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Simple compact quantum groups I ☆

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Abstract

The notion of simple compact quantum group is introduced. As non-trivial (noncommutative and noncocommutative) examples, the following families of compact quantum groups are shown to be simple: (a) The universal quantum groups $B_u(Q)$ for $Q \in GL(n, \mathbb{C})$ satisfying $Q\bar{Q} = \pm I_n$, $n \geqslant 2$; (b) The quantum automorphism groups $A_{aut}(B, \tau)$ of finite-dimensional C^*-algebras B endowed with the canonical trace τ when $\dim(B) \geqslant 4$, including the quantum permutation groups $A_{aut}(X_n)$ on n points ($n \geqslant 4$); (c) The standard deformations K_q of simple compact Lie groups K and their twists K_q^u, as well as Rieffel's deformation K_J.

Keywords: Simple quantum groups; Woronowicz C^*-algebras; Deformation quantization; Noncommutative geometry; Hopf algebras

1. Introduction

The theory of quantum groups saw spectacular breakthroughs in the 1980s when on the one hand Drinfeld [24] and Jimbo [27] discovered the quantized universal enveloping algebras of semisimple Lie algebras based on the work of the Faddeev school on the quantum inverse scattering method, and on the other hand Woronowicz [58–60] independently discovered quantum deformations of compact Lie groups and formulated the axioms for compact quantum groups. Further work of Rosso [38,40], Soibelman and Vaksman, Levendorskii [30,43,44] showed that "compact real forms" K_q of the Drinfeld–Jimbo quantum groups and their twists K_q^u are exam-

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ples of compact quantum groups in the sense of Woronowicz. Most notable of these is the work of Soibelman [43] based on his earlier joint work with Vaksman [44], in which a general Kirillov type orbit theory of representations of the quantum function algebras of deformed simple compact Lie groups was developed using the orbits of dressing transformations (i.e. symplectic leaves) in Poisson Lie group theory (see also the monograph [29] for more detailed treatment).

Starting in his PhD thesis [48], the author of the present article took a different direction from the above by viewing quantum groups as intrinsic objects and found in a series of papers (including [47] in collaboration with Van Daele) several classes of compact quantum groups that cannot be obtained as deformations of Lie groups. The most important of these are the universal compact quantum groups of Kac type $A_u(n)$ and their self-conjugate counterpart $A_o(n)$ [49], the more general universal compact quantum groups $A_u(Q)$ and their self-conjugate counterpart $B_u(Q)$ [47,50], where $Q \in GL(n, \mathbb{C})$, and the quantum automorphism groups $A_{aut}(B, tr)$ of finite-dimensional C^*-algebras B endowed with a tracial functional tr, including the quantum permutation groups $A_{aut}(X_n)$ on the space X_n of n points [53]. Further studies of these quantum groups reveal remarkable properties: (1) According to deep work of Banica [2–4], the representation rings (also called the fusion rings) of the quantum groups $B_u(Q)$ (when $Q\bar{Q}$ is a scalar) are all isomorphic to that of $SU(2)$ (see [2, Théorème 1]), and the representation rings of $A_{aut}(B, \tau)$ (when $\dim(B) \geqslant 4$, τ being the canonical trace on B) are all isomorphic to that of $SO(3)$ (see [4, Theorem 4.1]), and the representation ring of $A_u(Q)$ is almost a free product of two copies of $\mathbb{Z}$ (see [3, Théorème 1]); (2) The compact quantum groups $A_u(Q)$ admit ergodic actions on both finite and infinite injective von Neumann factors [54]; (3) The special $A_u(Q)$'s for positive Q and $B_u(Q)$'s for Q satisfying the property $Q\bar{Q} = \pm I_n$ are classified up to isomorphism using *respectively* the eigenvalues of Q (see [56, Theorem 1.1]) and polar decomposition of Q and eigenvalues of $|Q|$ (see [56, Theorem 2.4]), and the general $A_u(Q)$'s and $B_u(Q)$'s for arbitrary Q have explicit decompositions as free products of the former special ones (see [56, Theorems 3.1, 3.3 and Corollaries 3.2, 3.4]); (4) Certain quantum symmetry groups in the theory of subfactors were found by Banica [6,7] to fit in the theory of compact quantum groups; (5) The quantum permutation groups $A_{aut}(X_n)$ admit interesting quantum subgroups that appear in connection with other areas of mathematics, such as the quantum automorphism groups of finite graphs and the free wreath products discovered by Bichon [15,16]. See also [17] and [8–14] and the references therein for other interesting results related to the quantum permutation groups.

The purpose of this article is to initiate a study of simple compact quantum groups. It focuses on the introduction of a notion of simple compact quantum groups and first examples. It is shown that the compact quantum groups mentioned in the last two paragraphs are simple in generic cases. The paper is organized as follows.

In Section 2, we recall the notion of a normal quantum subgroup N of a compact quantum group G introduced in [48,49], on which the main notion of a simple compact quantum group in this paper depends. We prove several equivalent conditions for N to be normal, including one that stipulates that the quantum coset spaces G/N and $N\backslash G$ are identical. Further applications of these are contained in [57].

In Section 3 the notion of simple compact quantum groups is introduced. In the classical setting, the notion of a simple compact Lie group can be defined in two ways: one using Lie algebra and the other using the group itself. Though the universal enveloping algebras of simple Lie groups can be deformed into the quantized universal enveloping algebras [24,27], we have no analog of Lie algebras for general quantum groups. Hence we formulate the notion of a *simple compact quantum group* using group theoretical language so that our notion reduces precisely to the notion of a *simple compact Lie group* when the quantum group is a compact Lie group:

Definition 1.1. A compact matrix quantum group is called *simple* if it is connected and has no non-trivial connected normal quantum subgroups and no non-trivial representations of dimension one.

Here a compact quantum group G is called *connected* if the coefficients of every non-trivial irreducible representation of G generate an infinite-dimensional C^*-algebra. In the classical situation, the fact that a simple compact Lie group has no non-trivial representations of dimension one is a consequence of the deep Weyl dimension formula. It is not known if the postulate that a simple compact matrix quantum group has no non-trivial representations of dimension one follows from the other postulates in the definition, for we do not have a dimension formula for irreducible representations of a general simple compact quantum group except the specific examples studied in this paper.

After preparatory work in Sections 2 and 3, the main examples of this paper are studied in Sections 4 and 5. Recall [4] that *the canonical trace* τ on a finite-dimensional C^*-algebra B is the restriction of the unique tracial state on the algebra $L(B)$ of operators on B. In Section 4, we prove that $B_u(Q)$ and $A_{aut}(B,\tau)$ are simple:

Theorem 1.2. *(See Theorem 4.1.) Let $Q \in GL(n,\mathbb{C})$ be such that $Q\bar{Q} = \pm I_n$ and $n \geqslant 2$. Then $B_u(Q)$ is a simple compact quantum group.*

Theorem 1.3. *(See Theorem 4.7.) Let B be a finite-dimensional C^*-algebra with* $\dim(B) \geqslant 4$ *and τ its canonical trace. Then $A_{aut}(B,\tau)$ is a simple compact quantum group.*

The proofs of these two results rely heavily on the fundamental work of Banica [2,4] on the structure of fusion rings (i.e. representative rings) of these quantum groups, as well as the technical results on the correspondence between Hopf $*$-ideals and Woronowicz C^*-ideals and the reconstruction of a normal quantum subgroup from the identity in the quotient quantum group, which are developed in Section 4 and are of interest in their own right.

It is also shown in Section 4 that the closely related quantum group $A_u(Q)$ is not simple for any n and any $Q \in GL(n,\mathbb{C})$ (see Proposition 4.5).

The last Section 5 is devoted to the standard deformations K_q of simple compact Lie groups, their twists K_q^u [30,31,43], and Rieffel's quantum groups K_J [37], where $q \in \mathbb{R}\setminus\{0\}$, $u \in \bigwedge^2(i\mathfrak{t})$ and J is an appropriate skew-symmetric transformation on the direct sum $\mathfrak{t}\oplus\mathfrak{t}$ of Cartan subalgebra $\mathfrak{t}$ of the Lie algebra of K:

Theorem 1.4. *(See Theorems 5.1 and 5.6.) Let K be a connected and simply connected simple compact Lie group. Then both K_q and its twists K_q^u are simple compact quantum groups.*

Theorem 1.5. *(See Theorem 5.4.) Let K be a simple compact Lie group with a toral subgroup T of rank at least two. Then K_J is a simple compact quantum group.*

The proofs of Theorems 1.4 and 1.5 make use of the work of Lusztig and Rosso [32,39] on representations of quantized universal enveloping algebras, the work of Soibelman and Levendorskii [30,31,43] on quantum function algebras of K_q and K_q^u, and the work of Rieffel [37] and the author [51] on strict deformations of Lie groups and quantum groups, as well as the technical results in Section 4 mentioned earlier.

Classification of simple compact quantum groups and their irreducible representations up to isomorphism are two of the main goals in the study of compact quantum groups. Namely, one would like to develop a theory of simple compact quantum groups that parallels the Killing–Cartan theory and the Cartan–Weyl theory for simple compact Lie groups. To accomplish the first goal, one must first construct all simple compact quantum groups. Though we have given several infinite classes of examples of these in this article, it should be pointed out that the construction of simple compact quantum groups is far from being complete. In fact it is fair to say that we are only at the beginning stage for this task at the moment. One indication of this is that *all the simple compact quantum groups known so far have commutative representation rings, and these rings are order isomorphic to the representation rings of compact Lie groups* (we call such quantum groups *almost classical*). The universal compact matrix quantum groups $A_u(Q)$ have a "very" noncommutative representation ring, being close to the free product of two copies of the ring of integers, according to the fundamental work of Banica [3], where $Q \in GL(n, \mathbb{C})$ are positive, $n \geqslant 2$. However, $A_u(Q)$ are not simple quantum groups (see Section 4). Because of their universal property, $A_u(Q)$ should play an important role in the construction and classification of simple compact quantum groups with non-commutative representation rings. A natural and profitable approach seems to be to study quantum automorphism groups of appropriate quantum spaces and their quantum subgroups, such as those in [53–55] and the papers of Banica and Bichon and their collaborators [6]–[17]. In retrospect, both simple Lie groups and finite simple groups are automorphism groups, a similar approach for the theory of simple quantum groups should also play a fundamental role.

Convention and notation. We assume that all Woronowicz C^*-algebras (also called Woronowicz Hopf C^*-algebras) considered in this paper to be full unless otherwise explicitly stated, since morphisms between quantum groups are meaningful only for full Woronowicz C^*-algebras (cf. [49,52]). For a compact quantum group G, A_G, or $C(G)$, denote the underlying Woronowicz C^*-algebra and $\mathcal{A}_G$ denotes the associated canonical dense Hopf $*$-algebra of quantum representative functions on G. Sometimes we also call A_G a compact quantum group, referring to G. See [49,59] for more on other unexplained definitions and notations used in this paper.

2. The notion of normal quantum subgroups

Before making the notion of simple quantum groups precise, we recall the notion of normal quantum subgroups (of compact quantum groups) introduced in [48,49] and study their properties further. Let (N, π) be a quantum subgroup of a compact quantum group G with surjections $\pi : A_G \to A_N$ and $\hat{\pi} : \mathcal{A}_G \to \mathcal{A}_N$. The quantum group (N, π) should be more precisely called a closed quantum subgroup, but we will omit the word *closed* in this paper, since we do not consider non-closed quantum subgroups. Define

$$A_{G/N} = \{ a \in A_G \mid (id \otimes \pi)\Delta(a) = a \otimes 1_N \},$$
$$A_{N\backslash G} = \{ a \in A_G \mid (\pi \otimes id)\Delta(a) = 1_N \otimes a \},$$

where Δ is the coproduct on A_G, 1_N is the unit of the algebra A_N. We omit the subscript N in 1_N when no confusion arises. Similarly, we define

$$\mathcal{A}_{G/N} = \mathcal{A}_G \cap A_{G/N}, \quad \text{and} \quad \mathcal{A}_{N\backslash G} = \mathcal{A}_G \cap A_{N\backslash G}.$$

Note that G/N, $N\backslash G$ shall be denoted more precisely by $G/(N,\pi)$, $(N,\pi)\backslash G$ respectively, if there is a possible confusion. Let h_N be the Haar measure on N. Let

$$E_{G/N} = (id \otimes h_N\pi)\Delta, \qquad E_{N\backslash G} = (h_N\pi \otimes id)\Delta.$$

Then $E_{G/N}$ and $E_{N\backslash G}$ are projections of norm one (completely positive and completely bounded conditional expectations) from A_G onto $A_{N\backslash G}$ and $A_{G/N}$ respectively (cf. [34] as well as [54, Proposition 2.3 and Section 6]), and

$$\mathcal{A}_{G/N} = E_{G/N}(\mathcal{A}_G), \quad \text{and} \quad \mathcal{A}_{N\backslash G} = E_{N\backslash G}(\mathcal{A}_G).$$

From this, we see that the $*$-subalgebras $\mathcal{A}_{N\backslash G}$ and $\mathcal{A}_{G/N}$ are dense in $A_{G/N}$ and $A_{N\backslash G}$ respectively.

Assume N is a closed subgroup of an ordinary compact group G. Let π be the restriction morphism from $A_G := C(G)$ to $A_N := C(N)$. Let $C(G/N)$ and $C(N\backslash G)$ be continuous functions on G/N and $N\backslash G$ respectively. Then one can verify that

$$C(G/N) = A_{G/N} = E_{G/N}(A_G),$$
$$C(N\backslash G) = A_{N\backslash G} = E_{N\backslash G}(A_G).$$

Therefore we will use the symbols $C(G/N)$ and $A_{G/N}$ (respectively $C(N\backslash G)$ and $A_{N\backslash G}$; $C(G)$ and A_G) interchangeably for all quantum groups.

Proposition 2.1. *Let N be a quantum subgroup of a compact quantum group G. Then the following conditions are equivalent*:

(1) *$A_{N\backslash G}$ is a Woronowicz C^*-subalgebra of A_G.*
(2) *$A_{G/N}$ is a Woronowicz C^*-subalgebra of A_G.*
(3) *$A_{G/N} = A_{N\backslash G}$.*
(4) *For every irreducible representation u^λ of G, either $h_N\pi(u^\lambda) = I_{d_\lambda}$ or $h_N\pi(u^\lambda) = 0$, where h_N is the Haar measure on N, d_λ is the dimension of u^λ and I_{d_λ} is the $d_\lambda \times d_\lambda$ identity matrix.*

Proof. We only need to show that (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3). The proof of the implications (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) is similar.

(3) $\Rightarrow$ (4). In general one has

$$\Delta(\mathcal{A}_{N\backslash G}) \subseteq \mathcal{A}_{N\backslash G} \otimes \mathcal{A}_G, \qquad \Delta(\mathcal{A}_{G/N}) \subseteq \mathcal{A}_G \otimes \mathcal{A}_{G/N}.$$

Letting $\mathcal{B} = \mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N}$ one has

$$\Delta(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{B}.$$

For $\lambda \in \hat{G}$, let n_λ be the multiplicity of the trivial representation of N in the representation $\pi(u^\lambda)$. We claim that either $n_\lambda = d_\lambda$ or $n_\lambda = 0$.

Assume on the contrary that there is a $\lambda \in \hat{G}$ such that $1 < n_\lambda < d_\lambda$. Note that in general

$$E_{N\backslash G}\big(u_{ij}^{\lambda}\big) = (h_N\pi \otimes id)\Delta\big(u_{ij}^{\lambda}\big) = \sum_k h_N\pi\big(u_{ik}^{\lambda}\big)u_{kj}^{\lambda},$$

$$E_{G/N}\big(u_{ij}^{\lambda}\big) = (id \otimes h_N\pi)\Delta\big(u_{ij}^{\lambda}\big) = \sum_k h_N\pi\big(u_{kj}^{\lambda}\big)u_{ik}^{\lambda}.$$

Using unitary equivalence if necessary we choose u_{ij}^{λ} in such a way that the n_λ trivial representations of N appear on the upper left diagonal corner of $\pi(u^{\lambda})$. Then

$$E_{N\backslash G}\big(u_{ij}^{\lambda}\big) = \begin{cases} u_{ij}^{\lambda} & \text{if } 1 \leqslant i \leqslant n_\lambda,\ 1 \leqslant j \leqslant d_\lambda, \\ 0 & \text{if } n_\lambda < i \leqslant d_\lambda,\ 1 \leqslant j \leqslant d_\lambda, \end{cases}$$

$$E_{G/N}\big(u_{ij}^{\lambda}\big) = \begin{cases} u_{ij}^{\lambda} & \text{if } 1 \leqslant i \leqslant d_\lambda,\ 1 \leqslant j \leqslant n_\lambda, \\ 0 & \text{if } 1 \leqslant i \leqslant d_\lambda,\ n_\lambda < j \leqslant d_\lambda. \end{cases}$$

Since $\mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N} = \mathcal{B}$ and both $E_{N\backslash G}$ and $E_{G/N}$ are projections from $\mathcal{A}_G$ onto $\mathcal{B}$, we have

$$E_{N\backslash G} = E_{G/N}.$$

Then for $n_\lambda < j \leqslant d_\lambda$,

$$0 \neq u_{ij}^{\lambda} = E_{N\backslash G}\big(u_{ij}^{\lambda}\big) = E_{G/N}\big(u_{ij}^{\lambda}\big) = 0.$$

This is a contradiction.

(4) $\Rightarrow$ (3). Let $S(N)$ (or $S(N,\pi)$) be the subset of $\hat{G}$ consisting of those λ's such that $h_N\pi(u^{\lambda})$ is I_{d_λ}. Then a straightforward calculation using the fact that $E_{N\backslash G}$ and $E_{G/N}$ are projections of $\mathcal{A}_G$ onto $\mathcal{A}_{N\backslash G}$ and $\mathcal{A}_{G/N}$ respectively, one gets

$$\mathcal{A}_{N\backslash G} = \mathcal{A}_{G/N} = \bigoplus \big\{\mathbb{C}u_{ij}^{\lambda}\ \big|\ \lambda \in S(N),\ i, j = 1, \ldots, d_\lambda\big\}.$$

(4) $\Rightarrow$ (1). Let $S(N)$ be defined as in the proof of (4)$\Rightarrow$(3). It is clear that $A_{N\backslash G}$ is a Woronowicz C^*-subalgebra of A_G and that

$$\big\{u_{ij}^{\lambda}\ \big|\ \lambda \in S(N),\ i, j = 1, \ldots, d_\lambda\big\}$$

is a Peter–Weyl basis of the dense $*$-subalgebra $\mathcal{A}_{N\backslash G}$ of $A_{N\backslash G}$.

(1) $\Rightarrow$ (4). Let $G_1 = N\backslash G$. Then by Woronowicz's Peter–Weyl theorem for compact quantum groups, every irreducible representation u^{λ} of G is either an irreducible representation of G_1 or none of the coefficients u_{ij}^{λ} is in $\mathcal{A}_{G_1}$. That is

$$E_{N\backslash G}\big(u_{ij}^{\lambda}\big) = \begin{cases} u_{ij}^{\lambda} & \text{if } \lambda \in \hat{G}_1, \\ 0 & \text{if } \lambda \in \hat{G} \setminus \hat{G}_1. \end{cases}$$

By the definition of $E_{N\backslash G}$ and linear independence of the u_{ij}^{λ}'s, this implies that

$$h_N\pi\left(u^\lambda_{ik}\right)=\delta_{ik},\quad \lambda\in\hat{G}_1,\ i,k=1,\ldots,d_\lambda,$$
$$h_N\pi\left(u^\lambda_{ik}\right)=0,\quad \lambda\in\hat{G}\setminus\hat{G}_1.$$

This completes the proof of the proposition. □

Definition 2.2. A quantum subgroup N of a compact quantum group G is said to be *normal* if it satisfies the equivalent conditions of Proposition 2.1.

Remarks. (a) Condition (4) of Proposition 2.1 plays an important role in this paper. It is a reformulation of the following condition for a normal quantum subgroup N that appears near the end of [49, Section 2]: for every irreducible representation v of G, the multiplicity of the trivial representation of N in the representation $\pi(v)$ is either zero or the dimension of v. From the proof of the proposition we see that the counit of $\mathcal{A}_{G/N}$ is equal to the restriction morphism $\pi|_{\mathcal{A}_{G/N}}$.

(b) Note also that on p. 679 of [49] the following statement is found: "In general, a right quotient quantum group is different from the corresponding left quotient quantum group." Though in the purely algebraic setting of Hopf algebras, one needs to distinguish between left and right normal quantum subgroups, as indicated in Parshall and Wang [33, 1.5] (see also [1,42,46]), however, in view of Proposition 2.1 above, this cannot happen for normal quantum subgroups of compact quantum groups. Moreover, using Lemmas 4.2–4.4 below, it can be shown that the notion of normality defined in [33] when applied to compact quantum groups is equivalent to our notion of normality. As the main results of this paper do not depend on this equivalence, its proof and other applications are in [57].

(c) The notion of a normal quantum subgroup depends on the morphism π, which gives the "position" of the quantum group N in G. If (N,π_1) is another quantum subgroup of G with surjection $\pi_1: A_G\to A_N$, (N,π_1) may not be normal even if (N,π) is. This phenomenon already occurs in the group situation. For example a finite group can contain two isomorphic subgroups with one normal but the other not.

Examples. We show in (1) and (2) below that the identity group and the full quantum group G are both normal quantum subgroups of G under natural embeddings. These will be called the *trivial normal quantum subgroups*. See Sections 4, 5 and [57] for examples of non-trivial normal quantum subgroups.

(1) Let $N=\{e\}$ be the one element identity group. Let $\pi=\epsilon=$ counit of A_G be the morphism from A_G to A_N. Then by the counital property, one has

$$A_{G/N}=\left\{a\in A_G\ \middle|\ (id\otimes\epsilon)\Delta(a)=a\otimes 1\right\}=A_G.$$

That is $(\{e\},\epsilon)$ is normal and $G/(\{e\},\epsilon)=G$.

(2) Now let $N=G$ and let $\pi: A_G\to A_N$ be any isomorphism of Woronowicz C^*-algebras [49]. Let h be the Haar measure on G and $a\in A_{G/N}$. Since π is an isomorphism and $(id\otimes\pi)\Delta(a)=a\otimes 1$, one has $\Delta(a)=(id\otimes\pi)^{-1}(a\otimes 1)=a\otimes 1$. From the invariance of h one has

$$h(a)1=(1\otimes h)\Delta(a)=ah(1)=a.$$

Hence

$$A_{G/N}=\left\{a\in A_G\ \middle|\ (id\otimes\pi)\Delta(a)=a\otimes 1\right\}=\mathbb{C}1.$$

That is (G, π) is normal and $G/(G, \pi) \cong \{e\}$.

(3) We note that besides the embeddings in (2) it is possible to construct examples of compact quantum groups G with non-normal proper embeddings of G into G. In fact this can happen for compact groups already.

The following is a justification of the above notion of normal quantum subgroups.

Proposition 2.3. *Let $A = C(G)$ be a compact group. Let N be a closed subgroup of G. Let π be the restriction map from A to $A_N = C(N)$. Then (N, π) is normal in the sense above if and only if N is a normal subgroup of G in the usual sense.*

Proof. Under the Gelfand–Naimark correspondence which associates to every commutative C^*-algebra its spectrum, quotients of G by (ordinary) closed normal subgroups N correspond to Woronowicz C^*-subalgebras of $C(G)$, i.e.,

$$G/N \text{ corresponds to } C(G/N),$$

see [49, 2.6 and 2.12]. Since $A_{G/N} = C(G/N)$ for any closed subgroup N, the proposition follows from Proposition 2.1 above. $\square$

The following result gives a complete description of quantum normal subgroups of the compact quantum group dual of a discrete group Γ, whose proof is straightforward using e.g. [59] and Proposition 2.1.

Proposition 2.4. *Let $A_G = C^*(\Gamma)$. Let N be a quantum subgroup of G with surjection $\pi : A_G \to A_N$. Then N is normal, $L := \pi(\Gamma)$ is a discrete group and $A_N = C^*(L)$. Moreover, $A_{G/N} = C^*(K)$, where $K = \ker(\pi : \Gamma \to L)$.*

To distinguish two different quantum subgroups, we include the following result, which should be known to experts in the theory of C^*-algebras.

Proposition 2.5. *Let $\pi_k : A \to A_k$ be surjections of unital C^*-algebras with kernels I_k $(k = 1, 2)$. Let P_k be the pure state space of A_k. Then the following conditions are equivalent*:

(1) $\{\phi_1 \circ \pi_1 \mid \phi_1 \in P_1\} = \{\phi_2 \circ \pi_2 \mid \phi_2 \in P_2\}$ *as subsets of pure states of A.*
(2) $I_1 = I_2$.
(3) *There is an isomorphism $\alpha : A_1 \to A_2$ such that $\pi_2 = \alpha \circ \pi_1$.*

Proof. (1) $\Rightarrow$ (2). If $I_1 \neq I_2$, say, there is an $x \in I_1 \setminus I_2$. Then there is a pure state ϕ of A/I_2 such that $\phi(\pi_2(x)) \neq 0$, where we identify A_2 with A/I_2. But $\phi\pi_2$ is a pure state of $A/I_1 \cong A_1$ according to assumption (1). Hence we must have $\phi\pi_2(x) = 0$. This is a contradiction.

(2) $\Rightarrow$ (3). Let $I = I_1 = I_2$. Let π be the quotient map $A \to A/I$. Let $\tilde{\pi}_k$ be the homomorphism from A/I to A_k such that $\pi_k = \tilde{\pi}_k \pi$ $(k = 1, 2)$. Then $\tilde{\pi}_k$ are isomorphisms. Put $\alpha = \tilde{\pi}_2 \circ \tilde{\pi}_1^{-1}$. Then $\pi_2 = \alpha \circ \pi_1$.

(3) $\Rightarrow$ (1). This follows from $P_1 = P_2 \circ \alpha$. $\square$

The following proposition is an easy consequence of Proposition 2.1.

Proposition 2.6. *Let* (N_1, π_1) *be a normal quantum subgroup of* G. *Let*

$$\alpha : A_{N_1} \longrightarrow A_{N_2}$$

be an isomorphism of quantum groups. Then $(N_2, \alpha\pi_1)$ *is normal.*

In view of the above discussions, it is reasonable to have the following definition (cf. also remarks after Proposition 2.3).

Definition 2.7. Two quantum subgroups (π_1, H_1) and (π_2, H_2) of a quantum group G are said to have *the same imbedding* in G if π_1, π_2 satisfy the equivalent conditions of Proposition 2.5. When this happens, we denote $(H_1, \pi_1) = (H_2, \pi_2)$.

Geometrically speaking, two quantum subgroups (H_1, π_1) and (H_2, π_2) of a quantum group G are said to have *the same imbedding* in G if their "images" in G are the same.

3. Simple compact quantum groups

To avoid such a difficulty as the classification of finite groups up to isomorphism in developing the theory of simple compact quantum groups, we assume connectivity as a part of the postulates of the latter. We use representation theory to define the notion of connectivity:

Definition 3.1. We call a compact quantum group G_A *connected* if for each non-trivial irreducible representation $u^\lambda \in \hat{G}_A$, the C^*-algebra $C^*(u^\lambda_{ij})$ generated by the coefficients of u^λ is of infinite dimension.

In virtue of [26, (28.21)], we have

Proposition 3.2. *Let* G_A *be an ordinary compact group* (*i.e.* A_G *is commutative*). *Then* G_A *is connected as a topological space if and only if it is connected in the sense above.*

Definition 3.3. We call a compact quantum group G_A *simple* if it satisfies the following conditions (1)–(4):

(1) The Woronowicz C^*-algebra A_G is finitely generated;
(2) G_A is connected;
(3) G_A has no non-trivial connected normal quantum subgroups;
(4) G_A has no non-trivial representations of dimension one.

A (simple) quantum group is called *absolutely simple* if it has no non-trivial normal quantum subgroups. Similarly a finite quantum group is called *simple* if it has no non-trivial normal quantum subgroups.

Just as the notion of simple compact Lie groups excludes the torus groups, the above notion of simple quantum groups excludes abelian compact quantum groups in the sense of Woronowicz [59], i.e. quantum groups coming from group C^*-algebras $C^*(\Gamma)$ of discrete groups Γ (note that $C^*(\Gamma)$ is the algebra of continuous functions on the torus $\mathbb{T}^n$ when Γ is the discrete group $\mathbb{Z}^n$). This is important because it is impossible to classify discrete groups up to isomorphism. However, we do not know if condition (4) in Definition 3.3 (i.e., there is no non-trivial group-like elements) is superfluous, as is the case for simple compact Lie groups because of the Weyl dimension formula.

As a justification of this definition, we have the following proposition that shows that our notion of simple compact quantum groups recovers *exactly* the ordinary notion of simple compact Lie groups.

Proposition 3.4. *If G_A is a simple compact quantum group with A commutative, then the set $G := \hat{A}$ of Gelfand characters is a simple compact Lie group in the ordinary sense. Conversely, every simple compact Lie group in the ordinary sense is of this form.*

The proof Proposition 3.4 follows immediately from [49, Theorem 2.8] and Proposition 3.2 above. We remark that although it is easy as above to prove the characterization of the ordinary simple compact Lie group in terms of our notion of simple compact quantum groups when A_G is commutative, it has been highly non-trivial to prove the analogous characterization of ordinary differential manifolds in terms of the axioms of non-commutative manifolds, which is finally achieved in the recent work of Connes [21] (see references therein for earlier, presumably unsuccessful, attempts to such a characterization).

Note that a simple compact Lie group is not a direct product of proper connected subgroups. Also, a simple Lie group is not a semi-direct product. Similarly, the following general results are true for quantum groups (for proofs see [57]):

Proposition 3.5. *If G_A is a simple compact quantum group, then A_G is neither a tensor product, nor a crossed product by a non-trivial discrete group.*

To put in perspective the examples of simple compact quantum groups to be studied later, we introduce some properties for compact quantum groups. First we recall that the *representation ring* (also called the *fusion ring*) $R(G)$ of a compact quantum group G is an ordered algebra over the integers $\mathbb{Z}$ with positive cone (or semiring, which is also a basis) $R(G)_+ := \{\chi_u\}$ consisting of characters χ_u of irreducible representations $u \in \hat{G}$ of G, and structure constants $c_{uv}^w \in \mathbb{N} \cup \{0\}$ given by the rules

$$\chi_u \chi_v = \sum_{w \in \hat{G}} c_{uv}^w \chi_w,$$

where the product $\chi_u \chi_v$ is taken in the algebra $\mathcal{A}_G$.

Definition 3.6. Let G be a compact quantum group. We say that G has *property F* if each Woronowicz C^*-subalgebra of A_G is of the form $A_{G/N}$ for some normal quantum subgroup N of G. We say that G has *property FD* if each quantum subgroup of G is normal.

We say that G is *almost classical* if its representation ring $R(G)$ is order isomorphic to the representation ring of a compact group.

By Proposition 2.3, a compact group trivially has property F. We will give in Sections 4 and 5 non-trivial simple compact quantum groups that are almost classical and have property F. Among compact quantum groups, simple compact quantum groups that are almost classical or have property F are closest to ordinary simple compact Lie groups in regard to noncommutative geometry.

By Proposition 2.4, as the dual of discrete group Γ, a compact quantum group of the form $C^*(\Gamma)$ has property *FD*. When Γ is finite, $C^*(\Gamma)$ is equal to the dual of the function algebra $C(\Gamma)$. This explains the term *FD*.

A compact quantum group G is absolutely simple with property F if and only if every nontrivial representation v of G is faithful, i.e., $C^*(v_{ij}) = A_G$, cf. [54].

By a theorem of Handelman [25], the representation ring of a compact connected Lie group is a complete isomorphism invariant. But this fails for compact quantum groups, since the representation rings of a simple compact Lie group K and its standard deformation K_q are order isomorphic.

In [5], Banica uses the positive cone $R_+(G)$ of the representation ring $R(G)$ of a compact quantum group G to define what he calls an R_+ deformation. This is closely related to almost classical quantum groups.

It is clear that a quantum quotient group G/N of an almost classical quantum group G is also almost classical. But a quantum subgroup of an almost classical quantum group need not be almost classical. For example, the quantum permutation groups are almost classical (cf. [4,53] and remarks preceding Theorem 4.7), but according to Bichon [16], their quantum subgroups $A_2(\mathbb{Z}/m\mathbb{Z})$ are not almost classical if $m \geqslant 3$ (see Corollary 2.7 and the paragraph following Corollary 4.3 of [16]). However, for a compact quantum group with property F, we have the following general result.

Theorem 3.7. *Let G be a compact quantum group with property F. Then its quantum subgroups and quotient groups G/N (by normal quantum subgroups N) also have property F.*

As we will only use the definitions of quantum groups with property F (respectively property *FD*) but not the assertion in the theorem above, the details for the proof of the theorem is included in a separate paper [57].

The main goals/problems in the theory of simple compact quantum groups are: (1) to construct and classify (up to isomorphism if possible) simple compact quantum groups; (2) to construct and classify irreducible representations of simple compact quantum groups; (3) to analyze the structure of compact quantum groups in terms of simple ones; and (4) to develop applications of simple compact quantum groups in other areas of mathematics and physics. For these purposes, new techniques for compact quantum groups must be developed.

The above is a very difficult program at present. Even problem (1) of the program above is daunting. To obtain *clues* on the general problem (1), it is desirable to find and solve easier parts of it. For this purpose, we propose the following apparently easier problems.

Problem 3.8. (1) Construct and classify all simple compact quantum groups with property F (up to isomorphism if possible).

(2) Construct and classify all simple compact quantum groups that are almost classical (up to isomorphism if possible).

Problem 3.9. Construct simple compact quantum groups with property *FD*.

Simple quantum groups in Problems 3.8, 3.9 are most closest to groups known in mathematics. They should be easiest classes to classify. Therefore they should play a fundamental role in the main problems in the theory of simple compact quantum groups.

4. Simplicity of $B_u(Q)$ and $A_{aut}(B, \tau)$

To prove the main results in this and the next sections, we develop here two technical results, which are of interest in their own right: one on the correspondence between Hopf $*$-ideals and Woronowicz C^*-ideals; the other on the reconstruction of a normal quantum group from the identity in the quotient quantum group.

We first recall the construction of compact quantum group $B_u(Q)$ associated to a non-singular $n \times n$ complex scalar matrix Q (cf. [2,47,49,50]). The (noncommutative) C^*-algebra of functions on the quantum group $B_u(Q)$ is generated by noncommutative coordinate functions u_{ij} $(i, j = 1, \ldots, n)$ that are subject to the following relations:

$$u^*u = I_n = uu^*, \qquad u^t Q u Q^{-1} = I_n = Q u Q^{-1} u^t,$$

where $u = (u_{ij})_{i,j=1}^n$. When $Q\bar{Q}$ is a scalar multiple cI_n of the identity matrix I_n, the quantum group $B_u(Q)$ and the group $SU(2)$ have the same fusion rules for their irreducible representations, as shown by Banica [2], which implies that $B_u(Q)$ is an almost classical quantum group. Under the condition $Q\bar{Q} = \pm I_n$, the isomorphism classification of $B_u(Q)$ is determined by the author [56] using polar decomposition of Q and eigenvalues of $|Q|$ (see [56, Theorem 2.4]). For arbitrary Q, $B_u(Q)$ is a free product of its building blocks, involving both $B_u(Q_l)$'s and $A_u(P_k)$'s with $Q_l\bar{Q}_l$ being scalar matrices and P_k positive matrices (see [56, Theorem 3.3]). The precise definition of $A_u(Q)$ is recalled later in the paragraphs before Proposition 4.5. For positive matrix Q, $A_u(Q)$ is classified up to isomorphism in terms of the eigenvalues of Q (see [56, Theorem 1.1]); and for an arbitrary non-singular matrix Q, the general $A_u(Q)$ is a free product of $A_u(P_k)$'s with positive matrices P_k (see [56, Theorem 3.1]). In Bichon et al. [18], the same techniques as in [56] were used to classify the unitary fiber functors of the quantum groups $A_u(Q)$ and $B_u(Q)$ and their ergodic actions with full multiplicity. Note that for $n = 1$, $B_u(Q) = C(\mathbb{T})$ is the trivial 1×1 unitary group. We will concentrate on the non-trivial case $n \geqslant 2$. Note that the isomorphism class of $B_u(Q)$ depends on the normalized Q only if $Q\bar{Q}$ is a scalar matrix [56].

Theorem 4.1. *Let $Q \in GL(n, \mathbb{C})$ be such that $Q\bar{Q} = \pm I_n$. Then $B_u(Q)$ is an almost classical simple compact quantum group with property F. In fact it has only one normal subgroup of order* 2.

Proof. As noted above, the quantum group $B_u(Q)$ is almost classical because its representation ring is order isomorphic to the representation ring of the compact Lie group $SU(2)$ [2]. More precisely, according to [2] irreducible representations of the quantum group $B_u(Q)$ can be parametrized by r_k $(k = 0, 1, 2, \ldots)$ with r_0 trivial and $r_1 = (u_{ij})_{ij=1}^n$, so that the fusion rules for their tensor product representations (i.e., decomposition into irreducible representations) read

$$r_k \otimes r_l = r_{|k-l|} \oplus r_{|k-l|+2} \oplus \cdots \oplus r_{k+l-2} \oplus r_{k+l}, \quad k, l \geqslant 0.$$

We show that the quantum group $B_u(Q)$ is connected. If $k = 2m$ is even $(m > 0)$, then let $r_l = r_k$ in the above tensor product decomposition and do the same for the irreducible constituents repeatedly, one sees that the algebra $C^*(r_{2m})$ generated by the coefficients of the representation r_{2m} contains the coefficients of r_{2s} for all s. Hence r_{2m} generates an infinite-dimensional algebra:

$$C^*(r_{2m}) = C^*\big(\{r_{2s} \mid s \geqslant 0\}\big).$$

If $k = 2m+1$ is odd ($m \geqslant 0$), then let $r_l = r_k$ in the above tensor product decomposition, one sees that the representation r_2 appears therein. Apply the decomposition to $r_{2m+1} \otimes r_2$, one sees that $r_1 = (u_{ij})$ appears therein. Hence the algebra generated by the coefficients of r_{2m+1} is the same as the algebra generated by those of $r_1 = (u_{ij})$. We conclude from this analysis that *there is only one non-trivial Woronowicz C^*-subalgebra in $B_u(Q)$, the one $C^*(r_{2m})$ generated by coefficients of r_{2m}*, which is obviously infinite-dimensional as noted above, where m is any nonzero positive number. In particular, the quantum group $B_u(Q)$ is connected.

For rest of the proof, we show that the quantum group $B_u(Q)$ has only one normal quantum subgroup, although it has many quantum subgroups.

Note that the coordinate functions v_{ij} of the matrix group $N = \{I_n, -I_n\}$ satisfy the defining relations of $B_u(Q)$, hence there is a surjection π from the C^*-algebra $B_u(Q)$ to the C^*-algebra A_N of functions on N such that

$$\pi(u_{ij}) = v_{ij}, \quad i, j = 1, 2, \ldots, n.$$

It is clear that π is a morphism of quantum groups, hence (N, π) is a quantum subgroup of the quantum group $B_u(Q)$.

We show that (N, π) is actually a normal quantum subgroup. To see this, it suffices by Proposition 2.1 to show that

$$\pi(r_{2m}) = d_{2m} \cdot v_0, \qquad \pi(r_{2m+1}) = d_{2m+1} \cdot v_1,$$

where d_{2m} and d_{2m+1} are dimensions of the representations r_{2m} and r_{2m+1} respectively, v_0 and v_1 are the trivial and the non-trivial irreducible representations of N respectively ($v_1(\pm I_n) = \pm 1$). By the definition of π and v_1 the assertion is clearly true for $m = 0$. In general, suppose the assertion is true for m. Then $\pi(r_{2m+1}) \otimes \pi(r_1)$ is a multiple of v_0 since $v_1^2 = v_0$. From the decomposition of $r_{2m+1} \otimes r_1$, we get

$$\pi(r_{2m+1}) \otimes \pi(r_1) = \pi(r_{2m}) \oplus \pi(r_{2m+2}).$$

Hence $\pi(r_{2(m+1)}) = \pi(r_{2m+2})$ is a scalar multiple of v_0. Similarly, from

$$\pi(r_{2m+2}) \otimes \pi(r_1) = \pi(r_{2m+1}) \oplus \pi(r_{2m+3}),$$

we see that $\pi(r_{2(m+1)+1}) = \pi(r_{2m+3})$ is a multiple of v_1. Since v_0 and v_1 are one-dimensional representations, the multiples we obtained above must be d_{2m+2} and d_{2m+3} respectively. That is (N, π) is normal and

$$A_{G/N} = C^*(r_2) = C^*\big(\{r_{2s} \mid s \geqslant 0\}\big),$$

where for simplicity of notation, the symbol G in G/N refers to the quantum group $G_{B_u(Q)}$. The above also shows that this quantum group has property F.

We have to show that $B_u(Q)$ has no other normal quantum subgroups, which will imply that it has no connected normal quantum subgroups and is therefore a simple quantum group.

Let (N_1, π_1) be a non-trivial normal quantum subgroup of $B_u(Q)$. We show that $(N_1, \pi_1) = (N, \pi)$ in the sense of Definition 2.7, which will finish the proof of the theorem. Since $N_1 \neq 1$, by Definition 2.7 and Proposition 2.1 there exists an irreducible representation v of the quantum group $B_u(Q)$ such that $\pi_1(v)$ is not a scalar and therefore $E_{G/N_1}(v) = 0$. Hence by the proof of Proposition 2.1 and Woronowicz's Peter–Weyl theorem [59], $A_{G/N_1} = E_{G/N_1}(A_G) \neq A_G$.

Similarly, we claim that $A_{G/N_1} \neq \mathbb{C}1$, where 1 is the unit of A_G. To prove this, we need three lemmas. It is instructive to compare the second lemma (Lemma 4.3) with the ideal theory for C^*-algebras.

Lemma 4.2. *Let B_1 and B_2 be Woronowicz C^*-algebras with canonical dense Hopf $*$-algebras of "representative functions" $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Assume B_2 is full and $\psi : B_1 \to B_2$ is a morphism of Woronowicz C^*-algebras such that the induced morphism $\hat{\psi} : \mathcal{B}_1 \to \mathcal{B}_2$ is an isomorphism. Then B_1 is full and ψ is also an isomorphism.*

Remark. The above is false if the roles of B_1 and B_2 are exchanged, as seen by taking $B_1 = C^*(F_2)$ and $B_2 = C_r^*(F_2)$.

Proof of Lemma 4.2. Since $\mathcal{B}_1$ is dense in B_1, it suffices to show that $\|\hat{\psi}(a)\| = \|a\|$ for $a \in \mathcal{B}_1$.

Since ψ is a morphism of C^*-algebras, we have $\|\psi(a)\| \leqslant \|a\|$ and therefore the first inequality

$$\big\|\hat{\psi}(a)\big\| = \big\|\psi(a)\big\| \leqslant \|a\|.$$

Since B_2 is full, the norm on $\mathcal{B}_2$ is the universal C^*-norm (see [52]):

$$\big\|\hat{\psi}(a)\big\| = \sup\big\{\big\|\pi\big(\hat{\psi}(a)\big)\big\|\colon\ \pi \text{ is a } *\text{-representation of } \mathcal{B}_2\big\}.$$

Taking $\pi = \hat{\psi}^{-1}$ in the above, we obtain the second inequality

$$\big\|\hat{\psi}(a)\big\| \geqslant \big\|\hat{\psi}^{-1}\big(\hat{\psi}(a)\big)\big\| = \|a\|.$$

Combining the first the second inequalities finishes the proof of Lemma 4.2. □

Lemma 4.3 *(Hopf $*$-ideals vs. Woronowicz C^*-ideals).*

(1) *Let G be a compact quantum group. Let $\mathcal{I}$ be a Hopf $*$-ideal of $\mathcal{A}_G$. Then the norm closure $\overline{\mathcal{I}}$ in the C^*-algebra A_G is a Woronowicz C^*-ideal and $A_G/\overline{\mathcal{I}}$ is a full Woronowicz C^*-algebra. The Hopf $*$-algebra $\mathcal{A}_G/\mathcal{I}$ admits a universal C^*-norm and its completion under this norm is a Woronowicz C^*-algebra isomorphic to $A_G/\overline{\mathcal{I}}$.*

(2) *The map $f(\mathcal{I}) = \overline{\mathcal{I}}$ is a bijection from the set of Hopf $*$-ideals $\{\mathcal{I}\}$ of $\mathcal{A}_G$ onto the set of Woronowicz C^*-ideals $\{I\}$ of A_G such that A_G/I is full. The inverse g of f is given by $g(I) = I \cap \mathcal{A}_G$.*

Remarks. (a) Note that (2) and the last part of (1) in the lemma above are false if the Woronowicz C^*-algebra A_G or A_G/I is not full, as is shown by the following example. Let $A_G = C^*(F_2)$

be the group C^*-algebra of the free group F_2 on two generators. Let I be the kernel of the canonical map $\pi : C^*(F_2) \to C_r^*(F_2)$ where $C_r^*(F_2)$ is the reduced group C^*-algebra of F_2. Then $I \cap \mathcal{A}_G = 0$ but $I \neq \bar{0}$.

(b) This lemma strengthens the philosophy in [52] that the "pathology" associated with the ideals between 0 and the kernel of the morphism from the full Woronowicz C^*-algebra to reduced one such as $\pi : C^*(F_2) \to C_r^*(F_2)$ is not (quantum) group theoretical, but purely functional analytical, and $C^*(F_2)$ and $C_r^*(F_2)$ should be viewed as the same quantum group because the same dense Hopf $*$-subalgebra that completely determines the quantum group can be recovered from either the full or the reduced algebra. Similarly, for a general compact quantum group G, the totality of (*quantum*) *group theoretic* information is encoded in the *purely algebraic* object $\mathcal{A}_G$, any other (Hopf) algebra should be viewed as defining the same quantum group as $\mathcal{A}_G$ so long as $\mathcal{A}_G$ can be recovered from it. The advantage of working with the category of full C^*-algebras or the purely algebraic objects $\mathcal{A}_G$ is that morphisms can be easily defined for them, whereas it is not even possible to define a morphism from the one element group to the quantum group associated with the reduced algebra $C_r^*(F_2)$ if is viewed as a different quantum group than the one associated with the full algebra $C^*(F_2)$.

Proof of Lemma 4.3. Let $\mathcal{I}$ be as in (1). Let $\pi_1 : A_G \to A_G/\overline{\mathcal{I}}$ be the quotient map. Since $\mathcal{I}$ is a Hopf $*$-ideal, we have in particular (see Sweedler [45])

$$\Delta(\mathcal{I}) \subset \mathcal{A}_G \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{A}_G \subset \ker(\pi_1 \otimes \pi_1).$$

Therefore $\Delta(\overline{\mathcal{I}}) \subset \ker(\pi_1 \otimes \pi_1)$. That is, $\overline{\mathcal{I}}$ is a Woronowicz C^*-ideal and $A_G/\overline{\mathcal{I}}$ is a Woronowicz C^*-algebra (see [49, 2.9–2.11]). Denote $B_1 = A_G/\overline{\mathcal{I}}$ and let $\hat{\pi}_1$ be the induced morphism of the canonical dense Hopf-$*$-subalgebras $\hat{\pi}_1 : \mathcal{A}_G \to \mathcal{B}_1$.

We claim that $\ker \hat{\pi}_1 = \mathcal{I}$ and $\hat{\psi}_0 : \mathcal{A}_G/\mathcal{I} \to \mathcal{B}_1$, $\hat{\psi}_0 : [a] \mapsto \pi_1(a)$ is an isomorphism, where $[a] \in \mathcal{A}_G/\mathcal{I}$, $a \in \mathcal{A}_G$.

By [49,59], $\mathcal{A}_G$ is generated as an algebra by the coefficients u_{ij}^{λ} of irreducible unitary corepresentations u^{λ} of Hopf $*$-algebra $\mathcal{A}_G$. The images $[u_{ij}^{\lambda}]$ of u_{ij}^{λ} in the quotient Hopf $*$-algebra $\mathcal{A}_G/\mathcal{I}$ give rise to unitary corepresentation of $\mathcal{A}_G/\mathcal{I}$, and generate it as an *algebra* (not just as a $*$-algebra). Therefore $\mathcal{A}_G/\mathcal{I}$ is a compact quantum group algebra (CQG algebra) in the sense of Dijkhuizen and Koornwinder [23] (see also [28,52,60])—a more appropriate name for compact quantum group (CQG) algebra might be Woronowicz $*$-algebra (or compact Hopf $*$-algebra), since the quantum group C^*-algebra of a compact quantum group G is the C^*-algebra $C^*(G)$ dual to $C(G)$ according to [35].

Let $\mathcal{B}_2 = \mathcal{A}_G/\mathcal{I}$ and let B_2 be the closure of $\mathcal{B}_2$ in the universal C^*-norm. Then B_2 is a Woronowicz C^*-algebra. As the norm on A_G is universal, the composition

$$\mathcal{A}_G \longrightarrow \mathcal{A}_G/\mathcal{I} \longrightarrow B_2$$

is bounded and extends to a morphism of Woronowicz C^*-algebras $\rho : A_G \to B_2$. Since $\mathcal{I} \subset \ker(\rho)$, we have $\overline{\mathcal{I}} \subset \ker(\rho)$ and ρ factors through $B_1 = A_G/\overline{\mathcal{I}}$ via a C^*-algebra morphism ψ:

$$A_G \xrightarrow{\pi_1} B_1 \xrightarrow{\psi} B_2, \quad \rho = \psi\pi_1.$$

It is clear that $\rho(a) = [a]$ for $a \in \mathcal{A}_G$ and from this it can be checked that $\hat{\psi}$ and $\hat{\psi}_0$ are inverse morphisms, where $\hat{\psi} : \mathcal{B}_1 \to \mathcal{B}_2$ is the restriction of ψ to the dense Hopf $*$-subalgebra $\mathcal{B}_1$ and B_2. Hence $\hat{\psi}_0$ is an isomorphism as claimed.

From $\hat{\rho} = \hat{\psi}\hat{\pi}_1$ (since $\rho = \psi\pi_1$), it is easy to see that ψ is a morphism of Woronowicz C^*-algebras (see [49, 2.3]). Since $\hat{\psi} = \hat{\psi}_0^{-1}$ is an isomorphism and B_2 is full, by Lemma 4.2, B_1 is full and ψ is itself an isomorphism from B_1 to B_2. (We note in passing that since $A_G/\ker(\rho) \cong B_2$, we have $\bar{\mathcal{I}} = \ker(\rho)$.) This proves part (1) of the lemma.

To prove part (2) of the lemma, let $\mathcal{I}$ be as in (2) and $B_1 = A_G/\bar{\mathcal{I}}$. Then by (1) above and [49], B_1 is a Woronowicz C^*-algebra. Let $\mathcal{B}_1$ be the canonical dense Hopf $*$-algebra of B_1 and let $\hat{\pi}_1 : \mathcal{A}_G \to \mathcal{B}_1$ be the morphism associated with the quotient morphism π_1. Then clearly

$$\mathcal{I} \subset \bar{\mathcal{I}} \cap \mathcal{A}_G = gf(\mathcal{I}).$$

Conversely if $x \in \bar{\mathcal{I}} \cap \mathcal{A}_G$, then $x \in \ker(\hat{\pi}_1) = \mathcal{I}$. Hence $gf(\mathcal{I}) = \mathcal{I}$.

Next let I be as in (2). We show that $fg(I) = I$. Let $B_2 = A_G/I$—this is not the same B_2 as in (1) above. Let π_2 be the quotient morphism from A_G onto B_2 (compare with ρ above). Define $\mathcal{I} = g(I) = I \cap \mathcal{A}_G$. We need to show that $\bar{\mathcal{I}} = I$. The idea of proof is the same as that of the last part in (1).

Using the morphism $\hat{\pi}_2 : \mathcal{A}_G \to \mathcal{B}_2$ of dense Hopf $*$-algebras associated with π_2, we see that $\mathcal{I} = \ker(\hat{\pi}_2)$. Hence $\mathcal{I}$ is a Hopf $*$-ideal in $\mathcal{A}_G$ and $\mathcal{A}_G/\mathcal{I}$ is isomorphic to $\mathcal{B}_2$ under the natural map induced from $\hat{\pi}_2$, and by (1) above, $B_1 := A_G/\bar{\mathcal{I}}$ is a Woronowicz C^*-algebra. Since $\bar{\mathcal{I}} \subset I$, the morphism π_2 factors through B_1 via a morphism ψ of Woronowicz C^*-algebras:

$$A_G \xrightarrow{\pi_1} B_1 \xrightarrow{\psi} B_2, \quad \pi_2 = \psi\pi_1.$$

Besides being isomorphic to $\mathcal{B}_2$, $\mathcal{A}_G/\mathcal{I}$ is also isomorphic to $\mathcal{B}_1$ (under the morphism $\hat{\psi}_0$) according to the proof of (1) earlier. Hence the restriction $\hat{\psi}$ of ψ to the dense Hopf $*$-algebras is an isomorphism from $\mathcal{B}_1$ to $\mathcal{B}_2$. Since B_2 is full, by Lemma 4.2, ψ itself is an isomorphism, which means that $I = \bar{\mathcal{I}}$ (and $B_1 = B_2$). This completes the proof of Lemma 4.3. □

Lemma 4.4 *(Reconstruct N from G/N). Let (N, π) be a normal quantum subgroup of a compact quantum group G. Let $\hat{\pi}$ be the associated morphism from $\mathcal{A}_G$ to $\mathcal{A}_N$. Then,*

$$\ker(\hat{\pi}) = \mathcal{A}_{G/N}^+ \mathcal{A}_G = \mathcal{A}_G \mathcal{A}_{G/N}^+ = \mathcal{A}_G \mathcal{A}_{G/N}^+ \mathcal{A}_G,$$

where $\mathcal{H}^+$ denotes the augmentation ideal (i.e. kernel of the counit) for any Hopf algebra $\mathcal{H}$.

Remarks. (a) In the notation of Schneider [42], the result above can be restated as follows: the map Φ is the left inverse of Ψ, where $\Psi(\ker(\hat{\pi})) := \mathcal{A}_{G/N}$ and $\Phi(\mathcal{A}_{G/N}) := \mathcal{A}_G \mathcal{A}_{G/N}^+$. In the language of Andruskiewitsch and Devoto [1], the result above implies that the sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1,$$

or the sequence

$$0 \longrightarrow \mathcal{A}_{G/N} \longrightarrow \mathcal{A}_G \longrightarrow \mathcal{A}_N \longrightarrow 0$$

is exact. It is instructive to compare this with the purely algebraic situation in Parshall and Wang [33], where for a given normal quantum subgroup in the sense there, the existence of an exact sequence is not known and the uniqueness does not hold in general (cf. [33, 1.6 and 6.3]). Note that the notion of exact sequence of quantum groups in Schneider [42] is equivalent to that in Andruskiewitsch and Devoto [1] under certain faithful (co)flat conditions. Though a Hopf algebra is not faithfully flat over its Hopf subalgebras if it is not commutative or cocommutative (see Schauenburg [41]), we have the following

Conjecture 1. *Let G be a compact quantum group. Then the Hopf algebra A_G (respectively $\mathcal{A}_G$) is faithfully flat over its Hopf subalgebras.*

Similarly, A_G (respectively $\mathcal{A}_G$) is faithfully coflat over A_G/I (respectively $\mathcal{A}_N/\mathcal{I}$) for every Woronowicz C^*-ideal I (respectively Hopf $*$-ideal $\mathcal{I}$).

(b) It can be shown using Lemma 4.4 and Schneider [42] that the notion of normal quantum groups in this paper (or in [48,49]) and the one in Parshall and Wang [33] are equivalent for compact quantum groups. For more details, see [57].

Proof of Lemma 4.4. The proof is an adaption of the ones in Sweedler [45, 16.0.2] and of Childs [20, (4.21)] for finite-dimensional Hopf algebras to infinite-dimensional ones considered here. We sketch the main steps here for convenience of the reader.

It suffices to prove $\ker(\hat{\pi}) = \mathcal{A}_{G/N}^{+}\mathcal{A}_G$. The other equality $\ker(\hat{\pi}) = \mathcal{A}_G\mathcal{A}_{G/N}^{+}$ is proved similarly. From these it follows that $\ker(\hat{\pi}) = \mathcal{A}_G\mathcal{A}_{G/N}^{+}\mathcal{A}_G$.

Consider the right $\mathcal{A}_N$-comodule structures on $\mathcal{A}_N$ and $\mathcal{A}_G$ given respectively by

$$\Delta_N : \mathcal{A}_N \to \mathcal{A}_N \otimes \mathcal{A}_N, \quad \text{and} \quad (id \otimes \hat{\pi})\Delta_G : \mathcal{A}_G \to \mathcal{A}_G \otimes \mathcal{A}_N,$$

where Δ_N and Δ_G are respectively the coproducts of the Hopf algebras $\mathcal{A}_N$ and $\mathcal{A}_G$. Since $\mathcal{A}_N$ is cosemisimple by the fundamental work of Woronowicz [59] (see remarks in [49, 2.2]), it follows from of [22, Theorem 3.1.5] that every $\mathcal{A}_N$-comodule is projective. Furthermore, one checks that the surjection $\hat{\pi} : \mathcal{A}_G \to \mathcal{A}_N$ is a morphism of $\mathcal{A}_N$-comodules. Hence $\hat{\pi}$ has a comodule splitting $s : \mathcal{A}_N \to \mathcal{A}_G$ with $\hat{\pi}s = id_{\mathcal{A}_N}$.

Let $x \in \mathcal{A}_{G/N}^{+}$. By remark (a) following Definition 2.2, $\hat{\pi}(x) = 0$. Hence $\mathcal{A}_{G/N}^{+} \subset \ker(\hat{\pi})$ and therefore $\mathcal{A}_{G/N}^{+}\mathcal{A}_G \subset \ker(\hat{\pi})$.

Define a linear map ϕ on $\mathcal{A}_G$ by $\phi = (s\hat{\pi}) * S = m(s\hat{\pi} \otimes S)\Delta_G$, where m and S are respectively the multiplication map and antipode of $\mathcal{A}_G$. Then using the coassociativity of Δ_G and $\hat{\pi}s = id_{\mathcal{A}_N}$ along with the antipodal property of S, one verifies that $\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}$. Since $\ker(\hat{\pi}) \subset \mathrm{Im}(id - s\hat{\pi})$, to show $\ker(\hat{\pi}) \subset \mathcal{A}_{G/N}^{+}\mathcal{A}_G$, it suffices to show that $\mathrm{Im}(id - s\hat{\pi}) \subset \mathcal{A}_{G/N}^{+}\mathcal{A}_G$. Since $(\epsilon - id)\phi(\mathcal{A}_G) \subset \mathcal{A}_{G/N}^{+}$, the later follows from the identity

$$id - s\hat{\pi} = (\epsilon - id)\phi * id = m\big((\epsilon - id)\phi \otimes id\big)\Delta_G,$$

which one verifies using basic properties of the convolution product along with $\epsilon\phi = \epsilon$ and the splitting property of s. This proves Lemma 4.4. □

End of proof of Theorem 4.1. If $A_{G/N_1} = \mathbb{C}1$, we would have $\mathcal{A}_{G/N_1} = \mathbb{C}1$ and $\mathcal{A}_{G/N_1}^{+} = 0$. Let $\hat{\pi}_1$ be the morphism of Hopf algebras from $\mathcal{A}_G$ to $\mathcal{A}_{N_1}$ associated with π_1. Then by

Lemma 4.4,

$$\ker(\hat{\pi}_1) = \mathcal{A}^+_{G/N_1}\mathcal{A}_G = 0.$$

Since $\ker(\hat{\pi}_1)$ is dense in $\ker(\pi_1)$ by Lemma 4.3, we would have $\ker(\pi_1) = 0$. This contradicts the assumption that N_1 is a non-trivial quantum subgroup of G and therefore $A_{G/N_1} \neq \mathbb{C}1$.

Then A_{G/N_1} has to be the only non-trivial Woronowicz C^*-subalgebra of $B_u(Q)$, i.e. $A_{G/N_1} = C^*(\{r_{2m} \mid m \geqslant 0\})$ as noted near the beginning of the proof of Theorem 4.1. We infer from Proposition 2.1 that $\pi_1(r_{2m})$ is a multiple of the trivial representation of N_1 for any m. From

$$\pi_1(r_1) \otimes \pi_1(r_1) = \pi_1(r_0) \oplus \pi_1(r_2),$$

we see that $\pi_1(r_1) \otimes \pi_1(r_1)$ is a multiple of the trivial representation of N_1. That is

$$\sum_{ijkl} e_{ij} \otimes e_{kl} \otimes \tilde{u}_{ij}\tilde{u}_{kl} = I_n \otimes I_n \otimes 1,$$

where $\tilde{u}_{ij}$ are the (i,j)-entries of $\pi_1(r_1)$ and e_{ij} are matrix units. Hence

$$\tilde{u}_{ij}\tilde{u}_{kl} = 0, \quad \text{when } i \neq j, \text{ or } k \neq l;$$
$$\tilde{u}_{ii}\tilde{u}_{ll} = 1, \quad \text{for all } i, l.$$

Therefore $\tilde{u}_{ij} = 0$ for $i \neq j$ and A_{N_1} is commutative. That is, N_1 is an ordinary compact group. Now it is clear that $\tilde{u}_{ii} = \tilde{u}_{ll} = \tilde{u}_{ll}^{-1}$ for all i, l, which we denote by a. Since A_{N_1} is generated by a and N_1 is non-trivial, we conclude that N_1 is a group of order 2. The map α from A_N to A_{N_1} defined by $\alpha(v_{ij}) = \tilde{u}_{ij}$ is clearly an isomorphism such that $\pi_1 = \alpha\pi$. Hence $(N_1, \pi_1) = (N, \pi)$ by Definition 2.7.

For an example of a non-normal quantum subgroup (H, θ) of $B_u(Q)$, take a two-elements group $H = \{I_n, V\}$, where

$$V = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix},$$

and $\theta(u_{ij}) = w_{ij}$, the coordinate functions on H. □

Let us also recall the construction of the quantum groups $A_u(Q)$ closely related to $B_u(Q)$ [47,49,50]. For every non-singular matrix Q, the quantum group $A_u(Q)$ is defined in terms of generators u_{ij} $(i, j = 1, \ldots, n)$, and relations:

$$u^*u = I_n = uu^*, \qquad u^t Q\bar{u}Q^{-1} = I_n = Q\bar{u}Q^{-1}u^t.$$

According to Banica [3], when $Q > 0$, the irreducible representations of the quantum group $A_u(Q)$ are parameterized by the free monoid $\mathbb{N} * \mathbb{N}$ with generators α and β and anti-multiplicative involution $\bar{\alpha} = \beta$ (the neutral element is e with $\bar{e} = e$). The classes of u and

$\bar{u}$ are r_α and r_β respectively. Moreover, for each pair of irreducible representations r_x and r_y ($x, y \in \mathbb{N} * \mathbb{N}$), one has the following direct sum decomposition (fusion) rules:

$$r_x \otimes r_y = \sum_{x=ag,\, \bar{g}b=y} r_{ab}.$$

In [56], the special $A_u(Q)$'s with $Q > 0$ are classified up to isomorphism and the general $A_u(Q)$'s with arbitrary Q are shown to be free products of the special $A_u(Q)$'s. The following result was observed by Bichon through private communication (the proof given below was developed by the author):

Proposition 4.5. *The quantum groups $A_u(Q)$ are not simple for any $Q \in GL(n, \mathbb{C})$.*

Proof. To prove this, we first introduce the following notion. A quantum subgroup (N, π) of a compact quantum group G is said to be *in the center* of G if

$$(\pi \otimes id)\Delta = (\pi \otimes id)\sigma\Delta,$$

where $\sigma(a_1 \otimes a_2) = a_2 \otimes a_1$, $a_1, a_2 \in A_G$, and Δ is the coproduct of A_G.

Assume (N, π) is in the center of G. Then using the definitions of $A_{G/N}$ and $A_{N\backslash G}$ in Section 2, it is straightforward to verify that $A_{G/N} = A_{N\backslash G}$. By Proposition 2.1(3), (N, π) is normal in G. Namely, a quantum subgroup that is in the center of G is always normal, just as in the classical case.

Let $\mathbb{T}$ be the one-dimensional (connected) torus group and $t \in C(\mathbb{T})$ the function such that $t^*t = 1 = tt^*$. Then $C(\mathbb{T})$ is generated by t as a C^*-algebra: $C(\mathbb{T}) = C^*(t)$. Define the morphism $\pi : A_u(Q) \to C(\mathbb{T})$ by $\pi(u_{ij}) = \delta_{ij} t$ (note the special case $A_u(Q) = C(\mathbb{T})$ when $n = 1$). Then it is routine to verify that the connected group (T, π) is in the center of the quantum group $A_u(Q)$ (not viewed as an algebra) in the sense above and is therefore a normal subgroup therein. Hence $A_u(Q)$ is not simple. □

We remark that although $A_u(Q)$ is not simple, for $n \geqslant 2$ and $Q > 0$, it is very close to being normal, satisfying most of the axioms of a simple compact quantum group: its function algebra is finitely generated, it is connected, and its non-trivial irreducible representations are all of dimension greater than one (see Wang [56] for a computation of the dimension of its irreducible representations based on Banica [3]). In particular following problems should be accessible:

Problem 4.6.

(1) Study further the structure of $A_u(Q)$ for positive matrices $Q \in GL(n, \mathbb{C})$ and $n \geqslant 2$. Determine all of their simple quotient quantum groups. Alternatively,
(2) Construct simple compact quantum groups that are not almost classical.

A solution of part (1) of the above problem should also give a solution to part (2) and provide the first examples of simple compact quantum groups that are not almost classical because of the highly non-commutative representation ring of $A_u(Q)$ (note that all the simple quantum groups known so far are almost classical). It is worth noting that the determination of all simple quotient quantum groups of $A_u(Q)$ in the above problem is easier than the determination of all of their simple quantum subgroups, the latter being tantamount to finding all simple quantum groups because every compact matrix quantum group is a quantum subgroup of an appropriate $A_u(Q)$. These remarks also indicate that $A_u(Q)$ should play an important role in the theory of simple compact quantum groups.

Next we consider the quantum automorphism group $A_{aut}(B, tr)$ of a finite-dimensional C^*-algebra B endowed with a tracial functional tr (cf. [4,53]). This quantum group is defined to be the universal object in the category of compact quantum transformation groups of B that leave tr invariant. Note that the presence of a tracial functional tr is necessary for the existence of the universal object when B is non-commutative (see of [53, Theorem 6.1]). For an arbitrary finite-dimensional C^*-algebra B, the C^*-algebra $A_{aut}(B, tr)$ is described explicitly in [53] in terms of generators and relations. When $B = C(X_n)$ is the commutative C^*-algebra of functions on the space X_n of n points, the quantum automorphism group $A_{aut}(B) = A_{aut}(X_n)$ (also called the *quantum permutation group* on n letters) exists without the presence of a (tracial) functional and its description in terms of generators and relations is surprisingly simple. The C^*-algebra $A_{aut}(X_n)$ is generated by self-adjoint projections a_{ij} such that each row and column of the matrix $(a_{ij})_{i,j=1}^n$ add up to 1. That is,

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \quad i, j = 1, \dots, n,$$

$$\sum_{j=1}^n a_{ij} = 1, \quad i = 1, \dots, n,$$

$$\sum_{i=1}^n a_{ij} = 1, \quad j = 1, \dots, n.$$

For more general finite-dimensional C^*-algebras B, the description of $A_{aut}(B, tr)$ in terms of generators and relations is more complicated. We refer the reader to [53] for details.

Assume tr is the canonical trace τ on B (see [4, p. 772] or Section 1 for the definition). Then $A_{aut}(B, \tau)$ is an ordinary permutation group when the dimension of B is less than or equal to 3. However, when the dimension of B is greater than or equal to 4, $A_{aut}(B, \tau)$ is a non-trivial (noncommutative and noncocommutative) compact quantum group with an infinite-dimensional function algebra [53,54], and as Banica [4] showed, the algebra of symmetries of the fundamental representation of this quantum group is isomorphic to the infinite-dimensional Temply–Lieb algebras $TL(n)$ and the representation ring of $A_{aut}(B, \tau)$ is isomorphic to that of $SO(3)$. Hence $A_{aut}(B, \tau)$ is almost classical for all B. It is easy to see that for $B = C(X_n)$, the canonical trace τ is equal to the unique S_n-invariant state on B, where S_n acts on X_n by permutation. Hence by remark (2) following [53, Theorem 3.1], $A_{aut}(B, \tau)$ is the same as the quantum permutation group $A_{aut}(X_n)$.

We refer the reader to [4,53,54] for more on these quantum groups and [15–17] for interesting related results. Note that the description in [4] is not exactly as that in [53] but equivalent to it. We now prove

Theorem 4.7. *Let B be a finite-dimensional C^*-algebra with* $\dim(B) \geqslant 4$. *Then $A_{aut}(B, \tau)$ is an almost classical, absolutely simple compact quantum group with property F.*

Proof. The argument is similar to the one in Theorem 4.1. By Banica [4], the complete set of mutually inequivalent irreducible representations of the quantum group $A_{aut}(B, \tau)$ can be parametrized by r_k ($k \geqslant 0$, r_0 being the trivial one-dimensional representation). Under this parametrization the fusion rules of its irreducible representations are the same as those of $SO(3)$ and therefore it is almost classical:

$$r_k \otimes r_l = r_{|k-l|} \oplus r_{|k-l|+1} \oplus \cdots \oplus r_{k+l-1} \oplus r_{k+l}, \quad k, l \geqslant 0.$$

We claim that there are only two Woronowicz C^*-subalgebras in $A_{aut}(B, \tau)$, namely $\mathbb{C}1$ and $A_{aut}(B, \tau)$.

Let $A_1 \neq \mathbb{C}1$ be a Woronowicz C^*-subalgebra of $A_{aut}(B, \tau)$. Let v be a non-trivial irreducible representation of the compact quantum group of A_1. Then $v = r_k$ for some $k \neq 0$ and

$$r_k \otimes r_k = r_0 \oplus r_1 \oplus r_2 \oplus \cdots \oplus r_{2k-1} \oplus r_{2k}, \quad k, l \geqslant 0.$$

Hence the coefficients of each of the representations $r_1, r_2, \ldots, r_{2k}$ are in A_1. Similarly, from the decomposition of $r_{2k} \otimes r_{2k}$, we see that the coefficients of each of the representations $r_1, r_2, \ldots, r_{4k}$ are in A_1. Inductively, the coefficients of each of the representations $r_1, r_2, \ldots, r_{2^m k}$ are in A_1 ($m > 0$). Hence $A_1 = A_{aut}(B, \tau)$.

Let (π, N) be a normal quantum subgroup of $G = A_{aut}(B, \tau)$ different from the trivial one-element subgroup. Then there is a non-trivial irreducible representation $u^\lambda = (u^\lambda_{ij})$ such that $\pi(u^\lambda)$ is not a multiple of the trivial representation. Using the same argument as in the proof of Theorem 4.1 we have $A_{G/N} = E_{G/N}(A_G) \neq A_G$. Therefore we must have $A_{G/N} = \mathbb{C}1$. Then the argument near the end of the proof of Theorem 4.1 (i.e. the paragraph that follows the proof of Lemma 4.4) shows that $\ker(\pi) = 0$. That is, N is the same quantum group as G. □

Theorem 4.7 applies in particular to quantum permutation groups $A_{aut}(X_n)$ when $n \geqslant 4$. As Manin (private communication in July, 2002) pointed out to the author, the reason that these quantum groups are connected could be that there are so many more quantum symmetries that the originally $n!$ isolated permutations are connected together by them. Note however that their function algebras are generated by orthogonal projections a_{ij}, so these quantum groups are also disconnected, as observed by Bichon [16]. It would be interesting to find a satisfactory explanation of this paradox.

The proofs of the main results of this section do not need explicit description (models) of representations of the quantum groups $B_u(Q)$ and $A_{aut}(B, \tau)$ and $A_u(Q)$. Only the structures of their representation rings (i.e. fusion rules) are used. However, explicit constructions of models of irreducible representations of Lie groups are fundamental and have important applications in other branches of mathematics and physics. Moreover, just as the construction and classification of the representations of simple compact Lie groups is intimately intertwined with the classification of simple compact Lie groups, the same might hold true for simple compact quantum groups. In view of these, we believe an appropriate answer to the following problem should be important in the theory of compact quantum groups in general and the theory of simple compact quantum groups in particular. (Note that the model for the fundamental representation of the quantum group $A_u(Q)$ is used in [54] to construct ergodic actions on various von Neumann factors.)

Problem 4.8. Construct explicit models of the irreducible representations of the following quantum groups: $A_u(Q)$ for $Q > 0$; $B_u(Q)$ for $Q\bar{Q} = \pm I_n$; the quantum automorphism group $A_{aut}(B, \tau)$ of a finite-dimensional C^*-algebra B endowed with the canonical trace τ. Relate the results to the theory simple compact quantum groups if possible.

5. Simplicity of K_q, K_q^u and K_J

The compact real forms K_q of Drinfeld–Jimbo quantum groups and their twists K_q^u are studied in [43] and [30] respectively. See also [31] for a summary of [30,43] and [29] for more detailed treatment. Motivated by these works, Rieffel constructs in [37] a deformation K_J of compact Lie group K which contains a torus T and raises the question whether K_q^u can be obtained as a *strict deformation quantization* of K_q. This question is answered in the affirmative by the author in [51]. The purpose of this section is to show that the quantum groups K_q, K_q^u and K_J are simple in the sense of this paper, provided that the compact Lie group K is simple.

We first recall the notation of [30,31,43]. Let G be a connected and simply connected simple complex Lie group with Lie algebra $\mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, together with the corresponding decomposition $\Delta = \Delta_+ \cup \Delta_-$ of the root system and a fixed basis $\{\alpha_i\}_{i=1}^n$ for Δ_+. For each linear functional λ on $\mathfrak{h}$, H_λ denotes the element in $\mathfrak{h}$ corresponding to λ under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ determined by the Killing form $(\,,)$ on $\mathfrak{g}$. Note that if the reader keeps the context in mind, the symbols α and λ used in this context should not cause confusion with the same symbols used in this paper for other purposes. Let $\{X_\alpha\}_{\alpha\in\Delta} \cup \{H_i\}_{i=1}^n$ be a Weyl basis of $\mathfrak{g}$, where $H_i = H_{\alpha_i}$. This determines a Cartan involution ω_0 on $\mathfrak{g}$ with $\omega_0(X_\alpha) = -X_{-\alpha}$, $\omega_0(H_i) = -H_i$. Let $\mathfrak{k}$ be the compact real form of $\mathfrak{g}$ defined as the fixed points of ω_0 and K the associated compact real form of G. Put $\mathfrak{h}_\mathbb{R} = \bigoplus_{i=1}^n \mathbb{R}H_i$, $\mathfrak{t} = i\mathfrak{h}_\mathbb{R}$ and $T = \exp(\mathfrak{t})$, the later being the associated maximal torus of K.

Let $q = e^{h/4}$ ($h \in \mathbb{R} \setminus \{0\}$). For $n, k \in \mathbb{N}$, $n \geqslant k$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \dots [n-k+1]_q}{[k]_q[k-1]_q \dots [1]_q}.$$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ [24,27] is the complex associative algebra with generators $X_i^\pm$, $K_i^{\pm1}$ ($i = 1, \dots, n$) and defining relations

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i, \qquad K_i K_j = K_j K_i,$$
$$K_i X_j^\pm K_i^{-1} = q^{\pm(\alpha_i,\alpha_j)} X_j^\pm,$$
$$\left[X_i^+, X_j^-\right] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} \left(X_i^\pm\right)^k X_j^\pm \left(X_i^\pm\right)^{1-a_{ij}-k} = 0, \quad i \neq j,$$

where $q_i = q^{(\alpha_i,\alpha_i)}$.

On $U_q(\mathfrak{g})$ there is a Hopf algebra structure with coproduct

$$\Delta\left(K_i^{\pm1}\right) = K_i^{\pm1} \otimes K_i^{\pm1}, \qquad \Delta\left(X_i^\pm\right) = X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm,$$

and counit and antipode respectively

$$\varepsilon(X_i^{\pm}) = 0, \qquad \varepsilon(K_i^{\pm 1}) = 1, \qquad S(X_i^{\pm}) = -q_i^{\pm 1} X_i^{\pm}, \qquad S(K_i^{\pm 1}) = K_i^{\mp 1}.$$

Under the $*$-structure defined by

$$(X_i^{\pm})^* = X_i^{\mp}, \qquad K_i^* = K_i,$$

$U_q(\mathfrak{g})$ is a Hopf $*$-algebra.

Let $u = \sum_{k,l} c_{kl} H_k \otimes H_l \in \bigwedge^2 \mathfrak{h}_{\mathbb{R}}$. Then it can be shown (cf. [29]) that the following defines a new coproduct on $U_q(\mathfrak{g})$,

$$\Delta_u(\xi) = \exp(-ihu/2)\Delta(\xi)\exp(ihu/2),$$

where $X \in U_q(\mathfrak{g})$ and Δ is the original coproduct on $U_q(\mathfrak{g})$. The new Hopf $*$-algebra so obtained is denoted by $U_{q,u}(\mathfrak{g})$.

The function algebra $\mathcal{A}_{K_q}$ of the compact quantum group K_q is defined to be the subalgebra of the dual algebra $U_q(\mathfrak{g})^*$ consisting of matrix elements of finite-dimensional representations ρ of $U_q(\mathfrak{g})$ such that eigenvalues of the endomorphisms $\rho(K_i)$ are positive. The function algebra $\mathcal{A}_{K_q^u}$ of the compact quantum group K_q^u is defined to be the subalgebra of the dual algebra $U_{q,u}(\mathfrak{g})^*$ that has the same elements as $\mathcal{A}_{K_q}$, as well as the same $*$-structure, while the product of its elements is defined using Δ_u instead of Δ.

For each (algebraically) dominant integral weight $\lambda \in P_+$ of $(\mathfrak{g}, \mathfrak{h})$, define matrix elements $C^{\lambda}_{\mu,i;\nu,j}$ of the highest weight $U_q(\mathfrak{g})$ module $(L(\lambda), \rho_\lambda)$ as follows. Let $\{v_\nu^{(i)}\}$ be an orthonormal weight basis for the unitary $U_q(\mathfrak{g})$ module $L(\lambda)$. Then $C^{\lambda}_{\mu,i;\nu,j}$ is defined by

$$C^{\lambda}_{\mu,i;\nu,j}(X) = \langle \rho_\lambda(X) v_\nu^{(j)}, v_\mu^{(i)} \rangle,$$

where $X \in U_q(\mathfrak{g})$ and $\langle\,,\rangle$ is the inner product on $L(\lambda)$. The $C^{\lambda}_{\mu,i;\nu,j}$'s is a linear (Peter–Weyl) basis of both $\mathcal{A}_{K_q}$ and $\mathcal{A}_{K_q^u}$ when λ ranges through the set P_+ of dominant integral weights of $(\mathfrak{g}, \mathfrak{h})$.

Theorem 5.1. *Let K be a connected and simply connected simple compact Lie group. Then for each q, K_q is an almost classical simple compact quantum group with property F.*

Proof. First we recall that representations of K and K_q are in one to one correspondence via deformation and the decompositions of tensor products of irreducible representations are not altered under deformation (see Lusztig [32] and Rosso [39] or Chari and Pressley [19]). From this it follows immediately that K_q is almost classical.

Let ξ be the map that associates each irreducible representation v of K an irreducible representation $\xi(v)$ of K_q in this correspondence. This map defines an isomorphism of *vector spaces* from $\mathcal{A}_K$ to $\mathcal{A}_{K_q}$, which we also denote by ξ. It follows from this that K_q is connected and has no non-trivial representations of dimension one. Comparing decompositions of tensor products of representations of K and K_q we see that the ξ maps bijectively the set of Hopf subalgebras of $\mathcal{A}_K$ onto the set of Hopf subalgebras of $\mathcal{A}_{K_q}$.

Let ρ_q be the quotient morphism from A_{K_q} to the abelianization $A^{ab}_{K_q}$, which is by definition the quotient of A_{K_q} by the closed two-sided ideal of A_{K_q} generated by commutators $[a, b]$,

$a, b \in A_{K_q}$. According to [49], $A_{K_q}^{ab}$ is the algebra of continuous functions on the maximal compact subgroup $\hat{A}_{K_q}$ of K_q and ρ_q gives rise to the embedding of the quantum groups from $\hat{A}_{K_q}$ to K_q. It is shown in [43] that the maximal compact subgroup $\hat{A}_{K_q}$ is isomorphic to the maximal torus T of K.

The associated morphism $\hat{\rho}_q$ from $\mathcal{A}_{K_q}$ to $\mathcal{A}_T$ is given by

$$\hat{\rho}_q\big(C^{\lambda}_{\mu,i;\nu,j}\big)(t) = \delta_{ij}\delta_{\mu\nu}e^{2\pi\mu(x)},$$

where $t = \exp(x) \in T$, $x \in \mathfrak{t} = i\mathfrak{h}_{\mathbb{R}}$ (see [19, p. 438], but $\sqrt{-1}$ should not appear in the formula there). It is clear that one has the same formula as above for restriction morphism ρ from A_K to A_T:

$$\hat{\rho}\big(\xi^{-1}\big(C^{\lambda}_{\mu,i;\nu,j}\big)\big)(t) = \delta_{ij}\delta_{\mu\nu}e^{2\pi\mu(x)}, \quad \text{i.e.,} \quad \hat{\rho} = \hat{\rho}_q \circ \xi.$$

Let $N \subset K$ be a normal subgroup of K with surjections $\pi : A_K \to A_N$ and $\hat{\pi} : \mathcal{A}_K \to \mathcal{A}_N$. Then N is a finite subgroup of T and $A_N = \mathcal{A}_N$ is a finite-dimensional Hopf algebra. It is clear that $\pi = \rho_N \circ \rho$, where ρ_N is the restriction morphism from A_T to A_N. Define

$$\pi_q : A_{K_q} \longrightarrow A_N, \quad \text{by } \pi_q := \rho_N \circ \rho_q.$$

We claim that (N, π_q) is a normal subgroup of K_q. This follows immediately from the following identities, which one can easily verify using $\hat{\rho} = \hat{\rho}_q \circ \xi$ and $\hat{\pi} = \hat{\pi}_q \circ \xi$:

$$\mathcal{A}_{K_q/N} = \xi(\mathcal{A}_{K/N}), \quad \text{i.e.,}$$
$$\big\{a \in \mathcal{A}_{K_q} \bigm| (id \otimes \pi_q)\Delta(a) = a \otimes 1\big\} = \xi\big(\big\{a \in \mathcal{A}_K \bigm| (id \otimes \pi)\Delta(a) = a \otimes 1\big\}\big);$$
$$\mathcal{A}_{N\backslash K_q} = \xi(\mathcal{A}_{N\backslash K}), \quad \text{i.e.,}$$
$$\big\{a \in \mathcal{A}_{K_q} \bigm| (\pi_q \otimes id)\Delta(a) = 1 \otimes a\big\} = \xi\big(\big\{a \in \mathcal{A}_K \bigm| (\pi \otimes id)\Delta(a) = 1 \otimes a\big\}\big).$$

That is, every normal subgroup N of K gives rise to a normal subgroup (N, π_q) of K_q in the manner above.

Conversely, let (N', π') be a quantum normal subgroup of K_q. Then $\mathcal{A}_{K_q/N'}$ is a Hopf subalgebra of $\mathcal{A}_{K_q}$. Since every Hopf subalgebra of $\mathcal{A}_K$ is of the form $\mathcal{A}_{K/N}$ for some normal subgroup N of K (cf. [49]), by the correspondence between Hopf subalgebras of $\mathcal{A}_K$ and those of $\mathcal{A}_{K_q}$ noted near the beginning of the proof we have

$$\mathcal{A}_{K_q/N'} = \xi(\mathcal{A}_{K/N}) = \mathcal{A}_{K_q/N}$$

for some normal subgroup N of K. By Lemma 4.4, we have $\ker(\hat{\pi}_q) = \ker(\hat{\pi}')$. That is (N', π') and (N, π_q) is the same quantum subgroup of K_q (cf. Definition 2.7 and Lemma 4.3). Since normal subgroups N of K are finite, we conclude from the above that K_q has no non-trivial connected quantum normal subgroups. □

Examining the proof of Theorem 5.1, we formulate the following general result on the invariance of simplicity of compact quantum groups under deformation, which will be used to prove the simplicity of K_q^u and K_J.

Let G be an almost classical simple compact quantum group with property F and (H, ρ) a quantum subgroup. Assume all normal quantum subgroups of G are quantum subgroups of H. Let G_v be a family of compact quantum groups ("deformation" of G) indexed by a subset $\{v\}$ of a vector space that includes the origin. Suppose the family G_v satisfies the following conditions:

(C1) $G_0 = G$.
(C2) There is an isomorphism ξ of vector spaces from $\mathcal{A}_G$ to $\mathcal{A}_{G_v}$.
(C3) The coproduct is unchanged under deformation, i.e.,

$$\Delta_v\big(\xi(a)\big) = (\xi \otimes \xi)\Delta(a) \quad \text{for } a \in \mathcal{A}_G.$$

(C4) For any pair irreducible representations u^{λ_1} and u^{λ_2} of G, if

$$u^{\lambda_1} \otimes u^{\lambda_2} \cong u^{\gamma_1} \oplus u^{\gamma_2} \oplus \cdots \oplus u^{\gamma_l}$$

is a decomposition of $u^{\lambda_1} \otimes u^{\lambda_2}$ into direct sum of irreducible subrepresentations u^{γ_j} ($j = 1, 2, \ldots, l$), then

$$\xi\big(u^{\lambda_1}\big) \otimes \xi\big(u^{\lambda_2}\big) \cong \xi\big(u^{\gamma_1}\big) \oplus \xi\big(u^{\gamma_2}\big) \oplus \cdots \oplus \xi\big(u^{\gamma_l}\big)$$

is a decomposition of $\xi(u^{\lambda_1}) \otimes \xi(u^{\lambda_2})$ into direct sum of irreducible representations, where for instance $\xi(u^{\lambda_1})$ denotes the representation of G_v whose coefficients are images of coefficients of u^{λ_1}.
(C5) The quantum subgroup H is undeformed. The latter means that there is a morphism ρ_v of quantum groups from H to G_v such that

$$\rho_v\big(\xi(a)\big) = \rho(a) \quad \text{for } a \in \mathcal{A}_G.$$

Under the assumptions above, we have the following result. The proof is the same as that of Theorem 5.1 (H corresponds to T in Theorem 5.1).

Theorem 5.2. *For each $v \in \{v\}$, G_v is an almost classical simple quantum group with property F.*

Remarks. (a) Condition (C4) above is not the same as the requirement that

$$\xi\big(u^{\lambda_1} \otimes u^{\lambda_2}\big) = \xi\big(u^{\lambda_1}\big) \otimes \xi\big(u^{\lambda_2}\big).$$

The latter requirement together with conditions (C2) and (C3) imply that ξ is an isomorphism of quantum from G_v to G, which is not the case for the quantum groups under consideration here.

(b) We believe similar results on invariance of simplicity under deformation hold true without the property F assumption on G. But at the moment we do not know of any simple compact quantum groups that do not satisfy this property, though there are many non-simple quantum groups without this property.

Next we recall the construction in [37,51]. Let $A = A_G$ be a compact quantum group with coproduct Δ. Suppose that the quantum group G has a toral subgroup (T, ρ)—to obtain non-trivial deformation we assume that T has rank no less than 2. For any element t in T, denote by E_t the corresponding evaluation functional on $C(T)$. Assume that η is a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to T, where n is allowed to be different from the dimension of T. Define an action α of $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$ on the C^*-algebra A as follows:

$$\alpha_{(s,v)} = l_{\eta(s)} r_{\eta(v)},$$

where

$$l_{\eta(s)} = (E_{\eta(-s)}\rho \otimes id)\Delta, \qquad r_{\eta(v)} = (id \otimes E_{\eta(v)}\rho)\Delta.$$

For any skew-symmetric operator S on $\mathbb{R}^n$, one may apply Rieffel's quantization procedure [36] for the action α above to obtain a deformed C^*-algebra A_J whose product is denoted $\times_J$, where $J = S \oplus (-S)$. The family A_{hJ} ($h \in \mathbb{R}$) is a strict deformation quantization of A (see [36, Chapter 9]). In [51] the following result is obtained.

Theorem 5.3. *The deformation A_J is a compact quantum group containing T as a (quantum) subgroup; A_J is a compact matrix quantum group if and only if A is.*

We denote by G_J the quantum group for A_J. When G is a compact Lie group, the construction G_J above is the same as Rieffel's construction [37]. By [37, 5.2], G_J is an almost classical compact quantum group if G is a compact Lie group.

Combining Theorems 5.3 and 5.2, we obtain

Theorem 5.4. *Let K be a simple compact Lie group with a toral subgroup T. Then K_J of Rieffel* [37] *is an almost classical simple compact quantum group with property F.*

We note that unlike in Theorem 5.1, in the result above we do not need to assume K to be simply connected. This is because $\mathcal{A}_{K_q}$ is defined using irreducible representations of $U_q(\mathfrak{g})$ associated with all dominant integral weights P_+ of $(\mathfrak{g}, \mathfrak{h})$, so that $\mathcal{A}_{K_q}$ becomes the algebra of representative functions on a simply connected K when $q \to 1$. One could also start with a non-simply connected K in Theorem 5.1 too, but then one needs to modify the definition of the quantum algebra $\mathcal{A}_{K_q}$ by using irreducible representations of $U_q(\mathfrak{g})$ associated with analytically dominant integral weights only. This newly defined $\mathcal{A}_{K_q}$ is a Hopf subalgebra of the Hopf algebra defined originally. It is clear from the proof of Theorem 5.1 that its conclusion remains valid for this newly defined K_q.

Finally we consider K_q^u. To avoid confusion with the Killing form, we now use $s \oplus v$, instead of (s, v) used above, to denote an element of $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^n$. In the present setting, the space $\mathbb{R}^n$ is $\mathfrak{h}_\mathbb{R}$, with inner product $\langle\,,\rangle = (\,,)$, where $(\,,)$ is the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}_\mathbb{R}$. We will also use $\langle\,,\rangle$ to denote the inner product on $\mathfrak{h}_\mathbb{R} \oplus \mathfrak{h}_\mathbb{R}$. Noting that the compact abelian group T is also a subgroup of both K_q and K_q^u (see [30,43]). The map η there in this case is defined by $\eta(s) = \exp(2\pi i s)$. We can define as above an action of $\mathbb{R}^d$ on A_{K_q} by

$$\alpha_{s\oplus v} = l_{\exp(-2\pi i s)} r_{\exp(2\pi i v)}.$$

This action may be viewed as an action of $H = T \times T$ in the sense of [36]. For each ν in the weight lattice P of $\mathfrak{g}$, the element H_ν is in $\mathfrak{h}_\mathbb{R}$. We use the notation $H_\nu \oplus H_\mu$ to denote $H_\nu + H_\mu$ as an element of $\mathfrak{h}_\mathbb{R} \oplus \mathfrak{h}_\mathbb{R}$. Keep the notation of [36] for the spectral subspaces of the action α (see [36, 2.22]).

Let $\check{u}$ be the map on $\mathfrak{h}^*$ determined by u via the Killing form $(\,,\,)$ on $\mathfrak{g}$. Let

$$p = -(H_{\nu_1} \oplus H_{\mu_1}), \qquad q = -(H_{\nu_2} \oplus H_{\mu_2}),$$
$$J = \frac{h}{4\pi}\big(S_u \oplus (-S_u)\big),$$

where S_u is the skew-symmetric operator on $\mathfrak{h}_{\mathbb{R}}$ defined by

$$S_u(H_\nu) = \sum_{k,l} c_{kl}\,\nu(H_k)H_l.$$

Then one has

$$C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} \circ C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2} = \exp\left(\frac{ih}{2}\big((\mu_1, \check{u}\mu_2) - (\nu_1, \check{u}\nu_2)\big)\right) C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2}$$
$$= \exp\big(-2\pi i\langle p, Jq\rangle\big) C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2}$$

where $\circ$ on the left-hand side is the multiplication in $A_{K_q^u}$ and the right-hand side is the multiplication in A_{K_q}.

On the other hand one has from [36, 2.22] that

$$C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} \times_J C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2} = \exp\big(-2\pi i\langle p, Jq\rangle\big) C^{\lambda_1}_{\mu_1,i_1;\nu_1,j_1} C^{\lambda_2}_{\mu_2,i_2;\nu_2,j_2}.$$

This means that we have the following result [51].

Theorem 5.5. *The Hopf $*$-algebras $\mathcal{A}_{K_q^u}$ and $(\mathcal{A}_{K_q}, \times_J)$ are isomorphic.*

That is $K_q^u = (K_q)_J$ in the notation of Theorem 5.3, answering Rieffel's question [37] in the affirmative.

Combining Theorems 5.5, 5.1 and 5.2, we obtain the following

Theorem 5.6. *Let K be a connected and simply connected simple compact Lie group. Then for each (q, u), K_q^u is an almost classical simple compact quantum group with property F.*

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