rangle$ be the inner product on \mathfrak{u} defined by $\langle X, Y \rangle = -B(X, Y)$. Then \mathfrak{k} and \mathfrak{q} are orthogonal subspaces. We assume that the Riemannian metric g of M is normalized such that it agrees with $\langle \cdot, \cdot \rangle$ on $\mathfrak{q} = T_oM$.

We denote by \exp the exponential map $\mathfrak{u} \rightarrow U$ (which is surjective), and by Exp the map $\mathfrak{q} \rightarrow M$ given by $\text{Exp}(X) = \exp(X) \cdot o$. By identification of \mathfrak{q} with the tangent space T_oM , we thus identify Exp with the exponential map associated to the Riemannian connection.

The inner product on \mathfrak{u} determines an inner product on the dual space \mathfrak{u}^* in a canonical fashion. Furthermore, these inner products have complex bilinear extensions to the complexifications $\mathfrak{u}_\mathbb{C}$ and $\mathfrak{u}_\mathbb{C}^*$. All these bilinear forms are denoted by the same symbol $\langle \cdot, \cdot \rangle$.

Let $\mathfrak{a} \subset \mathfrak{q}$ be a maximal abelian subspace, \mathfrak{a}^* its dual space, and $\mathfrak{a}_\mathbb{C}^*$ the complexified dual space. Let Σ denote the set of non-zero (restricted) roots of \mathfrak{u} with respect to \mathfrak{a} , then $\Sigma \subset \mathfrak{a}_\mathbb{C}^*$ and all the elements of Σ are purely imaginary on \mathfrak{a} . The multiplicity of a root $\alpha \in \Sigma$ is denoted m_α . The corresponding Weyl group, generated by the reflections in the roots, is denoted W . Recall that it is naturally isomorphic with the factor group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ of the normalizer and the centralizer of \mathfrak{a} in K (see [13, Corollary VII.2.13]).

3. Fourier series

Let (π, V) be an irreducible unitary representation of U , and let

$$V^K = \{v \in V \mid \forall k \in K: \pi(k)v = v\},$$

then V^K is either 0 or 1-dimensional. In the latter case π is said to be a K -spherical representation.

Let $\mathfrak{h} \subset \mathfrak{u}$ be a Cartan subalgebra containing \mathfrak{a} , then $\mathfrak{h} = \mathfrak{h}_m \oplus \mathfrak{a}$, where $\mathfrak{h}_m = \mathfrak{h} \cap \mathfrak{k}$. Let Δ denote the set of roots of \mathfrak{u} with respect to \mathfrak{h} , then Σ is exactly the set of non-zero restrictions to \mathfrak{a} of elements of Δ . We fix a set $\Sigma^+ \subset \Sigma$ of positive restricted roots, and a compatible set $\Delta^+ \subset \Delta$ of positive roots. The set of dominant integral linear functionals on \mathfrak{h} is

$$\Lambda^+(\mathfrak{h}) = \left\{ \lambda \in \mathfrak{h}_\mathbb{C}^* \mid \forall \alpha \in \Delta^+: \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \right\},$$

where $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. Notice that since u is compact, all elements of Δ and $\Lambda^+(\mathfrak{h})$ take purely imaginary values on \mathfrak{h} .

Let $\Lambda^+(U) \subset \mathfrak{h}^*$ denote the set of highest weights of irreducible representations of U , then $\Lambda^+(U) \subset \Lambda^+(\mathfrak{h})$ with equality if and only if U is simply connected. Let $\Lambda_K^+(U)$ denote the subset of $\Lambda^+(U)$ which corresponds to K -spherical representations. We recall the following identification of $\Lambda_K^+(U)$, due to Helgason (see [12, p. 535]).

Theorem 3.1. *Let $\lambda \in \Lambda^+(U)$. Then $\lambda \in \Lambda_K^+(U)$ if and only if $\lambda|_{\mathfrak{h}_m} = 0$ and the restriction $\mu = \lambda|_{\mathfrak{a}}$ satisfies*

$$\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+, \tag{3.1}$$

for all $\alpha \in \Sigma^+$.

Furthermore, if $\mu \in \mathfrak{a}^*$ satisfies (3.1) for all $\alpha \in \Sigma^+$, then the element $\lambda \in \mathfrak{h}_\mathbb{C}^*$ defined by $\lambda|_{\mathfrak{h}_m} = 0$ and $\lambda|_{\mathfrak{a}} = \mu$ belongs to $\Lambda^+(\mathfrak{h})$. If this element λ belongs to $\Lambda^+(U)$, then it belongs to $\Lambda_K^+(U)$.

Let $\Lambda^+(U/K)$ denote the set of restrictions $\mu = \lambda|_{\mathfrak{a}}$ where $\lambda \in \Lambda_K^+(U)$, according to the preceding theorem this set is in bijective correspondence with $\Lambda_K^+(U)$. For each $\mu \in \Lambda^+(U/K)$ we fix an irreducible unitary representation (π_μ, V_μ) of U with highest weight λ , and we fix a unit vector $e_\mu \in V_\mu^K$. The spherical function on U/K associated with μ is the matrix coefficient

$$\psi_\mu(x) = (\pi_\mu(x)e_\mu, e_\mu), \quad x \in U,$$

viewed as a function on U/K . It is K -invariant on both sides, and it is independent of the choice of the unit vector e_μ . The spherical Fourier transform of a continuous K -invariant function f on $M = U/K$ is the function \tilde{f} on $\Lambda^+(U/K)$ defined by

$$\tilde{f}(\mu) = \int_M f(x) \overline{\psi_\mu(x)} dx,$$

where dx is the Riemannian measure on M , normalized with total measure 1. Notice that $\overline{\psi_\mu(gK)} = \psi_\mu(g^{-1}K)$ for $g \in U$, since π_μ is unitary. The spherical Fourier series for f is the series given by

$$\sum_{\mu \in \Lambda^+(U/K)} d(\mu) \tilde{f}(\mu) \psi_\mu \tag{3.2}$$

where $d(\mu) = \dim V_\mu$. The Fourier series converges to f in L^2 and, if f is smooth, absolutely and uniformly (see [12, p. 538]).

Furthermore, f is smooth if and only if the Fourier transform \tilde{f} is rapidly decreasing, that is, for each $k \in \mathbb{N}$ there exists a constant C_k such that

$$|\tilde{f}(\mu)| \leq C(1 + \|\mu\|)^{-k}$$

for all $\mu \in \Lambda^+(U/K)$ (see [19]).

4. Main theorem

For each $r > 0$ we denote by $B_r(0)$ the open ball in \mathfrak{q} centered at 0 and with radius r . The exponential image $\text{Exp } B_r(0)$ is the ball in M , centered at the origin and of radius r . Let $\bar{B}_r(0)$ and $\text{Exp } \bar{B}_r(0)$ denote the corresponding closed balls. We denote by $C_r^\infty(U/K)^K$ the space of K -invariant smooth functions on $M = U/K$ supported in $\text{Exp } \bar{B}_r(0)$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}_\mathbb{C}^*$.

Definition 4.1 (*Paley–Wiener space*). For $r > 0$ let $\text{PW}_r(\mathfrak{a})$ denote the space of holomorphic functions φ on $\mathfrak{a}_\mathbb{C}^*$ satisfying the following:

(a) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$|\varphi(\lambda)| \leq C_k (1 + \|\lambda\|)^{-k} e^{r \|\text{Re } \lambda\|}$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

(b) $\varphi(w(\lambda + \rho) - \rho) = \varphi(\lambda)$ for all $w \in W$, $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

We can now state the main theorem.

Theorem 4.2 (*The local Paley–Wiener theorem*). *There exists $R > 0$ such that the following holds for each $0 < r < R$:*

- (i) *Let $f \in C_r^\infty(U/K)^K$. Then the Fourier transform $\tilde{f}: \Lambda^+(U/K) \rightarrow \mathbb{C}$ of f extends to a function in $\text{PW}_r(\mathfrak{a})$.*
- (ii) *Let $\varphi \in \text{PW}_r(\mathfrak{a})$. There exists a unique function $f \in C_r^\infty(U/K)^K$ such that $\tilde{f}(\mu) = \varphi(\mu)$ for all $\mu \in \Lambda^+(U/K)$.*
- (iii) *The functions in the Paley–Wiener space $\text{PW}_r(\mathfrak{a})$ are uniquely determined by their values on $\Lambda^+(U/K)$.*

Thus the Fourier transform followed by the extension gives a bijection

$$C_r^\infty(U/K)^K \rightarrow \text{PW}_r(\mathfrak{a}).$$

Remark 4.3. It will be seen in the proof that each of the parts (i)–(iii) is valid for explicit values of the constant R . More precisely, part (i) is established under two conditions on R (see Section 6). The first is that it should be at most the injectivity radius of M , that is, the supremum of the values r for which the restriction of Exp to $B_r(0)$ is a diffeomorphism onto its image. The second condition for part (i) is that $R \leq \pi/(2\|\alpha\|)$ for all $\alpha \in \Sigma$. Furthermore, part (ii) will be proved with R equal to the injectivity radius of U (see Section 8). Finally, for part (iii) we need that $R \leq \pi/\|\mu\|$ for all the fundamental weights μ of U/K (see Section 7). The full theorem is thus valid with R equal to the minimum of all these numbers. There is however no reason to believe that this minimum is the optimal value of R . It would be desirable to have the theorem with R equal to the injectivity radius of M , but we have not been able to establish that.

5. The invariant differential operators

Let $\mathbb{D}(U/K)$ denote the algebra of U -invariant differential operators on U/K . It is commutative (see [12, Corollary II.5.4]). Recall that the Harish-Chandra homomorphism maps $\gamma: \mathbb{D}(U/K) \rightarrow S(\mathfrak{a}^*)^W$. It can be defined as follows. Let $\mathcal{U}(\mathfrak{u})$ denote the universal enveloping algebra of \mathfrak{u} . The algebra $\mathbb{D}(U/K)$ is naturally isomorphic with the quotient $\mathcal{U}(\mathfrak{u})^K / \mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\mathfrak{k}$, see [12, Theorem II.4.6]. It follows from [12, Theorem II.5.17] (by application to a symmetric pair of the non-compact type with Lie algebras $\mathfrak{g} = \mathfrak{k} + i\mathfrak{q}$ and \mathfrak{k}), that there exists an isomorphism of the quotient $\mathcal{U}(\mathfrak{u})^\mathfrak{k} / \mathcal{U}(\mathfrak{u})^\mathfrak{k} \cap \mathcal{U}(\mathfrak{u})\mathfrak{k}$ onto $S(\mathfrak{a}^*)^W$. The Harish-Chandra map results from composition of the two, using that $\mathcal{U}(\mathfrak{u})^K \subset \mathcal{U}(\mathfrak{u})^\mathfrak{k}$. We shall need the following fact.

Lemma 5.1. *The Harish-Chandra map γ is an isomorphism onto $S(\mathfrak{a}^*)^W$.*

Proof. Let K_0 denote the identity component of K . It follows from the description of γ above, that it suffices to prove equality between the quotients $\mathcal{U}(\mathfrak{u})^K / \mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\mathfrak{k}$ and $\mathcal{U}(\mathfrak{u})^{K_0} / \mathcal{U}(\mathfrak{u})^{K_0} \cap \mathcal{U}(\mathfrak{u})\mathfrak{k}$.

We shall employ [12, Corollary II.4.8], according to which the two quotients are in bijective linear correspondence with $S(\mathfrak{q})^K$ and $S(\mathfrak{q})^{K_0}$, respectively. It therefore suffices to prove identity between these two spaces.

Let $p \in S(\mathfrak{q})^{K_0}$ and let $k \in K$. By means of the Killing form we regard p as a polynomial function on \mathfrak{q} . The claimed identity amounts to $p \circ \text{Ad} k = p$. Notice that $p \circ \text{Ad} k \in S(\mathfrak{q})^{K_0}$, since k normalizes K_0 . According to [12, Corollary II.5.12], the elements of $S(\mathfrak{q})^{K_0}$ are uniquely determined by restriction to \mathfrak{a} . According to the lemma below, k is a product of elements from K_0 and $Z_K(\mathfrak{a})$, and hence it follows that $p \circ \text{Ad} k = p$ on \mathfrak{a} . \square

Lemma 5.2. *Each component of K contains an element from the centralizer $Z_K(\mathfrak{a})$.*

Proof. Let $k \in K$ be arbitrary. Then $\text{Ad} k$ maps \mathfrak{a} to a maximal abelian subspace in \mathfrak{q} , hence to $\text{Ad} k_0(\mathfrak{a})$ for some $k_0 \in K_0$. It follows that $k_0^{-1}k$ normalizes \mathfrak{a} . The description of the Weyl group cited in the end of Section 2 implies that $N_K(\mathfrak{a}) / Z_K(\mathfrak{a}) = N_{K_0}(\mathfrak{a}) / Z_{K_0}(\mathfrak{a})$, hence $k_0^{-1}k \in N_{K_0}(\mathfrak{a})Z_K(\mathfrak{a})$ and $k \in K_0Z_K(\mathfrak{a})$. \square

The spherical function ψ_μ satisfies the joint eigenequation

$$D\psi_\mu = \gamma(D, \mu + \rho)\psi_\mu, \quad D \in \mathbb{D}(U/K) \tag{5.1}$$

(see [4, Lemma 2.5]). It follows that

$$(Df)^\sim(\mu) = \overline{\gamma(D^*, \mu + \rho)} \tilde{f}(\mu)$$

where $D^* \in \mathbb{D}(U/K)$ is the adjoint of D .

In particular, the Laplace–Beltrami operator L on M belongs to $\mathbb{D}(U/K)$, and we have

$$\gamma(L, \lambda) = \langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle.$$

Since L is self-adjoint it follows that

$$(Lf)^\sim(\mu) = (\langle \mu + \rho, \mu + \rho \rangle - \langle \rho, \rho \rangle) \tilde{f}(\mu) \tag{5.2}$$

for all $f \in C^\infty(U/K)^K$.

6. The estimate of Opdam

In this section we prove part (i) of Theorem 4.2. The proof is based on the following result. Let $\tilde{\Omega}$ be the closure of

$$\Omega = \left\{ X \in \mathfrak{a} \mid \forall \alpha \in \Sigma: |\alpha(X)| < \frac{\pi}{2} \right\}.$$

Theorem 6.1 (Opdam). *For each $X \in \tilde{\Omega}$ the map*

$$\mu \mapsto \psi_\mu(\text{Exp } X), \quad \mu \in \Lambda^+(U/K),$$

has an analytic continuation to $\mathfrak{a}_\mathbb{C}^$, denoted $\lambda \mapsto \psi_\lambda(\text{Exp } X)$, with the following properties. There exists a constant $C > 0$ such that*

$$|\psi_\lambda(\text{Exp } X)| \leq C e^{\max_{w \in W} \text{Re } w\lambda(X)} \tag{6.1}$$

for all $\lambda \in \mathfrak{a}_\mathbb{C}^$, $X \in \tilde{\Omega}$. Furthermore, the map $X \mapsto \psi_\lambda(\text{Exp } X)$ is analytic, and*

$$\psi_{w(\lambda+\rho)-\rho}(\text{Exp } X) = \psi_\lambda(\text{Exp } X) \tag{6.2}$$

for all $w \in W$.

Proof. The existence of the analytic continuation follows from [17, Theorem 3.15], by identification of $\psi_\mu(\text{Exp } X)$ with $G(\mu + \rho, k; X)$, where G is the function appearing there. Recall that the root system R in [17] is 2Σ . For the shift by ρ and (6.2), see [4, Lemma 2.5]. It follows from [17, Theorem 6.1(2)] that the analytic extension satisfies (6.1). \square

Remark 6.2. An analytic extension of $\psi_\mu(\text{Exp } X)$ exists for X in the larger domain 2Ω . This was proved by Faraut (see [4, p. 418]) and by Krötz and Stanton (see [15]). However, the estimate (6.1) has not been obtained in this generality.

We can now derive Theorem 4.2(i). The following integration formula holds on $M = U/K$ (see [12, p. 190]), up to normalization of measures:

$$\int_M f(x) dx = \int_K \int_{A_*} f(ka \cdot o) \delta(a) da dk$$

where A_* is the torus $\exp \mathfrak{a}$ in U equipped with Haar measure, and where δ is defined by

$$\delta(\exp H) = \prod_{\alpha \in \Sigma^+} |\sin i\alpha(H)|^{m_\alpha}$$

for $H \in \mathfrak{a}$. It follows that

$$\tilde{f}(\mu) = \int_{A_*} f(a \cdot o) \psi_\mu(a^{-1} \cdot o) \delta(a) da.$$

Let $R > 0$ be sufficiently small, such that the restriction of \exp to $B_R(0)$ is injective, then if $r < R$ and f is K -invariant with support inside $\text{Exp } \tilde{B}_r(0)$, it follows that

$$\tilde{f}(\mu) = \int_{B_r(0) \cap \mathfrak{a}} f(\text{Exp } H) \psi_\mu(\text{Exp}(-H)) \delta(\text{exp } H) dH. \tag{6.3}$$

Assume in addition that $R \leq \pi/(2\|\alpha\|)$ for all $\alpha \in \Sigma$. Then $B_r(0) \cap \mathfrak{a} \subset \Omega$ for $r < R$, and it follows from Theorem 6.1 that $\mu \mapsto \tilde{f}(\mu)$ allows an analytic continuation to $\mathfrak{a}_\mathbb{C}^*$, given by the same formula (6.3), and denoted $\tilde{f}(\lambda)$, such that

$$|\tilde{f}(\lambda)| \leq C \max_{a \in A_*} \{ |f(a \cdot o) \delta(a)| \} e^{r \|\text{Re } \lambda\|} \tag{6.4}$$

where C is a constant depending on r , but not on f . The derivation of the polynomial decay of $\tilde{f}(\lambda)$ in (a) of Definition 4.1 is then easily obtained from the estimate (6.4), when applied to the function $L^m f$ with a sufficiently high power of L , by means of (5.2).

The Weyl group transformation property in part (b) of Definition 4.1 follows immediately from (6.2). Hence we can conclude that $\tilde{f}(\lambda)$ belongs to $\text{PW}_r(\mathfrak{a})$.

7. Uniqueness

In this section part (iii) of Theorem 4.2 is proved. The proof is based on the following simple generalization of Carlson’s theorem (see [3, p. 153]).

Lemma 7.1. *Let $f: \mathbb{C}^n \rightarrow \mathbb{C}$ be holomorphic. Assume:*

(i) *There exist a constant $c < \pi$, and for each $z \in \mathbb{C}^n$ a constant C such that*

$$|f(z + \zeta e_i)| \leq C e^{c|\zeta|}$$

for all $\zeta \in \mathbb{C}$, $i = 1, \dots, n$.

(ii) *$f(k) = 0$ for all $k \in (\mathbb{Z}^+)^n$.*

Then $f = 0$.

Proof. For $n = 1$ this is Carlson’s theorem. In general it follows by induction that $z \mapsto f(z, \kappa)$ is identically 0 on \mathbb{C}^{n-1} for each $\kappa \in \mathbb{Z}^+$. By a second application of Carlson’s theorem it then follows that $f(z, \zeta) = 0$ for all $(z, \zeta) \in \mathbb{C}^n$. \square

It follows that if X is sufficiently close to 0, then the analytic continuation $\lambda \mapsto \psi_\lambda(\text{Exp } X)$ in Theorem 6.1 is unique, when (6.1) is required. More precisely, let $\mu_1, \dots, \mu_n \in \mathfrak{a}_\mathbb{C}^*$ be such that $\Lambda^+(U/K) = \mathbb{Z}^+ \mu_1 + \dots + \mathbb{Z}^+ \mu_n$. If U is simply connected the elements $\mu_1, \dots, \mu_n \in \mathfrak{a}_\mathbb{C}^*$ are the fundamental weights determined by

$$\frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$$

where $\alpha_1, \dots, \alpha_n$ is a basis for the inmultipliable roots of Σ^+ . If U is not simply connected, the μ_i are suitable integral multiples of the fundamental weights, in order that they correspond to representations of U . If $\|X\| < \pi/\|\mu_i\|$ for all i the uniqueness of the analytic continuation now follows by application of Lemma 7.1 to the function $z \mapsto f(z_1\mu_1 + \dots + z_n\mu_n)$.

In the same fashion, if $R \leq \pi/\|\mu_i\|$ for all i , it follows from Lemma 7.1 that for $r < R$ the elements $\varphi \in \text{PW}_r(\mathfrak{a})$ are uniquely determined on $\Lambda^+(U/K)$, as claimed in Theorem 4.2(iii).

Notice that the minimal value of $\pi/\|\mu_i\|$ can be strictly smaller than the injectivity radius of U/K . See Remark 4.3.

8. The theorem of Gonzalez

In this section we treat the special case, where the symmetric space is the compact semisimple Lie group U itself, viewed as a symmetric space for the product group $U \times U$ with the action given by $(g, h) \cdot x = gxh^{-1}$. The stabilizer at e is the diagonal subgroup $\Delta = \{(x, x) \mid x \in U\}$ in $U \times U$, and the corresponding involution of $U \times U$ is $(x, y) \mapsto (y, x)$. The Δ -invariant functions on U are the class functions (also called central functions), that is, those for which $f(uxu^{-1}) = f(x)$ for all $u, x \in U$. In this case the local Paley–Wiener theorem was obtained by Gonzalez [11]. Let us recall his result.

As before, we denote by \mathfrak{h} a Cartan subalgebra of \mathfrak{u} , and by $\Lambda^+(\mathfrak{h}) \subset i\mathfrak{h}^*$ the set of dominant integral linear functionals. For $\mu \in \Lambda^+(U)$ we denote by χ_μ the character of π_μ , that is, $\chi_\mu(x)$ is the trace of $\pi_\mu(x)$ for $x \in U$. The function $d(\mu)^{-1}\chi_\mu$ is normalized so that its value at e is 1, and when U is viewed as a symmetric space, this class function is exactly the spherical function associated with π_μ . It is however more convenient to use the unnormalized function χ_μ in the definition of the Fourier transform, since it is a unit vector in L^2 (with the normalized Haar measure on U).

Following custom, we thus define the Fourier transform by

$$\hat{F}(\mu) = \langle F, \chi_\mu \rangle = \int_U F(u) \overline{\chi_\mu(u)} du, \quad \mu \in \Lambda^+(U),$$

for class functions $F \in L^2(U)^U$. The corresponding Fourier series is given by

$$\sum_{\mu \in \Lambda^+(U)} \hat{F}(\mu) \chi_\mu(x). \tag{8.1}$$

It converges to F in L^2 . If F is smooth it also converges absolutely and uniformly (see [12, p. 534]).

The theorem of Gonzalez [11] now reads as follows. Let $R > 0$ be the injectivity radius of U . If U is simply connected, this means that $R = 2\pi/\|\alpha\|$ where α is the longest root in Δ (see [13, p. 318]).

Theorem 8.1 (Gonzalez). *Let a class function $F \in C^\infty(U)^U$ be given, and let $0 < r < R$. Then F belongs to $C_r^\infty(U)^U$ if and only if the Fourier transform $\mu \mapsto \hat{F}(\mu)$ extends to a holomorphic function Φ on $\mathfrak{h}_\mathbb{C}^*$ with the following properties:*

(a) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$|\Phi(\lambda)| \leq C_k (1 + \|\lambda\|)^{-k} e^{r\|\operatorname{Re}\lambda\|}$$

for all $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$.

(b) $\Phi(w(\lambda + \rho) - \rho) = \det(w)\Phi(\lambda)$ for all $w \in W$, $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$.

Notice that as before the extension Φ is unique if r is sufficiently small. In that case, the Fourier transform, followed by holomorphic extension, is then a bijection onto the space of holomorphic functions satisfying (a) and (b).

9. Construction of K -invariant functions

The following result is important for the proof of Theorem 4.2.

Lemma 9.1. Let $F \in C^\infty(U)^U$ and define $f: U \rightarrow \mathbb{C}$ by

$$f(u) = \int_K F(ku) dk = \int_K F(uk) dk.$$

Then $f \in C^\infty(U/K)^K$ and

$$d(\mu)\tilde{f}(\mu) = \hat{F}(\lambda) \tag{9.1}$$

for all $\mu \in \Lambda^+(U/K)$, where $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is the extension of μ determined by

$$\lambda|_{\mathfrak{a}} = \mu, \quad \lambda|_{\mathfrak{h}_m} = 0.$$

Proof. The fact that $f \in C^\infty(U/K)^K$ is clear. From the uniform convergence of the Fourier series (8.1) it follows that

$$f(u) = \sum_{\lambda \in \Lambda^+(U)} \hat{F}(\lambda) \int_K \chi_\lambda(uk) dk.$$

By the lemma below we then obtain

$$f(u) = \sum_{\mu \in \Lambda^+(U/K)} \hat{F}(\lambda)\psi_\mu(u)$$

where λ is the extension of μ as above. The statement (9.1) now follows by comparison with (3.2). \square

Lemma 9.2. Let $\lambda \in \Lambda^+(U)$ and $\mu = \lambda|_{\mathfrak{a}}$. If $\lambda \in \Lambda_K^+(U)$ then

$$\int_K \chi_\lambda(uk) dk = \psi_\mu(u)$$

for all $u \in U$, and otherwise $\int_K \chi_\lambda(uk) dk = 0$.

Proof. (See also [12, p. 417].) The function $u \mapsto \int_K \chi_\lambda(uk) dk$ is a K -fixed vector in the right representation generated by χ_λ , which is equivalent with π_λ , hence it vanishes if $\lambda \notin \Lambda_K^+(U)$.

Assume $\lambda \in \Lambda_K^+(U)$, and choose an orthonormal basis v_1, \dots, v_d for the representation space V , such that v_1 is K -fixed. Then

$$\int_K \chi_\lambda(uk) dk = \int_K \sum_{i=1}^d \langle \pi_\lambda(u)\pi_\lambda(k)v_i, v_i \rangle dk.$$

Since the operator $\int_K \pi_\lambda(k) dk$ is the orthogonal projection onto V^K , it follows that

$$\int_K \chi_\lambda(uk) dk = \langle \pi_\lambda(u)v_1, v_1 \rangle = \psi_\mu(u)$$

as claimed. \square

Lemma 9.3. *Let $F \in C^\infty(U)^U$ and $f: U/K \rightarrow \mathbb{C}$ be as above. If $F \in C_r^\infty(U)^U$ for some $r > 0$ then $f \in C_r^\infty(U/K)^K$.*

Proof. Let $x \in M$ with $f(x) \neq 0$ and choose $X \in \mathfrak{q}$ such that the curve on M given by $t \mapsto \gamma(t) = \text{Exp}(tX)$ where $t \in [0, 1]$, is a minimal geodesic from o to x . The length of γ is $\|X\|$.

Let $x = u \cdot o$ where $u \in U$, then there exists $k \in K$ such that $F(ku) \neq 0$. Hence $ku = \exp Y$ where $Y \in \mathfrak{u}$ with $\|Y\| < r$. Let $Z = \text{Ad}(k^{-1})Y$, then $\|Z\| = \|Y\| < r$. The smooth curve $\xi(t) = \exp(tZ) \cdot o$, where $t \in [0, 1]$, also joins o to x . Hence it has length $\ell(\xi) \geq \|X\|$.

Let L_u denote left translation by u , then $\xi'(t) = dL_{\exp(tY)}(\xi'(0))$, and hence $\|\xi'(t)\| = \|\xi'(0)\|$ for all t . Let $Z_{\mathfrak{q}}$ denote the \mathfrak{q} component of Z in the orthogonal decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{q}$. Then $\xi'(0) = Z_{\mathfrak{q}}$, and we conclude that

$$\|X\| \leq \ell(\xi) = \int_0^1 \|\xi'(t)\| dt = \|Z_{\mathfrak{q}}\| \leq \|Z\| = \|Y\| < r.$$

Thus f has support in $\text{Exp } \bar{B}_r(0)$. \square

10. The result of Rais

The following result is due to M. Rais. Let $r > 0$ and recall that a holomorphic function φ on $\mathfrak{h}_\mathbb{C}^*$ is said to be of exponential type r if it satisfies (a) of Theorem 8.1. Let \tilde{W} denote the Weyl group of the root system Δ on \mathfrak{h} . Let $l = |\tilde{W}|$, and let P_1, \dots, P_l be a basis for $S(\mathfrak{h}^*)$ over $I(\mathfrak{h}^*) = S(\mathfrak{h}^*)^{\tilde{W}}$ (see [12, p. 360]).

Theorem 10.1. *For each holomorphic function ψ of exponential type r there exist unique \tilde{W} -invariant holomorphic functions ϕ_1, \dots, ϕ_l of exponential type r such that $\psi = P_1\phi_1 + \dots + P_l\phi_l$.*

Proof. See [8, Appendix B]. \square

In the following statement, we regard \mathfrak{a}^* as a subset of \mathfrak{h}^* , by trivial extension on \mathfrak{h}_m . Likewise \mathfrak{h}_m^* is regarded as a subspace by trivial extension on \mathfrak{a} . Then $\mathfrak{h}^* = \mathfrak{a}^* \oplus \mathfrak{h}_m^*$ holds as an orthogonal sum decomposition.

Corollary 10.2. *There exist a collection of polynomials $p_1, \dots, p_l \in S(\mathfrak{a}^*)^W$ with the following property. For each W -invariant holomorphic function φ on $\mathfrak{a}_{\mathbb{C}}^*$ of exponential type r , there exist \tilde{W} -invariant holomorphic functions ϕ_1, \dots, ϕ_l on $\mathfrak{h}_{\mathbb{C}}^*$ of exponential type r , such that*

$$\varphi = p_1(\phi_1|_{\mathfrak{a}_{\mathbb{C}}^*}) + \dots + p_l(\phi_l|_{\mathfrak{a}_{\mathbb{C}}^*}). \tag{10.1}$$

Proof. (See also [9, p. 30].) Notice that when ϕ_j is \tilde{W} -invariant, then $\phi_j|_{\mathfrak{a}^*}$ is W -invariant, since the normalizer in \tilde{W} of \mathfrak{a} maps surjectively onto W (see [12, p. 366]).

Fix a holomorphic function φ_m on $\mathfrak{h}_m^*_{\mathbb{C}}$ of exponential type r , with the value $\varphi_m(0) = 1$. Put $\psi(\lambda) = \varphi(\lambda_1)\varphi_m(\lambda_2)$, where λ_1 and λ_2 are the components of λ . Then ψ is of exponential type r , and we can apply Theorem 10.1. The restriction of ψ to \mathfrak{a}^* is exactly φ . Taking restrictions to \mathfrak{a}^* we thus obtain (10.1) with $p_j = P_j|_{\mathfrak{a}^*}$. The desired expression is obtained by averaging over W . \square

11. Proof of the main theorem

It remains to be seen that every function $\varphi \in \text{PW}_r(\mathfrak{a})$ is the extension of \tilde{f} for some $f \in C^\infty(U/K)^K$.

Thus let $\varphi \in \text{PW}_r(\mathfrak{a})$ be given. Let p_1, \dots, p_l and ϕ_1, \dots, ϕ_l be as in Corollary 10.2, applied to the W -invariant function $\lambda \mapsto \varphi(\lambda - \rho)$ on \mathfrak{a}^* . By Lemma 5.1 there exist $D_j \in \mathbb{D}(U/K)$ such that $\gamma(D_j^*, \lambda) = p_j(\lambda)$ for $\lambda \in i\mathfrak{a}^*$.

It follows from the Weyl dimension formula [12, p. 502] that $\mu \mapsto d(\mu)$ extends to a polynomial on \mathfrak{h}^* which satisfies the transformation property (b) of Theorem 8.1. Hence the function on $\mathfrak{h}_{\mathbb{C}}^*$ defined by $\Phi_j(\lambda) = d(\lambda)\phi_j(\lambda + \rho)$ satisfies both (a) and (b) in that theorem, and thus we can find $F_1, \dots, F_n \in C_r^\infty(U)^U$ such that

$$\hat{F}_j(\mu) = \Phi_j(\mu)$$

for all μ .

Let $f_j(uK) = \int_K F_j(uk) dk$ and define $f = \sum D_j f_j$. Then by Lemma 9.3 we have $f \in C_r^\infty(U/K)^K$, and it follows from (9.1) that

$$\begin{aligned} \tilde{f}(\mu) &= \sum_j \gamma(D_j, \mu + \rho) \tilde{f}_j(\mu) \\ &= \sum_j \gamma(D_j, \mu + \rho) d(\mu)^{-1} \hat{F}_j(\mu) \\ &= \sum_j p_j(\mu + \rho) \phi_j(\mu + \rho) = \varphi(\mu). \quad \square \end{aligned}$$

12. The sphere S^2

Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, then M can be realized as a homogeneous space for $U = \text{SU}(2)$ with the following action. Identify \mathbb{R}^3 with the space of Hermitian 2×2 matrices H with trace 0,

$$H = \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix},$$

then $u.H = uHu^{-1}$ for $u \in U$. The stabilizer of the point $o = (0, 0, 1) \in S^2$ is the set of diagonal elements in U , and the diagonal element

$$k_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

acts by rotation around the z -axis of angle θ .

A K -invariant function on M is determined by its values along the elements $(x, y, z) = (0, \sin t, \cos t)$, and it thus becomes identified as an even function of $t \in [-\pi, \pi]$. With the notation of above, the function is identified through the map $t \mapsto f(a_t \cdot o)$ where

$$a_t = \begin{pmatrix} \cos(t/2) & i \sin(t/2) \\ i \sin(t/2) & \cos(t/2) \end{pmatrix}.$$

The irreducible representations of U are parametrized by half integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, where π_l has dimension $2l + 1$, and the spherical representations are those for which l is an integer. The corresponding spherical functions are given by $\psi_l(a_t) = P_l(\cos t)$, where P_l is the l th Legendre polynomial.

The Fourier series of a K -invariant function on S^2 , identified as an even function on $[-\pi, \pi]$, is then the Fourier–Legendre series

$$\sum_{l=0}^{\infty} (2l + 1) \tilde{f}(l) P_l(\cos t)$$

where

$$\tilde{f}(l) = \frac{1}{2} \int_0^\pi f(t) P_l(\cos t) \sin t \, dt.$$

Our local Paley–Wiener theorem asserts the following for $r < \pi$:

An even function $f \in C^\infty(-\pi, \pi)$ is supported in $[-r, r]$ if and only if the Legendre transform $l \mapsto \tilde{f}(l)$ of f extends to an entire function g on \mathbb{C} of exponential type

$$|g(\lambda)| \leq C_k (1 + |\lambda|)^{-k} e^{r|\text{Im} \lambda|}$$

such that $g(\lambda - \frac{1}{2})$ is an even function of λ . The extension g with these properties is unique.

Moreover, every such function g on \mathbb{C} is obtained in this fashion from a unique function $f \in C_r^\infty(-\pi, \pi)$.

Essentially this is the result stated by Koornwinder (and attributed to Beurling) in [14, p. 158].

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