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Partial quasi-metrics

H.-P.A. Künzi[a,*], H. Pajoohesh[b], M.P. Schellekens[c]

[a] Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa
[b] Centre for Efficiency-Oriented Languages, Lancaster Hall, 6 Little Hanover Street, Cork, Ireland
[c] Department of Computer Science, National University of Ireland, Cork, Ireland

Abstract

In this article we introduce and investigate the concept of a partial quasi-metric and some of its applications. We show that many important constructions studied in Matthews's theory of partial metrics can still be used successfully in this more general setting. In particular, we consider the bicompletion of the quasi-metric space that is associated with a partial quasi-metric space and study its applications in groups and BCK-algebras.

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1. Introduction

Partial metrics were introduced by Matthews [10] in 1992. They generalize the concept of a metric space in the sense that the self-distance from a point to itself need not be equal to zero. They are useful in modelling partially defined information, which often appears in computer science. They are equivalent to weighted quasi-metrics, which will be discussed in Remark 9 below. Künzi and Vajner gave a sufficient condition for topologies to be partially metrizable.

Several variants of the concept of a partial metric have already been studied in the literature. O'Neill [12] allowed partial metrics to admit negative values.

Heckmann [2] omitted conditions (1a,1b) in the definition of a partial metric given below (see Remark 2) and obtained the concept of a weak partial metric. Moreover, by omitting condition (2) below in the definition of a partial metric, Waszkiewicz [21] introduced the concept of a partial semi-metric.

Later on those partial metrics whose range is a value quantale were investigated [5]. Value quantales are a generalization of the set of the extended positive real numbers and it was shown that every topology comes from a partial metric mapping into a value quantale.

Interesting representations of partial metrics were studied in [6,20]. In particular [6] describes connections between partial metric spaces and metric spaces with a base point.

* Corresponding author. Tel.: +27 21 6503187; fax: +27 21 6502334.
E-mail addresses: kunzi@maths.uct.ac.za (H.-P.A. Künzi), hp1@cs.ucc.ie (H. Pajoohesh), m.schellekens@cs.ucc.ie (M.P. Schellekens).

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Further applications of partial metrics to problems in Theoretical Computer Science were discussed in [15,19,18]. By dropping the symmetry condition in the definition of a partial metric, in this article we study another variant of partial metrics, namely partial quasi-metrics.

2. Preliminaries

We recall that given a set X, a function $q: X \times X \to [0, \infty)$ is called a *quasi-metric* if and only if

(i) $x = y$ iff $q(x, y) = 0 = q(y, x)$ whenever $x, y \in X$,

(ii) $q(x, z) \leqslant q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

Here $[0, \infty)$ denotes the set of non-negative reals.

If a function $q: X \times X \to [0, \infty)$ only satisfies condition (ii) and possesses the additional property that $q(x, x) = 0$ whenever $x \in X$, then we speak of a *quasi-pseudometric* on X.

Example 1. Let X be a set and let $f : X \to [0, \infty)$ be an arbitrary function. Set $q_f(x, y) = \max\{f(y) - f(x), 0\}$ whenever $x, y \in X$. Then q_f is a quasi-pseudometric on X. Observe that q_f is a quasi-metric if and only if f is one-to-one.

The *topology* $\tau(q)$ induced by a quasi-(pseudo)metric q on X is determined by the base consisting of all ε-balls $B^q_\varepsilon(x) = \{y \in X : q(x, y) < \varepsilon\}$ where $x \in X$ and $\varepsilon \in (0, \infty)$. If q is a quasi-metric, then the partial order $\leqslant$ on X given by $x \leqslant y$ iff $q(x, y) = 0$ is called the *specialization order*.

Remark 1. A quasi-metric space (X, q) with some fixed base point $\phi \in X$ gives rise to another quasi-metric q' on X defined by $q'(x, y) = q(\phi, x) + q(x, y) - q(\phi, y)$ whenever $x, y \in X$.

Proof. The triangle inequality is clearly satisfied by q'. We only verify the nontrivial implication of condition (i): Suppose that for $x, y \in X$ we have $q(\phi, x) + q(x, y) - q(\phi, y) = 0$ and $q(\phi, y) + q(y, x) - q(\phi, x) = 0$. Then $q(\phi, y) + q(y, x) + (q(x, y) - q(\phi, y)) = 0$. Thus $q(y, x) + q(x, y) = 0$ and $q(y, x) = q(x, y) = 0$. Hence $x = y$. $\square$

Let us note that ϕ is the smallest element of (X, q') with respect to the specialization order on X defined by $x \leqslant y$ iff $q'(x, y) = 0$, since $q'(\phi, y) = 0$ whenever $y \in X$. So the induced quasi-metric topology as well as the specialization order of (X, q) and (X, q') are distinct in general. Observe that $q = q'$ iff ϕ is the smallest element of (X, q) with respect to its specialization order. In particular our construction applied to (X, q', ϕ) where ϕ is chosen again as base point yields $(q')' = q'$.

We are now ready to introduce our main new concept.

Definition 1. A *partial quasi-metric* on a set X is a function $p: X \times X \to [0, \infty)$ such that:

(1a) $p(x, x) \leqslant p(x, y)$ whenever $x, y \in X$,

(1b) $p(x, x) \leqslant p(y, x)$ whenever $x, y \in X$,

(2) $p(x, z) + p(y, y) \leqslant p(x, y) + p(y, z)$ whenever $x, y, z \in X$,

(3) $x = y$ iff $(p(x, x) = p(x, y)$ and $p(y, y) = p(y, x))$ whenever $x, y \in X$.

If p satisfies all these conditions except possibly (1b), we shall speak of a *lopsided partial quasi-metric*.

Remark 2. If p is a partial quasi-metric on X satisfying

(4) $p(x, y) = p(y, x)$ whenever $x, y \in X$, then p is called a *partial metric* on X.

Lemma 1. *Any lopsided partial quasi-metric p on X also satisfies the condition*:

(3′) $x = y$ iff $(p(x, x) = p(y, x)$ *and* $p(y, y) = p(x, y))$ *whenever* $x, y \in X$.

Proof. In order to prove the nontrivial implication, suppose that for $x, y \in X$, $p(y, x) = p(x, x)$ and $p(x, y) = p(y, y)$. Then $p(y, x) - p(y, y) + p(x, y) - p(x, x) = 0$ and thus by (1a) $p(y, x) = p(y, y)$ and $p(x, y) = p(x, x)$. Therefore $x = y$ by (3). $\square$

Remark 3. Thus in particular every partial quasi-metric p on X also satisfies condition (3′). Similarly as in the preceding proof we see that if a function $p: X \times X \to [0, \infty)$ fulfills conditions (1a) and (1b), then it satisfies (3) iff it satisfies (3′).

Lemma 2. (*a*) *Each quasi-metric p on X is a partial quasi-metric on X with* $p(x, x) = 0$ *whenever* $x \in X$.

(*b*) *If p is a* (*partial*) *quasi-metric on X, then its conjugate* $p^{-1}(x, y) = p(y, x)$ *whenever* $x, y \in X$ *is a* (*partial*) *quasi-metric on X*.

(*c*) *If p is a* (*partial*) *quasi-metric on X, then* p^+ *defined by* $p^+(x, y) = p(x, y) + p^{-1}(x, y)$ *whenever* $x, y \in X$ *is a* (*partial*) *metric on X*.

Proof. The arguments are straightforward. As an illustration we prove for the statement (c) the nontrivial part of condition (3). Let $x, y \in X$ such that $p^+(x, x) = p^+(x, y)$ and $p^+(y, y) = p^+(y, x)$. Then $0 = (p(x, y) - p(x, x)) + (p(y, x) - p(x, x))$ and $0 = (p(y, x) - p(y, y)) + (p(x, y) - p(y, y))$. By (1a) and (1b) the four expressions in brackets must be equal to 0. Thus by (3) or (3′) we conclude that $x = y$. □

Remark 4. An arbitrary quasi-metric space (X, q) equipped with an arbitrary (so-called weight) function $w : X \to [0, \infty)$ will be called a *quasi-metric space with weight*. (It should be stressed that no condition of compatibility is assumed at this stage.)

The following construction describes a bijection between quasi-metrics with weight and lopsided partial quasi-metrics on X that will be used throughout this article. In the following we shall refer to this correspondence often by terms like the *quasi-metric or weight associated with a given* (*lopsided*) *partial quasi-metric*, *the* (*lopsided*) *partial quasi-metric associated with a given quasi-metric with weight* and similar self-explanatory expressions.

If p is a lopsided partial quasi-metric on X, then $q(x, y) = p(x, y) - p(x, x)$ whenever $x, y \in X$ and $w(x) = p(x, x)$ whenever $x \in X$ yield a quasi-metric space (X, q) with weight w, which we denote by (X, q, w).

If (X, q, w) is a quasi-metric space with weight, then $p(x, y) = q(x, y) + w(x)$ whenever $x, y \in X$ is a lopsided partial quasi-metric on X.

Next we define a compatibility condition between quasi-metric and weight that will be crucial for the following investigations.

Definition 2. A *quasi-*(*pseudo*)*metric space with compatible weight* on a set X is a triple (X, q, w) where $q: X \times X \to [0, \infty)$ is a quasi-(pseudo)metric on X and $w: X \to [0, \infty)$ is a function satisfying $w(y) \leqslant q(x, y) + w(x)$ whenever $x, y \in X$.

Remark 5. Let (X, q, w) be a quasi-metric space with weight. Then clearly w is a compatible weight on X if and only if $\hat{q}$ defined by $\hat{q}(x, y) = q(x, y) + w(x) - w(y)$ whenever $x, y \in X$ is a quasi-metric on X.

Hence for each partial quasi-metric p on a set X we obtain besides its associated quasi-metric $p(x, y) - p(x, x)$ also the quasi-metric $\hat{q}(x, y) = p(x, y) - p(y, y)$ where $x, y \in X$.

Indeed we shall see in Theorem 1 below that $\hat{q}$ is equal to $\overline{q}^{-1}$ where $\overline{q}$ is the quasi-metric associated with the conjugate partial quasi-metric p^{-1}. The weight $w(x) = p(x, x)$ $(x \in X)$ need not be compatible with $\hat{q}$, as Example 6 below shows in which $\hat{\rho} = (\overline{\rho})^{-1} = \rho^{-1}$.

Example 2. Let (X, q) be a quasi-metric space with weight $w: X \to [0, \infty)$. Then w is a compatible weight on the quasi-metric space (X, q') where q' denotes the quasi-metric on X defined by $q'(x, y) = q(x, y) + q_w(x, y)$ whenever $x, y \in X$ (compare Example 1).

Remark 6. According to Example 2 for any lopsided partial quasi-metric p with associated quasi-metric q and weight w on a set X we obtain a partial quasi-metric p' on X by setting $p'(x, y) = q(x, y) + q_\omega(x, y) + w(x) = p(x, y) + q_\omega(x, y)$ whenever $x, y \in X$. (Here $w(x) = p(x, x)$ whenever $x \in X$.)

Remark 7. (a) If w is a weight compatible with a given quasi-metric on X, then for any non-negative constant k, $w(x) + k$ whenever $x \in X$, is also a compatible weight; of course, the constant weight function 0 is compatible with any quasi-metric on X.

(b) Let (X, q) be any quasi-metric space. Suppose that A is a nonempty subset of X. For any $x \in X$ set $q(A, x) = \inf\{q(a, x): a \in A\}$ and define $w(x) = q(A, x)$ whenever $x \in X$. Then w is a compatible weight on (X, q).

(c) Let (X, q) be a quasi-metric space with specialization order defined as above by $x \leqslant y$ iff $q(x, y) = 0$. Then any compatible weight $w : X \to [0, \infty)$ is decreasing.

(d) If w is a compatible weight on a quasi-metric space (X, q) and x_0 is the maximum with respect to the specialization order on (X, q), then $w'(x) = w(x) - w(x_0)$ is a compatible weight on (X, q) vanishing at x_0.

Proof. The statements in (a) are obvious and the assertion (b) immediately follows from the inequality $q(A, y) \leqslant q(A, x) + q(x, y)$, which holds for any $x, y \in X$. Similarly, for (c) we have $w(y) - w(x) \leqslant q(x, y) = 0$ provided that $x \leqslant y$ and, thus, w is decreasing.

(d) By assumption clearly $w'(x_0) = 0$. Since w is decreasing, $w'(x) \geqslant 0$ whenever $x \in X$. Furthermore $w'(y) - w'(x) = w(y) - w(x) \leqslant q(x, y)$ whenever $x, y \in X$ and hence w' is compatible on (X, q). □

Proposition 1. *Let (X, q, w) be a quasi-metric space (X, q) with compatible weight w. Then $w' = 1/(1 + w)$ is a compatible weight on the quasi-metric space (X, q^{-1}).*

If w is bounded by the non-negative real number m, then (X, q^{-1}) is a quasi-metric space with compatible weight w'' where w'' is defined by $w''(x) = m - w(x)$ whenever $x \in X$.

Proof. We have $w'(y) - w'(x) = 1/(1 + w(y)) - 1/(1 + w(x)) \leqslant w(x) - w(y) \leqslant q^{-1}(x, y)$ whenever $x, y \in X$. Hence the assertion holds.

Suppose now that w is bounded. Since, by compatibility of w, we have $w(x) - w(y) \leqslant q(y, x) = q^{-1}(x, y)$ whenever $x, y \in X$, we conclude that $w''(y) - w''(x) \leqslant q^{-1}(x, y)$ whenever $x, y \in X$. Therefore the statement is verified. □

Proposition 2. *If (X, q, w) is a quasi-metric space with compatible weight, then $(X, \hat{q}, \hat{w})$ where*

(a) $\widehat{q}(x, y) = \min\{q(x, y), 1\}$ *and* $\widehat{w}(x) = \min\{w(x), 1\}$ *whenever $x, y \in X$, and*

(b) $\widehat{q}(x, y) = \frac{q(x,y)}{1+q(x,y)}$ *and* $\widehat{w}(x) = \frac{w(x)}{1+w(x)}$ *whenever $x, y \in X$, is also a quasi-metric space with compatible weight.*

Proof. For either case it is well known and easy to see that $\widehat{q}$ is a quasi-metric on X. Related arguments show that given $x, y \in X$, $w(y) \leqslant q(x, y) + w(x)$ implies in case (a) that $\min\{w(y), 1\} \leqslant \min\{q(x, y), 1\} + \min\{w(x), 1\}$ and in case (b) by monotonicity that $w(y)/(1 + w(y)) \leqslant (q(x, y) + w(x))/(1 + q(x, y) + w(x)) \leqslant q(x, y)/(1 + q(x, y)) + w(x)/(1 + w(x))$. Consequently the statements are verified. □

Lemma 3. *Let (X, q) be a quasi-metric space and fix $\phi \in X$. Then $x \mapsto q(\phi, x)$ is maximal (with respect to the pointwise order on the set of compatible weight functions) among the compatible weights w on (X, q) that vanish at ϕ.*

Proof. The assertion follows immediately, since a compatible weight function $w: X \to [0, \infty)$ vanishing at ϕ satisfies the inequality $w(x) = w(x) - w(\phi) \leqslant q(\phi, x)$ whenever $x \in X$ and $x \mapsto q(\phi, x)$ is a compatible weight vanishing at ϕ. □

Recall that a weight w is an *upper weighting function* on a quasi-metric space (X, q) in the sense of the third author if $q(x, y) \leqslant w(y)$ whenever $x, y \in X$ (compare [16,17]).

Proposition 3. *Let (X, q, w) be a quasi-metric space with weight. Then w is a compatible upper weighting function vanishing at ϕ on X if and only if $w(x) = q(\phi, x)$ whenever $x \in X$ and ϕ is the maximum on (X, q) with respect to the specialization order.*

Proof. If $w(x) = q(\phi, x)$ whenever $x \in X$ and ϕ is the maximum on (X, q) with respect to the specialization order, then w is a compatible weight function vanishing at ϕ and $q(x, y) \leqslant q(x, \phi) + q(\phi, y) = 0 + w(y)$ whenever $x, y \in X$ so that w is upper weighting.

In order to prove the converse suppose that w is a compatible upper weighting function on (X, q) vanishing at ϕ. Then $q(x, \phi) \leqslant w(\phi) = 0$ whenever $x \in X$ so that ϕ is the maximum on (X, q) with respect to the specialization order. Furthermore $w(x) = w(x) - w(\phi) \leqslant q(\phi, x) \leqslant w(x)$ and thus $w(x) = q(\phi, x)$ whenever $x \in X$. □

Theorem 1. *Under the bijective correspondence described above each partial quasi-metric p corresponds exactly to a quasi-metric q with compatible weight w.*

Note that if the partial quasi-metric p corresponds to the pair (q, w) under the described bijection, then the partial quasi-metric p^{-1} corresponds to $(\overline{q}, w)$, and the partial metric p^+ corresponds to $(\tilde{q}, 2w)$ where $\overline{q}(x, y) = q^{-1}(x, y) + w(y) - w(x)$ and $\tilde{q}(x, y) = q^+(x, y) + w(y) - w(x)$ whenever $x, y \in X$.

It follows that $q^+ = (\overline{q})^+$ and $2 \cdot q^+ = (\tilde{q})^+$. Furthermore $q + \overline{q} = \tilde{q}$.

Proof. The first assertion is evident. Let $x, y \in X$. Then $\overline{q}(x, y) = p^{-1}(x, y) - p^{-1}(x, x) = p(y, x) - p(x, x) = q(y, x) + p(y, y) - p(x, x) = q^{-1}(x, y) + w(y) - w(x)$. Similarly, $\tilde{q}(x, y) = p^+(x, y) - p^+(x, x) = p(x, y) - p(x, x) + p^{-1}(x, y) - p^{-1}(x, x) = q(x, y) + \overline{q}(x, y) = q^+(x, y) + w(y) - w(x)$. The remaining statements are obvious. □

Remark 8. Let p be a partial quasi-metric on a set X. Similarly as in the case of partial metrics we can introduce ε-balls of points and use them to define topologies on X. For the associated quasi-metric q we obtain the following ε-balls at $x \in X$ where $\varepsilon \in (0, \infty)$:

$B_\varepsilon^{q_p}(x) = \{y \in X: q(x, y) < \varepsilon\} = \{y \in X: p(x, y) - p(x, x) < \varepsilon\}$.

Similarly, we get for the quasi-metric q^{-1} the following ε-balls at x:

$B_\varepsilon^{q_p^{-1}}(x) = \{y \in X: q^{-1}(x, y) < \varepsilon\} = \{y \in X: q(y, x) < \varepsilon\} = \{y \in X: p(y, x) - p(y, y) < \varepsilon\}$.

For the quasi-metric $\overline{q}$ associated with the partial quasi-metric p^{-1} on X we similarly obtain its ε-balls at $x \in X$ as follows:

$B_\varepsilon^{\overline{q_p}}(x) = \{y \in X: \overline{q}(x, y) < \varepsilon\} = \{y \in X: p(y, x) - p(x, x) < \varepsilon\}$.

Finally for the conjugate quasi-metric $(\overline{q})^{-1}$ of $\overline{q}$ we get the following ε-balls at $x \in X$:

$B_\varepsilon^{\overline{q_p}^{-1}}(x) = \{y \in X: (\overline{q})^{-1}(x, y) < \varepsilon\} = \{y \in X: \overline{q}(y, x) < \varepsilon\} = \{y \in X: p(x, y) - p(y, y) < \varepsilon\}$.

In each of the four cases the collection of all these balls yields a base for a topology on X, which as usual, we shall denote by $\tau(q)$, $\tau(q^{-1})$, $\tau(\overline{q})$ and $\tau((\overline{q})^{-1})$, respectively. As quasi-metric topologies they are all T_0-topologies. Our next example shows that they can all be different from one another. However observe that if p is a partial metric, then $q = \overline{q}$ and so the first and third topology, namely $\tau(q)$ and $\tau(\overline{q})$, as well as the second and fourth topology, namely $\tau(q^{-1})$ and $\tau((\overline{q})^{-1})$, are necessarily the same.

Example 3. Let (X, q) be a quasi-metric space. Choose a base point $\phi \in X$. Moreover let $w(x) = q(\phi, x)$ whenever $x \in X$ be the compatible weight function on (X, q) (compare Remark 7(b)).

We now note that under these assumptions the quasi-metric q' considered in Remark 1 is indeed equal to the quasi-metric $(\overline{q})^{-1}$ that we have introduced in Theorem 1. In particular we have $\overline{q}(x, \phi) = q(\phi, x) + q(\phi, \phi) - q(\phi, x) = 0$ whenever $x \in X$.

To be more specific, let q be the so-called Sorgenfrey quasi-metric on the set $\mathbb{R}$ of the reals, that is, $q(x, y) = \min\{y - x, 1\}$ if $y \geqslant x$, and $q(x, y) = 1$ if $y < x$. Furthermore choose $\phi = 0$. So we consider the partial quasi-metric $p(x, y) = q(x, y) + q(0, x)$ on $\mathbb{R}$ where $x, y \in \mathbb{R}$.

It is well-known that $\tau(q)$ and $\tau(q^{-1})$ are two distinct Hausdorff topologies. In particular they both determine the trivial specialization order, namely equality. On the other hand, $\overline{q}(1, 0) = 0$, but $\overline{q}(0, 1) = q(1, 0) + q(0, 1) - q(0, 0) = 2$. Hence the point 1 is in the $\tau(\overline{q})$-closure of $\{0\}$, but the point 1 is not in the $\tau((\overline{q})^{-1})$-closure of $\{0\}$. We conclude that the studied four topologies are different from one another.

Lemma 4. *Note that if (X, q, w) is a quasi-metric space with compatible weight, then for any quasi-metric q' on X such that $q'(x, y) \geqslant q(x, y)$ whenever $x, y \in X$, (X, q', w) is also a quasi-metric space with compatible weight.*

Proof. The assertion is obvious, since $w(y) - w(x) \leqslant q(x, y) \leqslant q'(x, y)$ whenever $x, y \in X$. □

The preceding observation suggests many examples of partial quasi-metrics that are not partial metrics: just apply it to the metric $q' = q^+$ and a given nonconstant (compatible) weight w.

For instance we can also take any *metric* space (X, m) with cardinality at least two and the compatible weight function $x \mapsto m(\phi, x)$ for some base point $\phi \in X$. Set now $p(x, y) = m(x, y) + m(\phi, x)$ whenever $x, y \in X$. Then p is a partial quasi-metric and if $\psi \in X$ is distinct from ϕ, we have $p(\psi, \phi) - p(\phi, \psi) = m(\phi, \psi) \neq 0$.

Remark 9 (*Compare also [8]*). If (X, q, w) is a quasi-metric space with compatible weight, then the associated partial quasi-metric p is a partial metric if and only if q and w satisfy the weight condition due to Matthews, that is,

$$q(x, y) + w(x) = q(y, x) + w(y) \quad \text{whenever } x, y \in X.$$

Quasi-metric spaces (X, q) equipped with such a weight function w are called *weighted*. Of course, any weight ω satisfying the latter equality is compatible with the quasi-metric q on X.

Example 4. Let (X, p) be a partial quasi-metric space. Then the quasi-metric $\widetilde{q}$ associated with the partial metric p^+ (see Theorem 1) is weighted by the weight $2p(x, x)$ $(x \in X)$.

Definition 3. A partial quasi-metric space (X, p) is called *weightable* if its associated quasi-metric q is weightable (that is, there is a weight w, possibly different from $p(x, x)$ $(x \in X)$, such that q is weighted by w).

Example 5. Let $|\cdot|$ be the usual norm on the set $\mathbb{R}$ of the reals. The partial quasi-metric $p(x, y) = |y - x| + |x|$ where $x, y \in \mathbb{R}$ is weightable, since the associated quasi-metric $q(x, y) = |y - x|$ is a metric and, hence weightable by the constant weight function 0.

So in particular each weightable quasi-metric space (X, q) is weightable when considered as a partial quasi-metric space. It is known that there are weightable quasi-metrics q whose conjugate q^{-1} is not weightable (see [8, Example 5] for a stronger result). Furthermore, each partial metric p on X is weightable, as the weight $p(x, x)$ $(x \in X)$ witnesses.

3. Examples

The following example is crucial in the theory of partial (quasi-)metrics.

Example 6. Let $X = [0, \infty)$ and $\rho(x, y) = \max\{y - x, 0\}$ and $\lambda(x) = x$ whenever $x, y \in X$. Then (X, ρ, λ) is a quasi-metric space with compatible weight. The associated partial quasi-metric $p(x, y) = \rho(x, y) + \lambda(x)$ whenever $x, y \in X$ is indeed a partial metric.

Note that in this example the weight λ is not compatible with the quasi-metric space (X, ρ^{-1}). Hence ℓ defined by $\ell(x, y) = \rho(y, x) + \lambda(x)$ whenever $x, y \in [0, \infty)$ is a lopsided partial quasi-metric on X that is not a partial quasi-metric. The method of Remark 6 applied to ℓ yields the partial quasi-metric $\ell'(x, y) = |y - x| + \lambda(x)$ where $x, y \in [0, \infty)$.

Proof. We have $y - x \leqslant \rho(x, y)$ whenever $x, y \in X$, but for instance $3 - 2 \nleqslant \rho^{-1}(2, 3) = \rho(3, 2) = 0$. It is readily checked that $\rho(x, y) + \lambda(x) = \rho(y, x) + \lambda(y)$ whenever $x, y \in X$. Hence the assertions are verified. $\square$

We call a map $f : (X_1, p_1) \to (X_2, p_2)$ between two (partial) quasi-metric spaces (X_1, p_1) and (X_2, p_2) *non-expansive* provided that $p_2(f(x), f(y)) \leqslant p_1(x, y)$ whenever $x, y \in X_1$.

Remark 10. Note that $w : X \to [0, \infty)$ is a compatible weight function on the quasi-metric space (X, q) if and only if $w : (X, q) \to ([0, \infty), \rho)$ is a non-expansive map.

Proof. The compatibility of w with q means that $\rho(w(x), w(y)) = \max\{w(y) - w(x), 0\} \leqslant q(x, y)$ whenever $x, y \in X$. Hence the assertion follows. $\square$

Example 7 (*Compare [2, p. 79]*). Let (X, q) be a quasi-metric space. Consider its set of so-called *formal balls* $S = \{(x, r): x \in X, r \in [0, \infty)\}$. Set $q'((x, r), (y, s)) = q(x, y) + \rho(r, s)$ whenever $x, y \in X$ and $r, s \in [0, \infty)$. Then q' is a quasi-metric on S. Furthermore $w'((x, r)) = r$ whenever $x \in X$ and $r \in [0, \infty)$ clearly determines a compatible weight on S. Therefore we can define the partial quasi-metric $p(s_1, s_2) = q'(s_1, s_2) + w'(s_1)$ $(s_1, s_2 \in S)$ on S.

In order to discuss our next example we have to recall some well-known concepts from the theory of BCK-algebras (see [3]).

Important properties of the set-theoretic difference and the logical implication motivated the axioms of a BCK-algebra, as can readily be seen from the following definition.

A universal algebra $(A, \cdot, 0)$ (of type $(2, 0)$) is called a BCK-*algebra* (see e.g. [13]) if the following identities and quasi-identities hold:

(1) $((xy)(xz))(zy) = 0$
(2) $(x(xy))y = 0$
(3) $xx = 0$
(4) $0x = 0$
(5) $xy = yx = 0$ implies that $x = y$.

It is well known that A also satisfies the identities

(6) $(xy)z = (xz)y$
(7) $x0 = x$.

With a BCK-algebra A a partially ordered set $(A, \leqslant)$ is associated, where $x \leqslant y$ iff $xy = 0$.

Example 8 (*See [3, Example 1]*). Let A be an arbitrary nonempty set, and let X be the set of all real-valued functions defined on A. For $f, g \in X$ we define $g \star f$ by the two conditions: $(g \star f)(x) = 0$ if $g(x) \leqslant f(x)$, and $(g \star f)(x) = g(x) - f(x)$ if $f(x) < g(x)$.

By the definition of $\star$, X is a BCK-algebra that is commutative, that is, $f \star (f \star g) = g \star (g \star f)$ whenever $f, g \in X$.

Let A be a BCK-algebra. In [13] a function $n : A \to \mathbb{R}$ is called a *pseudo-norm* on A if

(8) $n(x) \leqslant n(xy) + n(y)$ whenever $x, y \in A$,
(9) $n(0) = 0$.

The pseudo-norm n is said to be a *norm* if

(10) $n(x) = 0$ implies $x = 0$ whenever $x \in A$.

For our purposes the following lemma proved by Raftery and Sturm is very useful.

Lemma 5 (*Raftery and Sturm [13, Lemma 1]*). *A pseudo-norm n on a BCK-algebra A is a non-negative increasing function satisfying*

(11) $n(xz) \leqslant n(xy) + n(yz)$ *whenever* $x, y, z \in A$.

As Raftery and Sturm remark [13], if xy is interpreted as set-theoretic difference $x - y$ $(x, y \in A)$ and 0 as the empty set, then a pseudo-norm on A has some of the properties of a measure.

Proposition 4. *Let n be a* (*pseudo*)*-norm on a BCK-algebra A. Then $d(x, y) = n(yx)$ is a quasi-*(*pseudo*)*metric on A and $w(x) = n(x)$ is a compatible weight on A.*

Proof. Note that d satisfies the triangle inequality by (11). Furthermore $d(x, x) = 0$ whenever $x \in A$. If n is a norm and $d(x, y) = 0 = d(y, x)$, then $n(xy) = 0 = n(yx)$. Thus $xy = 0 = yx$ and therefore $x = y$ by (5). Hence d is a quasi-metric. It remains to show that $n(y) \leqslant d(x, y) + n(x)$. But this just means $n(y) \leqslant n(yx) + n(x)$, which holds according to (8). □

Remark 11. Let us assume that in Proposition 4, n is a norm on A.

(a) Then obviously $d(x, y) = 0$ iff $y \leqslant x$. (Hence the usual order of the BCK-algebra A is equal to the inverse of the specialization order determined by d on A.)

Also $lm = st$ for $l, m, s, t \in A$ implies that $d(m, l) = d(t, s)$.

(b) Furthermore $p(x, y) = n(yx) + n(x)$ is a partial quasi-metric on the BCK-algebra A. So this partial quasi-metric is a partial metric if and only if $n(xy) + n(y) = n(yx) + n(x)$ whenever $x, y \in A$.

Proposition 5. *If d is a quasi-metric on a BCK-algebra A satisfying the two conditions formulated in part* (*a*) *of Remark* 11, *then $n(x) = d(0, x)$ whenever $x \in A$ is a norm on A.*

Proof. We have $n(0) = d(0, 0) = 0$. Furthermore $d(0, x) \leqslant d(0, y) + d(y, x)$ whenever $x, y \in A$. But since $(xy)0 = xy$, we get $d(0, xy) = d(y, x)$. Thus $n(x) \leqslant n(y) + d(0, xy) = n(y) + n(xy)$. Finally if $n(x) = 0$ for some $x \in A$, then $d(0, x) = 0$. Note that $0x = 00$ and thus $d(x, 0) = d(0, 0) = 0$. Since d is a quasi-metric, we conclude that $x = 0$. Hence n is a norm on A. □

It should be mentioned that in [13] Raftery and Sturm study the completion of pseudo-normed BCK-algebras with respect to the sum pseudometric $d^+ = d + d^{-1}$. Hence their results are related to our study in the next section which deals with the bicompletion of a quasi-metric space (see e.g. [1]).

Example 9. Let $(E, \cdot)$ be a group. A quasi-norm on E is a non-negative real-valued function $\|\cdot\|$ on E such that for all $x, y \in E$:

(i) ($\|x\| = 0$ and $\|x^{-1}\| = 0$) iff $x = e$ (where e denotes the neutral element of $(E, \cdot)$);

(ii) $\|x \cdot y\| \leqslant \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is then called a *quasi-normed group*.

To illustrate this concept (compare [22, Example 3, p. 239]), let [0, 1] be the unit interval of the set of the reals equipped with its usual topology and let G be the group of homeomorphisms on [0, 1] with the usual composition of functions as group operation. For each $f \in G$ set $\|f\| = \max\{|f(x) - x| : x \in [0, 1]\}$. It is well known that $\|\cdot\|$ defines a quasi-norm on G.

Let $(E, \|\cdot\|)$ be a quasi-normed group. Then $(E, d, \|\cdot\|)$ where $d(x, y) := \|y \cdot x^{-1}\|$ whenever $x, y \in E$ is a quasi-metric space with compatible weight $\|\cdot\|$.

Proof. It is well known and easy to check that $d(x, y) = \|y \cdot x^{-1}\|$ whenever $x, y \in E$ is a quasi-metric on E. We have $\|y\| = \|(y \cdot x^{-1}) \cdot x\| \leqslant \|y \cdot x^{-1}\| + \|x\|$. Hence $\|y\| - \|x\| \leqslant d(x, y)$ whenever $x, y \in E$ and we are done. □

For similar investigations on quasi-normed monoids and studies about their bicompletion (compare with our next section) we refer the reader to [14]. A referee also points out some connections with older work on distance functions on semigroups (see [4,9]).

Example 10. Let Y be the set of all the sequences over the alphabet of non-negative integers that are finally constant equal to 0. For each $x \in Y$ we put $q(x, x) = 0$. For distinct $x, y \in Y$ we set $q(x, y) = \min\{\sum_{k=x_i}^{y_i-1} 1/(k + 1), 1\}$ if the ith-coordinate is the first coordinate where x and y differ and $x_i < y_i$; finally set $q(x, y) = 1$ otherwise.

It is readily checked that q is a quasi-metric on Y (compare [7, Example 1]). If $\underline{0}$ denotes the constant zero sequence, then we consider the compatible weight w on Y defined by $w(x) = q(\underline{0}, x)$ where $x \in Y$. It follows that p defined by $p(x, y) = q(x, y) + w(x)$ whenever $x, y \in Y$ is a partial quasi-metric on Y.

4. Main applications

Finally we shall apply partial quasi-metrics to construct completions and to prove the existence of fixed points.

Definition 4. A sequence $(x_n)_{n\in\mathbb{N}}$ in a partial quasi-metric space (X, p) is called a *Cauchy sequence* if $\lim_{n\to\infty,m\to\infty} p(x_n, x_m)$ exists (that is, there is an $a \in [0, \infty)$ such that for each $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that for any positive integers $n, m \geqslant N_0$ we have that $|p(x_n, x_m) - a| < \varepsilon$).

Remark 12. We note that a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, p) iff it is a Cauchy sequence in (X, q^+) where q is the quasi-metric associated with p.

In this paper a partial quasi-metric space (X, p) is called *complete* if its associated quasi-metric q is *bicomplete*, that is, each Cauchy sequence in (X, q^+) converges in the metric space (X, q^+).

It is well known that each quasi-metric space (X, d) has an (up to isometry) unique bicompletion $(\widetilde{X}, \widetilde{d})$ constructed as follows. Denote by Y the set of all Cauchy sequences in the metric space (X, d^+). Define $d' : Y \times Y \to [0, \infty)$ as follows: For each pair $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in Y put $d'((x_n)_n, (y_n)_n) = \lim_n d(x_n, y_n)$. Now let

$$R = \{((x_n)_n, (y_n)_n) \in Y \times Y : (d')^+((x_n)_n, (y_n)_n) = 0\}.$$

Then R is an equivalence relation on Y. Denote by $\widetilde{X}$ the quotient Y/R. Define now $\widetilde{d}: \widetilde{X} \times \widetilde{X} \to [0, \infty)$ in the following way. For each pair $[(x_n)_n], [(y_n)_n]$ in $\widetilde{X}$, let $\widetilde{d}([(x_n)_n], [(y_n)_n]) = d'((x_n)_n, (y_n)_n)$. In fact,

$$\widetilde{d}([(x_n)_n], [(y_n)_n]) = \lim_n d(x_n, y_n).$$

It is well known that $(\widetilde{X}, \widetilde{d})$ yields the bicompletion of (X, d) (see e.g. [1]). The point $x \in X$ is identified with $[(\underline{x})]$ in $\widetilde{X}$ where $(\underline{x})$ denotes the sequence all terms of which are equal to x. Furthermore X is dense in $\widetilde{X}$ with respect to the topology $\tau((\widetilde{d})^+)$.

Theorem 2. *A partial quasi-metric space* (X, p) *is complete iff* (X, p^{-1}) *is complete iff* (X, p^+) *is complete.*

Proof. Denote the quasi-metric associated with p (resp. p^{-1}, resp. p^+) by q (resp. $\overline{q}$, resp. $\tilde{q}$), as we did in Theorem 1.

According to the latter theorem we know that $(\overline{q})^+ = q^+$ and $(\tilde{q})^+ = 2 \cdot q^+$. Hence the statements directly follow from the definition of completeness of a partial quasi-metric space. □

Remark 13 (*Compare [11] for weighted quasi-metrics resp. partial metrics*). Suppose that the partial quasi-metric space (X, p) corresponds to the quasi-metric space (X, q, w) with compatible weight and that $(\widetilde{X}, \widetilde{q})$ is the bicompletion of (X, q).

We observe that $w: X \to [0, \infty)$ has a well-defined extension $\widetilde{w}$ to the bicompletion $\tilde{X}$, which exists, since w is compatible with q: If $x \in \widetilde{X}$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence in X such that $(\widetilde{q})^+(x, x_n) \to 0$, then by the compatibility of w and q on X, $(w(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the usual metric on $[0, \infty)$. Hence we can set $\widetilde{w}(x) = \lim_{n\to\infty} w(x_n)$ where the limit is taken with respect to the usual topology on $[0, \infty)$. Obviously the value $\widetilde{w}(x)$ is well defined, that is, remains unchanged if we choose another Cauchy sequence equivalent to $(x_n)_{n\in\mathbb{N}}$ in X. Clearly, the weight $\widetilde{w}$ on the bicompletion $\widetilde{X}$ is compatible with the quasi-metric $\widetilde{q}$.

By definition, the completion $(\widetilde{X}, \widetilde{p})$ of the partial quasi-metric space (X, p) is the partial quasi-metric space that corresponds to the quasi-metric space $(\widetilde{X}, \widetilde{q})$ with its compatible weight $\widetilde{w}$. It is known that if p is a partial metric, then $\widetilde{p}$ is a partial metric (see [11]). We also remark that $\widetilde{p^{-1}} = (\widetilde{p})^{-1}$ and $\widetilde{p^+} = (\widetilde{p})^+$.

Note that for these arguments we use the facts that $\widetilde{q}$ and $\widetilde{\omega}$ are continuous functions and X is dense in $\widetilde{X}$ provided that $\widetilde{X}$ is endowed with the topology $\tau((\widetilde{q})^+)$.

We conclude this article with some remarks on fixed points. Let X be a set equipped with some (partial) quasi-metric $p: X \times X \to [0, \infty)$. We say that a map $f: (X, p) \to (X, p)$ is a *contraction* if there is a positive real constant $k < 1$ such that $p(f(x), f(y)) \leqslant kp(x, y)$ whenever $x, y \in X$.

Remark 14. Let $f: (X, q) \to (X, q)$ be a contraction defined on a bicomplete quasi-metric space (X, q). Then, noting that $f: (X, q^+) \to (X, q^+)$ is a contraction, we see by Banach's Fixed Point Theorem that f has a unique fixed point $x_0 \in X$.

For any $k \in [0, \infty)$, $w(x) = k$ whenever $x \in X$, is a compatible weight on (X, q). Hence for the associated partial quasi-metric p defined by $p(x, y) = q(x, y) + w(x)$ $(x, y \in X)$, we get that $p(x_0, x_0) = k$.

Similarly as in [10], we can obtain additional information on the weight of the fixed point if we assume that the map f satisfies the contraction inequality with respect to the partial quasi-metric itself.

Theorem 3. *Let* $f: (X, p) \to (X, p)$ *be a contraction defined on a complete partial quasi-metric space* (X, p). *Then* f *has a unique fixed point, which has weight 0.*

Proof. Note first that $f: (X, p^+) \to (X, p^+)$ is a contraction. Furthermore by Theorem 2 p is complete if and only if the partial metric p^+ is complete. According to a result due to Matthews [10, Theorem 5.3] f has a unique fixed point a, which satisfies $p^+(a, a) = 0$. Hence $p(a, a) = 0$. □

5. Conclusions and further work

We introduced the concept of a partial quasi-metric space and showed that many results dealing with partial metrics [10] can be generalized to this larger class of spaces. Partial quasi-metrics naturally arise in the theory of normed

BCK-algebras and the theory of quasi-normed groups. In both areas they have mainly been used to introduce an associated (bi)completion of the underlying algebraic structure. We argued that in general partial quasi-metrics possess an associated (bi)completion, which generalizes the known (bi)completion of partial metric spaces. The (bi)completion of partial metric spaces has been studied in detail in an article of Oltra et al. [11].

Given a quasi-metric space (X, q), we can equip the set $\mathcal{W}$ of all compatible weight functions on (X, q) with the extended quasi-metric

$$Q(w_1, w_2) = \sup\{\max\{w_1(x) - w_2(x), 0\}: x \in X\},$$

where $w_1, w_2 \in \mathcal{W}$. It might be interesting to consider the compatible extended weight $W = Q(\underline{0}, w)$ on the space $(\mathcal{W}, Q)$. Of course, $\underline{0}$ denotes the weight that equals to 0 on the whole of X.

Note that $x \mapsto w_x$ where $w_x(y) = q(x, y)$ whenever $y \in X$ defines an isometric embedding of the quasi-metric space (X, q) into the extended quasi-metric space $(\mathcal{W}, Q)$. Hence it may be enlightening to know more about the structure of the extended partial quasi-metric space associated with $(\mathcal{W}, Q, W)$.

Also it might be interesting to study partial quasi-metrics on more general algebras than BCK-algebras and to investigate how partial quasi-metrics defined on a substructure, like for instance ideals, can be extended to some given full structure.

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References

[1] A. Di Concilio, Spazi quasimetrici e topologie ad essi associate, Rend. Accad. Sci. Fis. Mat. Napoli (4) 38 (1971) 113–130.
[2] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7 (1999) 71–83.
[3] K. Iséki, S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon 23 (1978) 1–26.
[4] R. Kopperman, Lengths on semigroups and groups, Semigroup Forum 25 (1982) 345–360.
[5] R. Kopperman, S. Matthews, H. Pajoohesh, Partial metrizability in value quantales, Appl. General Topology 5 (2004) 115–127.
[6] R. Kopperman, S. Matthews, H. Pajoohesh, What do partial metrics represent?, Notes distributed at the 19th Summer Conference on Topology and its Applications, University of Cape Town, 2004.
[7] H.-P.A. Künzi, Nontransitive quasi-uniformities in the Pervin quasi-proximity class, Proc. Amer. Math. Soc. 130 (2002) 3725–3730.
[8] H.-P. A. Künzi, V. Vajner, Weighted quasi-metrics, in: Proceedings of the 8th Summer Conference on Topology and its Applications, Ann. New York Acad. Sci. 728 (1994) 64–77.
[9] D. Marxen, Uniform semigroups, Math. Ann. 202 (1973) 27–36.
[10] S.G. Matthews, Partial metric topology, in: Proceedings of the 8th Summer Conference on Topology and its Applications, Ann. New York Acad. Sci. 728 (1994) 183–197.
[11] S. Oltra, S. Romaguera, E.A. Sánchez-Pérez, Bicompleting weightable quasi-metric spaces and partial metric spaces, Rend. Circ. Mat. Palermo (2) 51 (2002) 151–162.
[12] S.J. O'Neill, Partial metrics, valuations, and domain theory, in: Proceedings of 11th Summer Conference on Topology and its Applications, Ann. New York Acad. Sci. 806 (1996) 304–315.
[13] J.G. Raftery, T. Sturm, On completions of pseudo-normed BCK-algebras and pseudo-metric universal algebras, Math. Japon. 33 (1988) 919–929.
[14] S. Romaguera, E.A. Sánchez-Pérez, O. Valero, Quasi-normed monoids and quasi-metrics, Publ. Math. Debrecen 62 (2003) 53–69.
[15] S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, Appl. General Topology 3 (2002) 91–112.
[16] M.P. Schellekens, Complexity spaces: lifting and directedness, Topology Proc. 22 (1997) 403–425.
[17] M.P. Schellekens, Extendible spaces, Appl. General Topology 3 (2002) 169–184.
[18] M.P. Schellekens, A characterization of partial metrizability: domains are quantifiable, Theoret. Comput. Sci. 305 (2003) 409–432.
[19] M.P. Schellekens, The correspondence between partial metrics and semivaluations, Theoret. Comput. Sci. 315 (2004) 135–149.
[20] P. Vitolo, The representation of weighted quasi-metric spaces, Rend. Istit. Mat. Univ. Trieste 31 (1999) 95–100.
[21] P. Waszkiewicz, The local triangle axiom in topology and domain theory, Appl. General Topology 4 (2003) 47–70.
[22] A. Wilansky, Topology for Analysis, Ginn, London, 1970.