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Finitistic dimension and restricted flat dimension ☆

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Abstract

We investigate the relations between finitistic dimensions and restricted flat dimensions (introduced by Foxby [L.W. Christensen, H.-B. Foxby, A. Frankild, Restricted homological dimensions and Cohen–Macaulayness, J. Algebra 251 (1) (2002) 479–502]). In particular, we show the following result. (1) If T is a selforthogonal left module over a left noetherian ring R with the endomorphism ring A, then $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{id}({}_RT) + \mathrm{rfd}(T_A)$. (2) If ${}_RT$ is classical partial tilting, then $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{rfd}(T_A)$. (3) If $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m = R$ are Artin algebras with the same identity such that, for each $0 \leqslant i \leqslant m-1$, $\mathrm{rad}\, A_i$ is a right ideal in A_{i+1} and $\mathrm{rfd}({A_{i+1}}_{A_i}) < \infty$ (e.g., ${A_{i+1}}_{A_i}$ is of finite projective dimension, or finite Gorenstein projective dimension, or finite Tor-bound dimension), then $\mathrm{fdim}({}_RR) < \infty$ implies $\mathrm{fdim}({}_AA) < \infty$. As applications, we disprove Foxby's conjecture [H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (2004) 167–193] on restricted flat dimensions by providing a counterexample and give a partial answer to a question posed by Mazorchuk [V. Mazorchuk, On finitistic dimension of stratified algebras, arXiv:math.RT/0603179, 6.4].

Keywords: Finitistic dimension; Restricted flat dimension; Selforthogonal module; Endomorphism ring; Idealized extension

Introduction

Let R be an associative ring with identity. We denote by R-Mod (resp., Mod-R) the category of all left (resp., right) R-modules and by R-mod (resp., mod-R) the category of all left (resp., right) modules possessing finitely generated projective resolutions. The left little (resp., big) fini-

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tistic (projective) dimension of R, denoted by $\mathrm{fdim}({}_RR)$ (resp., $\mathrm{Fdim}({}_RR)$), is defined as the supremum of the projective dimensions of all modules in R-mod (resp., R-Mod) of finite projective dimension. Obviously, $\mathrm{fdim}({}_RR) \leqslant \mathrm{Fdim}({}_RR)$. We denote by $\mathrm{fdim}(R_R)$ and $\mathrm{Fdim}(R_R)$ the corresponding right finitistic dimensions of R.

It is known that $\mathrm{Fdim}({}_RR)$ coincides with the Krull dimension of R in case R is commutative and noetherian and that $\mathrm{fdim}({}_RR) = \mathrm{depth}\, R$ in case R is commutative local and noetherian. So in the latter case both dimensions are finite, but they coincide if and only if R is Cohen–Macaulay. There are also examples of commutative noetherian rings with $\mathrm{Fdim}({}_RR) = \mathrm{fdim}({}_RR) = \infty$.

In case R is an Artin algebra, it is known that the first finitistic dimension conjecture, which stated that $\mathrm{Fdim}({}_RR) = \mathrm{fdim}({}_RR)$, fails in general and the differences can even be arbitrarily big. However, the second finitistic dimension conjecture, which states that $\mathrm{fdim}({}_RR) < \infty$, is still open. This conjecture is also related to many other homological conjectures and attracts many algebraists, see for instance [3,19,21].

In this note, we will investigate the finitistic dimension in terms of the restricted flat dimension.

Let A be a ring, and let T_A be a right A-module. Following [6], T_A is said to have (big) restricted flat dimension at most m if for each $i > m$ the functor $\mathrm{Tor}_i^A(T, -)$ vanishes on the category of modules of finite flat dimension. The little restricted flat dimension is defined correspondingly by considering only modules of finite flat dimension which admit a projective resolution with finitely generated projectives. Obviously, $\mathrm{rfd}(T_A) \leqslant \mathrm{Rfd}(T_A) \leqslant \mathrm{fd}(T_A)$ by definition, where $\mathrm{fd}(T_A)$ denotes the flat dimension of T_A. Moreover, restricted flat dimensions are also smaller than Gorenstein flat dimension and $*$-syzygy dimension (see Lemma 1.1). It was conjectured by Foxby that the (big) restricted flat dimension of a module is equal to its Gorenstein flat dimension whenever the latter is finite [11]. The conjecture was proved in case that A is commutative noetherian [11] and that A is a coherent ring with finite Gorenstein weak dimension [4].

We find that the restricted flat dimension is a useful tool to describe the finitistic dimension. For example, we obtain that the left little finitistic dimension of an Artin algebra A is equal to the little restricted flat dimension of $(DA)_A$, where D denotes the usual duality in Artin algebras (Corollary 2.7).

Recall that $T \in R$-Mod is selfsmall provided that $\mathrm{Hom}_R(T, T)^{(X)} \simeq \mathrm{Hom}_R(T, T^{(X)})$ canonically, for any X. ${}_RT$ is selforthogonal if $T \in \mathrm{KerExt}_R^{i\geqslant 1}(T, -)$, i.e., T belongs to the category of all modules M such that $\mathrm{Ext}_R^i(T, M) = 0$ for all $i \geqslant 1$. ${}_RT$ is said to be coproduct-selforthogonal if $T^{(X)} \in \mathrm{KerExt}_R^{i\geqslant 1}(T, -)$ for all X.

One of our main results states as follows. The idea comes from [17], where the global dimension of endomorphism rings is estimated in terms of the flat dimension.

Theorem 0.1. *Let R be a ring and $T \in R$-Mod with $A = \mathrm{End}_R T$.*

(1) *If T is selfsmall and coproduct-selforthogonal, then* $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA) \leqslant \mathrm{id}(\mathrm{Add}_R T) + \mathrm{Rfd}(T_A)$, *where* $\mathrm{id}(\mathrm{Add}_R T)$ *denotes the supremum of injective dimensions of modules in* $\mathrm{Add}_R T$.

(1′) *If T is selforthogonal, then* $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{id}({}_RT) + \mathrm{rfd}(T_A)$.

(2) *If ${}_RT$ is selfsmall and coproduct-selforthogonal and of finite projective dimension, then* $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{Rfd}(T_A)$.

(2′) *If* ${}_RT$ *is selforthogonal and of finite projective dimension, then* $\mathrm{fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{rfd}(T_A)$. *If additionally* ${}_RT \in R\text{-}\mathrm{mod}$, *then* $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{rfd}(T_A)$.

As applications, we show that, if ${}_RT$ is a classical fdim-test tilting module with $A = \mathrm{End}_R T$, then $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR)$ (Proposition 2.8). This gives a partial answer to a question posed by Mazorchuk [14, 6.4]. The question asks if there is any relation between $\mathrm{fdim}({}_RR)$ and $\mathrm{fdim}({}_AA)$, where $A = \mathrm{End}_R T$ and ${}_RT$ is a classical fdim-test tilting module over a standardly stratified algebra R such that its Ringel dual is a properly stratified algebra [9, Sections 5 and 6].

We also obtain the following corollary which contains [20, Theorems 1.2], where the conclusion is proved under assumptions that T_A is of finite $*$-syzygy dimension (or finite Gorenstein projective dimension or finite projective dimension).

Corollary 0.2. *Let R be a ring and $T \in R\text{-}\mathrm{Mod}$ with $A = \mathrm{End}_R T$.*

(1) *If* ${}_RT$ *is selfsmall and projective with* $\mathrm{Rfd}(T_A)$ *finite, then* $\mathrm{Fdim}({}_RR) < \infty$ *implies that* $\mathrm{Fdim}({}_AA) < \infty$.
(2) *If* ${}_RT$ *is projective with* $\mathrm{rfd}(T_A)$ *finite, then* $\mathrm{Fdim}({}_RR) < \infty$ *implies that* $\mathrm{fdim}({}_AA) < \infty$. *If furthermore* ${}_RT$ *is finitely generated, then* $\mathrm{fdim}({}_RR) < \infty$ *implies that* $\mathrm{fdim}({}_AA) < \infty$.

We give examples to show that, in general, the restricted flat dimension of a module may be strictly smaller than its $*$-syzygy dimension (Gorenstein projective dimension, projective dimension). In particular, we give a counterexample to Foxby's conjecture.

Another main result of the note concerns the finitistic dimension of some fixed subrings. The first part of the following result extends [13, Theorem 6], while the second part can be compared with [18, Theorem 3.1].

Theorem 0.3.

(1) *Let A be a subring of a ring R such that A is an A–A bimodule direct summand of R. Then* $\mathrm{Fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{Rfd}(R_A)$.
(2) *Assume that $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_m = R$ are Artin algebras with the same identity such that, for each $0 \leqslant i \leqslant m-1$,* $\mathrm{rad}\, A_i$ *is a right ideal in A_{i+1} and* $\mathrm{rfd}({A_{i+1}}_{A_i}) < \infty$ *(e.g., ${A_{i+1}}_{A_i}$ is of finite projective dimension or finite Gorenstein projective dimension), then* $\mathrm{fdim}({}_RR) < \infty$ *implies* $\mathrm{fdim}({}_AA) < \infty$.

1. Preliminaries

Throughout this paper, all rings will be associated with non-zero identity. By a category, we mean a full subcategory closed under isomorphisms.

In the following, we fix R to be a ring and $T \in R\text{-}\mathrm{Mod}$ with the endomorphism ring A. Then ${}_RT_A$ is an R–A bimodule. We denote by $\mathrm{Add}_R T$ (resp., $\mathrm{add}_R T$) the class of modules isomorphic to direct summands of (resp., finite) direct sums of copies of ${}_RT$.

Let $\mathcal{C} \subseteq R\text{-}\mathrm{Mod}$ be a category and $M \in R\text{-}\mathrm{Mod}$. We denote by $\mathcal{C}\text{-}\dim({}_RM)$ the minimal integer m such that there is an exact sequence $0 \to T_m \to \cdots \to T_0 \to M \to 0$ with each $T_i \in \mathcal{C}$ and call it the $\mathcal{C}$-dimension of ${}_RM$. Note that, for some ${}_RM$, the $\mathcal{C}$-dimension of ${}_RM$ may not exist. In the latter case, we denote $\mathcal{C}\text{-}\dim({}_RM) = \infty$. The category of all modules $M \in R\text{-}\mathrm{Mod}$ such that $\mathcal{C}\text{-}\dim({}_RM) < \infty$ is denoted by $\widehat{\mathcal{C}}$.

We define $\mathrm{Fdim}({}_R T)$ to be the supremum of the $\mathrm{Add}_R\, T$-dimensions of all modules in R-Mod of finite $\mathrm{Add}_R\, T$-dimension. Similarly, $\mathrm{fdim}({}_R T)$ is denoted to be the supremum of the $\mathrm{add}_R\, T$-dimensions of all modules in R-Mod of finite $\mathrm{add}_R\, T$-dimension. It is easy to see that $\mathrm{Fdim}({}_R R)$ (resp., $\mathrm{fdim}({}_R R)$) is just the left big (resp., little) finitistic dimension of R defined in the introduction.

Similarly to the notion $\mathrm{KerExt}_R^{i\geqslant 1}(T,-)$ defined in the introduction, we denote by $\mathrm{KerTor}_{i\geqslant 1}^A(T,-)$ the category of all modules $M \in A$-Mod such that $\mathrm{Tor}_i^A(T,M)=0$ for all $i \geqslant 1$.

It is well known that $(T\otimes_A \text{-}, \mathrm{Hom}_R(T,-))$ is a pair of adjoint functors and there are the following canonical homomorphisms for any R-module M and any A-module N:

$$\rho_M : T\otimes_A \mathrm{Hom}_R(T,M) \to M, \quad \text{by } t\otimes f \to f(t);$$
$$\sigma_N : N \to \mathrm{Hom}_R(T, T\otimes_A N), \quad \text{by } n \to [t \to t\otimes n].$$

Moreover, for any $M \in R$-Mod, the composition

$$\mathrm{Hom}_R(T,M) \xrightarrow{\sigma_{\mathrm{Hom}_R(T,M)}} \mathrm{Hom}_R\big(T, T\otimes_A \mathrm{Hom}_R(T,M)\big) \xrightarrow{\mathrm{Hom}_R(T,\rho_M)} \mathrm{Hom}_R(T,M)$$

is the identity. Similarly, the composition $(T\otimes_A \sigma_N)\circ \rho_{T\otimes_A N} = 1_{T\otimes_A N}$, for any $N \in A$-Mod (see for instance [16]).

Throughout the paper, we denote by $\mathrm{pd}({}_R T)$ (resp., $\mathrm{id}({}_R T)$, $\mathrm{fd}(T_A)$) the projective (resp., injective, flat) dimension of the module ${}_R T$ (resp., ${}_R T$, T_A).

Let A be a ring and $T \in$ Mod-A. T_A is said to be Gorenstein projective provided there is an exact sequence of projective modules $\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$ such that $T \simeq \mathrm{Im}(P_1 \to P_0)$ and such that $\mathrm{Hom}_A(-,Q)$ leaves the sequence exact whenever Q_A is a projective module. T is said to be Gorenstein flat provided there is an exact sequence of flat modules $\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $T \simeq \mathrm{Im}(F_1 \to F_0)$ and such that $-\otimes_A I$ leaves the sequence exact whenever ${}_A I$ is an injective module. T_A is said to be a $*$-syzygy module if, for all d, T_A is always isomorphic to a direct summand of a d-syzygy of a projective resolution of some right A-module.

The Gorenstein projective (resp., Gorenstein flat, $*$-syzygy) dimension of T_A is denoted by $\mathrm{Gpd}(T_A)$ (resp., $\mathrm{Gfd}(T_A)$, $*\text{-}\mathrm{sd}(T_A)$).

Lemma 1.1. *Let A be a ring and $T_A \in$ Mod-A.*

(1) $\mathrm{Rfd}(T_A) \leqslant *\text{-}\mathrm{sd}(T_A) \leqslant \mathrm{Gpd}(T_A) \leqslant \mathrm{pd}(T_A)$. *If* $\mathrm{pd}(T_A) < \infty$, *then* $\mathrm{Gpd}(T_A) = \mathrm{pd}(T_A)$.
(2) $\mathrm{Rfd}(T_A) \leqslant \mathrm{Gfd}(T_A) \leqslant \mathrm{fd}(T_A)$. *If A is coherent and* $\mathrm{fd}(T_A) < \infty$, *then* $\mathrm{Gfd}(T_A) = \mathrm{fd}(T_A)$.

Proof. (1) By definitions and [11, Proposition 2.27].

(2) The first part is shown in [11, Theorem 3.19]. The second part is easily obtained from [11, Theorem 3.6] and the dual part of [11, Proposition 2.27], together with relations between injective and flat modules. □

We will see in the next section that both inequalities $\mathrm{Rfd}(T_A) \leqslant *\text{-}\mathrm{sd}(T_A)$ and $\mathrm{Rfd}(T_A) \leqslant \mathrm{Gfd}(T_A)$ can be strict. It is not known to us if $\mathrm{Gfd}(T_A) = \mathrm{fd}(T_A)$ over any ring whenever the latter is finite.

Finally, we recall the definitions of tilting modules.

Let R be a ring and ${}_RT \in R$-Mod. We say ${}_RT$ is tilting if (i) $\mathrm{pd}({}_RT) < \infty$, (ii) ${}_RT$ is coproduct-selforthogonal and (iii) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some n with each $T_i \in \mathrm{Add}_R T$ (see for instance [2]). ${}_RT$ is classical tilting if (i) $\mathrm{pd}({}_RT) < \infty$ and ${}_RT \in R$-mod, (ii) ${}_RT$ is selforthogonal and (iii) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some n with each $T_i \in \mathrm{add}_R T$ (see for instance [10,15]). We say ${}_RT$ is classical partial tilting if it satisfies the condition (i) and (ii) in the definition of classical tilting modules.

Lemma 1.2. *Let R be a ring and ${}_RT \in R$-Mod with $A = \mathrm{End}_R T$.*

(1) *If ${}_RT$ is tilting, then there is $T' \simeq T^{(X)}$ for some X such that $R \simeq \mathrm{End}(T'_{A'})$ and $T'_{A'}$ is classical partial tilting, where $A' = \mathrm{End}_R T'$. In particular, $R \simeq \mathrm{End}(T_A)$.*
(2) *If ${}_RT$ is classical tilting, then T_A is classical tilting too. Moreover, $\mathrm{pd}({}_RT) = \mathrm{pd}(T_A)$.*

Proof. (1) By definition, there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some n with each $T_i \in \mathrm{Add}_R T$. Now take $T^{(X)} \simeq (\bigoplus T_i) \oplus T''$ for some T''. It is easy to see that $T' = T^{(X)}$ satisfies the conditions (ii) and (iii) in the definition of classical tilting modules. It follows from [15, Proposition 1.4(2)], that $R \simeq \mathrm{End}(T'_{A'})$ and $T'_{A'}$ is classical tilting. In particular, by [1, 14.2], we have that $R \simeq \mathrm{BiEnd}(T^{(X)}) \simeq \mathrm{BiEnd(T)}\ (= \mathrm{End}(T_A))$ canonically.

(2) See for instance [15, Theorem 1.5]. □

2. Finitistic dimension of endomorphism rings

We first note the following relations between finitistic dimensions and restricted flat dimensions.

Lemma 2.1. *Assume that A is a ring and $T_A \in \mathrm{Mod}$-A.*

(1) $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA)$.
(2) $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA)$.

Proof. Clearly we need only to show that if $\mathrm{Fdim}({}_AA) = n < \infty$ (resp., $\mathrm{fdim}({}_AA) = n < \infty$), then $\mathrm{Rfd}(T_A) \leqslant n$ (resp., $\mathrm{rfd}(T_A) \leqslant n$).

(1) By the definition of the big restricted flat dimension, it is sufficient to show that $\mathrm{Tor}^A_{n+1}(T, M) = 0$ for any ${}_AM$ with finite flat dimension. Since $\mathrm{Fdim}({}_AA) = n < \infty$, we obtain that every module with finite flat dimension has finite projective dimension, by [12, Proposition 6]. It follows that $\mathrm{pd}({}_AM) \leqslant n$. Hence $\mathrm{Tor}^A_{n+1}(T, M) = 0$.

(2) Note that the flat dimension coincides with the projective dimension for every ${}_AM \in A$-mod. So the conclusion follows by definition. □

Lemma 2.2. *Let R be a ring and ${}_RT \in R$-Mod.*

(1) *If ${}_RT$ is selfsmall and coproduct-selforthogonal, then*
  (i) $\mathrm{Add}_R T$-$\mathrm{dim}({}_RM) = \mathrm{pd}({}_A\mathrm{Hom}_R(T, M))$, *for any $M \in \widehat{\mathrm{Add}_R T}$.*
  (ii) $\mathrm{Fdim}({}_RT) \leqslant \mathrm{id}(\mathrm{Add}_R T)$.

(2) *If $_RT$ is selforthogonal, then*
 (i) $\mathrm{add}_R T\text{-}\dim({}_RM) = \mathrm{pd}({}_A\mathrm{Hom}_R(T, M))$, *for any* $M \in \widehat{\mathrm{add}_R T}$.
 (ii) $\mathrm{fdim}({}_RT) \leqslant \mathrm{id}({}_RT)$.

Proof. See for instance [17, Lemmas 2.1 and 2.2]. □

Lemma 2.3. *Let R be a ring and $_RT \in R\text{-}\mathrm{Mod}$ with $A = \mathrm{End}_R T$.*

(1) *If $_RT$ is selfsmall and coproduct-selforthogonal, then, for any $Y \in A\text{-}\mathrm{Mod}$ such that $\mathrm{Tor}^A_{i\geqslant 1}(T, Y) = 0$ and $\mathrm{pd}({}_AY) < \infty$, it holds that $Y \simeq \mathrm{Hom}_R(T, T \otimes_A Y)$ canonically and $\mathrm{Add}_R T\text{-}\dim(T \otimes_A Y) < \infty$.*
(2) *If $_RT$ is selforthogonal, then, for any $Y \in A\text{-}\mathrm{mod}$ such that $\mathrm{Tor}^A_{i\geqslant 1}(T, Y) = 0$ and $\mathrm{pd}({}_AY) < \infty$, it holds that $Y \simeq \mathrm{Hom}_R(T, T \otimes_A Y)$ canonically and $\mathrm{add}_R T\text{-}\dim(T \otimes_A Y) < \infty$.*

Proof. (1) Since $\mathrm{pd}({}_AY) < \infty$, we can take a finite projective resolution of $_AY$, say,

$$0 \to P_y \to \cdots \to P_1 \to P_0 \to Y \to 0$$

with each P_i projective. Note $_AY \in \mathrm{KerTor}^A_{i\geqslant 1}(T, -)$ and $\mathrm{KerTor}^A_{i\geqslant 1}(T, -)$ is closed under kernels of epimorphisms, so we obtain the following exact sequence, by applying the functor $T \otimes_A -$:

$$0 \to T \otimes_A P_y \to \cdots \to T \otimes_A P_0 \to T \otimes_A Y \to 0.$$

Denote $T_i := T \otimes_A P_i$ for $0 \leqslant i \leqslant y$. Then each $T_i \in \mathrm{Add}_R T$. It follows that $\mathrm{Add}_R T$-$\dim(T \otimes_A Y) < \infty$. Moreover, by applying the functor $\mathrm{Hom}_R(T, -)$, we obtain the following exact sequence:

$$0 \to \mathrm{Hom}_R(T, T \otimes_A P_y) \to \cdots \to \mathrm{Hom}_R(T, T \otimes_A P_0) \to \mathrm{Hom}_R(T, T \otimes_A Y) \to 0,$$

as T is coproduct-selforthogonal and $\mathrm{KerExt}^{i\geqslant 1}_R(T, -)$ is closed under cokernels of monomorphisms. Since T is also selfsmall, we have that $P_i \simeq \mathrm{Hom}_R(T, T \otimes_A P_i)$ canonically, for each i. It follows that $_AY \simeq {}_A\mathrm{Hom}_R(T, T \otimes_A Y)$ canonically.

(2) Similarly. □

The following is one of our main results.

Theorem 2.4. *Let R be a ring and $_RT \in R\text{-}\mathrm{Mod}$ with $A = \mathrm{End}_R T$.*

(1) *If $_RT$ is selfsmall and coproduct-selforthogonal, then $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RT) + \mathrm{Rfd}(T_A)$.*
(2) *If $_RT$ is selforthogonal, then $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RT) + \mathrm{rfd}(T_A)$.*

Proof. (1) If $\mathrm{Fdim}({}_RT)$ or $\mathrm{Rfd}(T_A)$ is infinite, then we have nothing to say in this case. So we assume that $\mathrm{Fdim}({}_RT) = r < \infty$ and $\mathrm{Rfd}(T_A) = t < \infty$. Obviously we need only to show that $\mathrm{Fdim}({}_AA) \leqslant r + t$.

Let ${}_AY \in A$-Mod with $\mathrm{pd}({}_AY) < \infty$. By taking the projective resolution of ${}_AY$, we obtain an exact sequence

$$0 \to P_s \xrightarrow{f_s} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Y \to 0$$

with P_i projective for each $0 \leqslant i \leqslant s$. Denote by Y_i the ith syzygy, for each i. We claim now $\mathrm{pd}({}_AY_t) \leqslant r$ and so $\mathrm{pd}({}_AY) \leqslant t + r$. Then the conclusion will be followed from the arbitrarity of the choice of ${}_AY$.

Indeed, since $\mathrm{Rfd}(T_A) = t$, we easily obtain that ${}_AY_t \in \mathrm{KerTor}^A_{i\geqslant 1}(T, -)$. It is obvious that $\mathrm{pd}({}_AY_t) < \infty$. Hence, by Lemma 2.3, we have that ${}_AY_t \simeq {}_A\mathrm{Hom}_R(T, T \otimes_A Y_t)$ canonically and $\mathrm{Add}_R\,T$-$\dim(T \otimes_A Y_t) < \infty$. It follows that $\mathrm{Add}_R\,T$-$\dim(T \otimes_A Y_t) \leqslant \mathrm{Fdim}({}_RT) = r$. Now, by Lemma 2.2, we obtain that $\mathrm{pd}({}_AY_t) = \mathrm{pd}({}_A\mathrm{Hom}_R(T, T \otimes_A Y_t)) \leqslant r$, as desired.

(2) Similarly. □

Immediately by Lemma 2.2 and Theorem 2.4, we have the following corollary which extends [17, Theorem 2.5] where the global dimension is considered.

Corollary 2.5. *Let R be a ring and ${}_RT \in R$-Mod with $A = \mathrm{End}_R\,T$.*

(1) *If ${}_RT$ is selfsmall and coproduct-selforthogonal, then* $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA) \leqslant \mathrm{id}(\mathrm{Add}_R\,T) + \mathrm{Rfd}(T_A)$*, where* $\mathrm{id}(\mathrm{Add}_R\,T)$ *denotes the supremum of injective dimensions of modules in* $\mathrm{Add}_R\,T$.
(2) *If ${}_RT$ is selforthogonal, then* $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{id}({}_RT) + \mathrm{rfd}(T_A)$.

More special case of Theorem 2.4 is the following.

Corollary 2.6. *Let R be a ring and ${}_RT \in R$-Mod with $A = \mathrm{End}_R\,T$.*

(1) *If ${}_RT$ is selfsmall and coproduct-selforthogonal with* $\mathrm{Add}_R\,T$ *closed under cokernels of monomorphisms, then* $\mathrm{Fdim}({}_AA) = \mathrm{Rfd}(T_A)$.
(2) *If ${}_RT$ is selforthogonal with* $\mathrm{add}_R\,T$ *closed under cokernels of monomorphisms, then* $\mathrm{fdim}({}_AA) = \mathrm{rfd}(T_A)$.

Proof. It is easy to see that $\mathrm{Fdim}({}_RT) = 0$ in (1) and $\mathrm{fdim}({}_RT) = 0$ in (2). □

Of course, (selfsmall) injective modules over left noetherian rings satisfy assumptions in the above corollary (2) (resp., (1)). In particular, if A is an Artin algebra, then ${}_A(DA)$ is injective with $A \simeq \mathrm{End}_A(DA)$, where D is the usual duality in Artin algebras. Hence we have the following.

Corollary 2.7. *If A is an Artin algebra, then*

(1) $\mathrm{Fdim}({}_AA) = \mathrm{Rfd}(DA)_A$.
(2) $\mathrm{fdim}({}_AA) = \mathrm{rfd}(DA)_A$.

Let R be a left noetherian ring and ${}_RT \in R$-Mod. We say ${}_RT$ is fdim-test tilting if ${}_RT$ is tilting such that $\mathrm{KerExt}^{i\geqslant 1}_R(T, -) = \mathrm{KerExt}^{i\geqslant 1}_R(\mathcal{P}^{<\infty}, -)$, where $\mathcal{P}^{<\infty}$ is the category of modules in R-mod of finite projective dimension. The fdim-test tilting modules were studied in [3], where it was proved that $\mathrm{fdim}({}_RR) < \infty$ if and only if there exists a fdim-test tilting module ${}_RT$ [3, Theorem 2.6]. In this case, $\mathrm{fdim}({}_RR) = \mathrm{pd}({}_RT)$. If R is an Artin algebra, then there exists a classical fdim-test tilting module ${}_RT$ if and only if $\mathcal{P}^{<\infty}$ is contravariantly finite in R-mod [3, Theorem 4.2].

Proposition 2.8. *Let R be an Artin algebra. Assume that there is a classical fdim-test tilting module ${}_RT$ with $A = \mathrm{End}_R T$. Then $\mathrm{add}_R T$ is closed under cokernels of monomorphisms. In particular,* $\mathrm{fdim}({}_AA) = \mathrm{rfd}(T_A) \leqslant \mathrm{pd}({}_RT) = \mathrm{fdim}({}_RR)$.

Proof. Let $0 \to T_0 \to T_1 \to M \to 0$ be exact with $T_0, T_1 \in \mathrm{add}_R T$. Then ${}_RM \in R$-mod and $\mathrm{pd}({}_RM) < \infty$, that is, $M \in \mathcal{P}^{<\infty}$. Since ${}_RT$ is fdim-test tilting, we have that ${}_RT \in \mathrm{KerExt}_R^{i\geqslant 1}(T, -) = \mathrm{KerExt}_R^{i\geqslant 1}(\mathcal{P}^{<\infty}, -) \subseteq \mathrm{KerExt}_R^{i\geqslant 1}(M, -)$. It follows that $\mathrm{Ext}_R^1(M, T) = 0$. Hence the above exact sequence splits and ${}_RM \in \mathrm{add}_R T$, i.e., $\mathrm{add}_R T$ is closed under cokernels of monomorphisms.

The remained part follows from Lemma 1.2(2) and Corollary 2.6. □

Remark 2.9. Following [9], if R is a standardly stratified algebra such that its Ringel dual is a properly stratified algebra, then there is a classical fdim-test tilting module ${}_RT$. It was asked in [14, 6.4] if there is any relation between $\mathrm{fdim}({}_RR)$ and $\mathrm{fdim}({}_AA)$, where $A = \mathrm{End}_R T$. The above result gives a partial answer to this question.

We now apply Theorem 2.4 to (coproduct-)selforthogonal modules of finite projective dimension.

Proposition 2.10. *Let R be a ring and ${}_RT \in R$-*Mod *with $A = \mathrm{End}_R T$.*

(1) *If ${}_RT$ is selfsmall and coproduct-selforthogonal and of finite projective dimension, then* $\mathrm{Rfd}(T_A) \leqslant \mathrm{Fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{Rfd}(T_A)$.
(2) *If ${}_RT$ is selforthogonal and of finite projective dimension, then* $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{rfd}(T_A)$. *If additionally ${}_RT \in R$-*mod, *i.e., ${}_RT$ is classical partial tilting, then* $\mathrm{rfd}(T_A) \leqslant \mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{rfd}(T_A)$.

Proof. (1) By Theorem 2.4, it is sufficient to show that $\mathrm{Fdim}({}_RT) \leqslant \mathrm{Fdim}({}_RR)$. Obviously, we may assume that $\mathrm{Fdim}({}_RR) = t < \infty$.

For any $M \in \widehat{\mathrm{Add}_R T}$, we have an exact sequence

$$0 \to T_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$$

with each $T_i \in \mathrm{Add}_R T$. Since $\mathrm{pd}({}_RT) < \infty$, $\mathrm{pd}({}_RM) < \infty$ too. Hence $\mathrm{pd}({}_RM) \leqslant t$. Moreover, we have that each $\mathrm{Im}\, f_i \in \mathrm{KerExt}_R^{i\geqslant 1}(T, -)$, as T is coproduct-selforthogonal. Suppose now $m > t$, then we obtain that $\mathrm{Ext}_R^1(\mathrm{Im}\, f_t, \mathrm{Im}\, f_{t+1}) \simeq \mathrm{Ext}_R^{t+1}(M, \mathrm{Im}\, f_{t+1}) = 0$ by the dimension shifting. It follows that $\mathrm{Im}\, f_t \in \mathrm{Add}_R T$ and consequently, $\mathrm{Add}_R T\text{-}\dim({}_RM) \leqslant t$. Hence $\mathrm{Fdim}({}_RT) \leqslant \mathrm{Fdim}({}_RR)$ by definition.

(2) Similarly. □

In particular, we obtain the following corollary which contains [20, Theorems 1.2], where the conclusion is proved under assumptions that T_A is of finite $*$-syzygy dimension (or Gorenstein projective dimension or projective dimension).

Corollary 2.11. *Let R be a ring and ${}_RT \in R$-Mod with $A = \mathrm{End}_R T$.*

(1) *If ${}_RT$ is selfsmall and projective, then* $\mathrm{Fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{Rfd}(T_A)$.
(2) *If ${}_RT$ is projective, then* $\mathrm{fdim}({}_AA) \leqslant \mathrm{Fdim}({}_RR) + \mathrm{rfd}(T_A)$. *If additionally ${}_RT$ is finitely generated, then* $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{rfd}(T_A)$.

We note that, in general, the restricted flat dimension may be strictly smaller than the $*$-syzygy dimension (Gorenstein projective dimension, projective dimension), as the following example shows.

Example 2.12. There exists a finite dimensional algebra A satisfying the following statement.

(1) There is a right A-module T_A such that $\mathrm{rfd}(T_A) = \mathrm{Rfd}(T_A) < \mathrm{Gpd}(T_A) = \mathrm{pd}(T_A) < \infty$.
(2) There is a right A-module U_A such that $\mathrm{rfd}(U_A) = \mathrm{Rfd}(U_A) < *\text{-}\mathrm{sd}(U_A)$ strictly.

Proof. By [8], for any arbitrary finite numbers m and n, there is a finite dimensional algebra A with $\mathrm{fdim}({}_AA) = \mathrm{Fdim}({}_AA) = m$ and $\mathrm{fdim}(A_A) = \mathrm{Fdim}(A_A) = n$. Let us take $m = 0$ and $n > 0$. Thus $\mathrm{rfd}(T_A) = \mathrm{Rfd}(T_A) = 0$ for all right A-module T, by Lemma 2.1.

For (1), we can take $T_A \in \mathrm{mod}\text{-}A$ to be of projective dimension exactly n. Combining with Lemma 1.1, we obtain that $0 = \mathrm{rfd}(T_A) = \mathrm{Rfd}(T_A) < \mathrm{Gpd}(T_A) = \mathrm{pd}(T_A) < \infty$.

For (2), we take $U_A = (DA)_A$. We need only to show that $(DA)_A$ is not a $*$-syzygy module and it then follows that $0 = \mathrm{rfd}(U_A) = \mathrm{Rfd}(U_A) < *\text{-}\mathrm{sd}(U_A)$ strictly. Suppose U_A is a $*$-syzygy module, then there is a monomorphism $U_A \to A'$ with A'_A projective by definition. Since $U_A = (DA)_A$ is injective, we obtain that U_A is a direct summand of A' and hence is projective. Consequently, every injective right module is projective. Hence A is self-injective by the Faith–Walker Theorem. It then follows $\mathrm{fdim}(A_A) = 0$, a contradiction. □

Remark 2.13. It was conjectured in [6] that, for any ring A, if $\mathrm{Gfd}(T_A)$ is finite, then $\mathrm{Gfd}(T_A) = \mathrm{Rfd}(T_A)$. The conjecture was proved in case that A is commutative noetherian [11] and that A is a coherent ring with finite Gorenstein weak dimension [4]. However, by the above example, one sees that the conjecture fails in general (note that $\mathrm{Gfd}(T_A) = \mathrm{pd}(T_A)$ in case $T_A \in \mathrm{mod}\text{-}A$, A is noetherian and $\mathrm{pd}(T_A) < \infty$, by Lemma 1.1).

Recall that for an Artin algebra R, ${}_RT \in R\text{-}\mathrm{mod}$ is classical cotilting if (i) $\mathrm{id}({}_RT) < \infty$, (ii) ${}_RT$ is selforthogonal and (iii) there is an exact sequence $0 \to T_n \to \cdots \to T_0 \to {}_R(DR) \to 0$ for some n with each $T_i \in \mathrm{add}_R T$, where D is the usual duality in Artin algebras. It is easy to see that ${}_RT$ is classical tilting if and only if $(DT)_R$ is classical cotilting. Note that all classical tilting modules are classical cotilting when the global dimension of R is finite.

Let R be an Artin algebra. A well-known result in the classical tilting theory is that $\mathrm{gd}\, R - \mathrm{pd}_R\, T \leqslant \mathrm{gd}\, A \leqslant \mathrm{gd}\, R + \mathrm{pd}_R\, T$ in case T is classical tilting with $A = \mathrm{End}_R\, T$, where $\mathrm{gd}\, R$ and $\mathrm{gd}\, A$ denote respectively the global dimensions of R and A, see for instance [10] or [15]. The result was improved in [7] where it was shown that $\mathrm{id}_R\, T \leqslant \mathrm{gd}\, A \leqslant \mathrm{pd}_R\, T + \mathrm{id}_R\, T$. The following proposition is a generalization of the latter result to finitistic dimension.

Proposition 2.14. *Let R, A be Artin algebras and $T \in R$-mod with $A = \mathrm{End}_R T$. If ${}_R T$ is classical tilting and cotilting, then*

$$\max\{\mathrm{fdim}({}_R R) - \mathrm{id}({}_R T), \mathrm{id}({}_R T)\} \leqslant \mathrm{fdim}({}_A A) \leqslant \mathrm{pd}({}_R T) + \mathrm{id}({}_R T).$$

Proof. Since $\mathrm{rfd}(T_A) \leqslant \mathrm{pd}(T_A) = \mathrm{pd}({}_R T)$, the upper-bound part follows from Corollary 2.5. Consider now the classical tilting and cotilting module ${}_A(DT)_R$, then we have that $\mathrm{fdim}({}_R R) \leqslant \mathrm{fdim}({}_A A) + \mathrm{rfd}((DT)_R) \leqslant \mathrm{fdim}({}_A A) + \mathrm{pd}((DT)_R)$ by Proposition 2.10. It follows that $\mathrm{fdim}({}_R R) - \mathrm{pd}((DT)_R) \leqslant \mathrm{fdim}({}_A A)$. Note that $\mathrm{pd}((DT)_R) = \mathrm{id}({}_R T)$, so we have that $\mathrm{fdim}({}_R R) - \mathrm{id}({}_R T) \leqslant \mathrm{fdim}({}_A A)$. Note also that $\mathrm{fdim}({}_A A) \geqslant \mathrm{pd}({}_A(DT)) = \mathrm{pd}((DT)_R) = \mathrm{id}({}_R T)$, so the lower-bound part follows. □

3. Finitistic dimension of fixed subrings

The section concerns the finitistic dimension of some fixed subrings.

Let A be a ring and $T \in \mathrm{Mod}\text{-}A$. We define the Tor-bound dimension of T_A, denoted by $\mathrm{Tbd}(T_A)$, to be the minimal nonnegative integer n such that $\mathrm{Tor}_p^A(T, M) = 0$ for all $p \geqslant n + 1$ whenever $\mathrm{Tor}_p^A(T, M) = 0$ for all p sufficiently large. The Tor-bound dimension of T_A is nothing but the minimal bound on the vanishing of $\mathrm{Tor}^A(T, -)$, see [13].

The following lemma shows that the Tor-bound dimension is a refinement of flat dimension.

Lemma 3.1. *Let A be a ring and $T \in \mathrm{Mod}\text{-}A$. Then $\mathrm{Rfd}(T_A) \leqslant \mathrm{Tbd}(T_A) \leqslant \mathrm{fd}(T_A)$. If $\mathrm{fd}(T_A) < \infty$, then $\mathrm{Tbd}(T_A) = \mathrm{fd}(T_A)$.*

Proof. The inequality is obvious. If $\mathrm{fd}(T_A) < \infty$, then $\mathrm{Tor}_p^A(T, M) = 0$ for all $p \geqslant \mathrm{fd}(T_A) + 1$ and all modules ${}_A M$. It follows that $\mathrm{Tor}_p^A(T, M) = 0$ for all $p \geqslant \mathrm{Tbd}(T_A) + 1$ and all modules ${}_A M$, by the definition of the Tor-bound dimension. The latter is indeed equivalent to say that $\mathrm{fd}(T_A) \leqslant \mathrm{Tbd}(T_A)$. □

Combining the above result and Example 2.12, we see that the restricted flat dimension is often strictly smaller than the Tor-bound dimension. In view of this point, most results in [13] can be extended. For example, the following proposition is a generalization of [13, Theorem 6].

Proposition 3.2. *Let A be a subring of a ring R such that A is an A–A bimodule direct summand of R. Then*

(1) $\mathrm{Fdim}({}_A A) \leqslant \mathrm{Fdim}({}_R R) + \mathrm{Rfd}(R_A)$.
(2) $\mathrm{fdim}({}_A A) \leqslant \mathrm{fdim}({}_R R) + \mathrm{rfd}(R_A)$.

Proof. The proof is just similar to that of [13, Theorem 6]. □

In the rest, we concentrate ourself on Artin algebras.

Let A, R be both Artin algebras. Following [18], we say that R is a left (resp., right) idealized extension of A if $A \subseteq R$ has the same identity and $\mathrm{rad}\, A$ is a left (resp., right) ideal in R. The following result is very important in studying the finitistic dimension conjecture in Artin algebras in connection with idealized extensions.

Lemma 3.3. *(See [18, Erratum, Lemma 0.1].) Let R be a left (resp., right) idealized extension of A. If $X \in A\text{-}\mathrm{mod}$ (resp., $X \in \mathrm{mod}\text{-}A$), then $\Omega_A^2(X) \in R\text{-}\mathrm{mod}$ (resp., $\Omega_A^2(X) \in \mathrm{mod}\text{-}R$). Hereafter, the symbol $\Omega_A^i(X)$ denotes the ith syzygy of the A-module X.*

Let A, R be both k-algebras such that R is a left idealized extension of A, where k is the residue field of a discrete valuation ring. It was shown in [18] that, if $\mathrm{fdim}({}_RR) < \infty$ implies $\mathrm{fdim}({}_AA) < \infty$ for all A, R satisfying the above assumptions, then $\mathrm{fdim}({}_AA) < \infty$ for all k-algebras A.

Proposition 3.4. *Let R be a left idealized extension of A. Then $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{pd}({}_AR) + \max\{\mathrm{rfd}(R_A), 2\}$. In particular, if both ${}_AR$ and R_A are of finite projective dimension, then $\mathrm{fdim}({}_AA)$ is finite provided that $\mathrm{fdim}({}_RR)$ is finite.*

Proof. Let $t = \max\{\mathrm{rfd}(R_A), 2\}$ and $r = \mathrm{fdim}({}_RR)$. Then, for any ${}_AM$ of finite projective dimension, we obtain that ${}_AM_t := \Omega_A^t(M) \in \mathrm{KerTor}_{i\geqslant 1}^A(R, -)$ by the dimension shifting, since $t \geqslant \mathrm{rfd}(R_A)$. Let $0 \to A_m \to \cdots \to A_0 \to M_t \to 0$ be exact with each ${}_AA_i$ projective. Then we have an induced exact sequence $0 \to {}_RR \otimes_A A_m \to \cdots \to {}_RR \otimes_A A_0 \to {}_RR \otimes_A M_t \to 0$. Obviously, ${}_RR \otimes_A A_i \in \mathrm{add}_R R$ for all $i \geqslant 0$. It follows that $\mathrm{pd}({}_RR \otimes_A M_t) \leqslant \mathrm{fdim}({}_RR) = r$. Consequently, we have an exact sequence $0 \to R_r \to \cdots \to R_0 \to {}_RR \otimes_A M_t \to 0$ with each ${}_RR_i$ projective. It restricts to an exact sequence in A-modules $0 \to R_r \to \cdots \to R_0 \to {}_AR \otimes_A M_t \to 0$, since $A \subseteq R$ has the same identity. Thus we have that $\mathrm{pd}({}_AR \otimes_A M_t) \leqslant \mathrm{pd}({}_AR) + r$. Since $t \geqslant 2$, ${}_AM_t$ is also a left R-module by Lemma 3.3. Consequently, ${}_AM_t \simeq {}_A\mathrm{Hom}_R({}_RR_A, M_t)$. Note that the canonical homomorphism ${}_A\mathrm{Hom}_R({}_RR_A, M_t)$ $(\simeq {}_AM_t) \to \mathrm{Hom}_R({}_RR_A, {}_RR \otimes_A \mathrm{Hom}_R({}_RR_A, M_t))$ $(\simeq \mathrm{Hom}_R({}_RR_A, {}_RR \otimes_A M_t) \simeq {}_AR \otimes_A M_t)$ is a split monomorphism, so that ${}_AM_t$ is a direct summand of ${}_AR \otimes_A M_t$. It follows that $\mathrm{pd}({}_AM_t) \leqslant \mathrm{pd}({}_AR) + r$. Therefore, $\mathrm{pd}({}_AM) \leqslant \mathrm{pd}({}_AR) + r + t$. □

Corollary 3.5. *Assume that $A_0 \subseteq \cdots \subseteq A_m$ are Artin algebras such that A_{i+1} is a left idealized extension of A_i, for each $0 \leqslant i \leqslant m-1$. If ${}_{A_i}A_{i+1}$ and ${A_{i+1}}_{A_i}$ are of finite projective dimension, for all $0 \leqslant i \leqslant m-1$, then $\mathrm{fdim}({}_{A_0}A_0)$ is finite provided that $\mathrm{fdim}({}_{A_m}A_m)$ is finite.*

The following result can be compared with [19, Theorem 3.1], which states that, if $A_0 \subseteq \cdots \subseteq A_m$ are Artin algebras such that, for each $0 \leqslant i \leqslant m-1$, A_{i+1} is a left idealized extension of A_i and ${}_{A_i}A_{i+1}$ is of finite projective dimension, then $\mathrm{fdim}({}_{A_0}A_0)$ is finite provided that $\mathrm{gd}(A_m)$ is finite, where $\mathrm{gd}(A_m)$ denotes the global dimension of A_m.

Theorem 3.6. *Let R be a right idealized extension of A. Then $\mathrm{fdim}({}_AA) \leqslant \mathrm{fdim}({}_RR) + \mathrm{rfd}(R_A) + 2$. In particular, if R_A is of finite projective dimension (or finite $*$-syzygy dimension or finite Gorenstein projective dimension or finite Tor-bound dimension), then $\mathrm{fdim}({}_AA)$ is finite provided that $\mathrm{fdim}({}_RR)$ is finite.*

Proof. Let $t = \mathrm{rfd}(R_A)$ and $r = \mathrm{fdim}({}_RR)$. Similarly as in the proof of Proposition 3.4, for any ${}_AM$ of finite projective dimension, we obtain that ${}_AM_t := \Omega_A^t(M) \in \mathrm{KerTor}_{i\geqslant 1}^A(R, -)$ and that $\mathrm{pd}({}_RR \otimes_A M_t) \leqslant \mathrm{fdim}({}_RR) = r$.

Claim. *$\Omega_A^{-2}(Y) \in R\text{-}\mathrm{mod}$, for any ${}_AY \in A\text{-}\mathrm{mod}$, where $\Omega_A^{-2}(Y)$ denotes the second cosyzygy in the minimal injective resolution of ${}_AY$.*

Proof of the Claim. Note that $(DY)_A \in \text{mod-}A$, where D is the usual duality in Artin algebras. Thus $\Omega_A^2(DY) \in \text{mod-}R$ by Lemma 3.3. It follows that $\Omega_A^{-2}(Y) \simeq D\Omega_A^2(DY) \in R\text{-mod}$. □

Now, for any ${}_AY \in A\text{-mod}$, we obtain that $\mathrm{Ext}_A^{r+3}(M_t, Y) \simeq \mathrm{Ext}_A^{r+1}(M_t, \Omega_A^{-2}(Y)) \simeq \mathrm{Ext}_R^{r+1}({}_RR \otimes_A M_t, \Omega_A^{-2}(Y)) = 0$, by the above arguments and [5, Chapter VI, Proposition 4.1.3, p. 118]. It follows that $\mathrm{pd}({}_AM_t) \leqslant r+2$ and consequently, $\mathrm{pd}({}_AM) \leqslant t+r+2$. □

Corollary 3.7. *Assume that* $A_0 \subseteq \cdots \subseteq A_m$ *are Artin algebras such that, for each* $0 \leqslant i \leqslant m-1$, A_{i+1} *is a right idealized extension of* A_i *and* $\mathrm{rfd}(A_{i+1_{A_i}}) < \infty$ *(e.g.,* $A_{i+1_{A_i}}$ *is of finite projective dimension, or finite* $*$*-syzygy dimension, or finite Gorenstein projective dimension, or finite Tor-bound dimension), then* $\mathrm{fdim}({}_{A_0}A_0)$ *is finite provided that* $\mathrm{fdim}({}_{A_m}A_m)$ *is finite.*

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