Existence of monotone positive solutions to a third order two-point generalized right focal boundary value problem

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Abstract

In this paper, we are concerned with the third order two-point generalized right focal boundary value problem

\[ x'''(t) + f(t, x(t)) = 0, \quad a < t < b, \]
\[ x(a) = x'(a) = x''(b) = 0. \]

A few new results are given for the existence of at least one, two, three and infinitely many monotone positive solutions of the above boundary value problem by using the Krasnosel’kii fixed-point theorem in cones, and the Leggett–Williams fixed-point theorem.

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1. Introduction

In this paper, we are concerned with the third order two-point generalized right focal boundary value problem

\[ x'''(t) + f(t, x(t)) = 0, \quad a < t < b, \quad (1.1) \]
\[ x(a) = x'(a) = x''(b) = 0, \quad (1.2) \]

where

\[ f \in C([a, b] \times \mathbb{R}, [0, +\infty)). \quad (1.3) \]

Recently, many researchers have studied the existence of positive solutions for various types of nonlinear differential, integral and difference equations, see [1–8], and the references therein. In 2003, Henderson and Yin [9]
considered the existence and uniqueness of solutions for third order two-point conjugate boundary value problems. In 1998, by means of the Leggett–Williams fixed-point theorem, Anderson [10] gave the existence at least three positive solutions to the third order three-point boundary value problem
\begin{align}
-x'''(t) + f(x(t)) &= 0, \quad 0 \leq t \leq 1, \\
x(0) &= x'(0) = x''(1) = 0,
\end{align}
where \( t_0 \in [\frac{1}{2}, 1) \). In 2002, by using the Krasnosel’skii fixed-point theorem and the five functional fixed-point theorem, Anderson and Davis [11] established the existence of multiple positive solutions for the third order three-point right focal boundary value problem
\begin{align}
x'''(t) &= f(t, x(t)), \quad t_1 \leq t \leq t_3, \\
x(t_1) &= x'(t_2) = x''(t_3) = 0,
\end{align}
where
\begin{align}
t_2 \in (t_1, t_3) \quad \text{and} \quad t_2 - t_1 > t_3 - t_2.
\end{align}
In 2003, by applying the Krasnosel’skii and Leggett–Williams fixed-point theorems, Anderson [12] continued to study the existence of positive solutions for the third order three-point generalized right focal boundary value problem (1.6) under boundary conditions (1.8) and
\begin{align}
x(t_1) = x'(t_2) = 0, \quad Ax(t_3) + Bx''(t_3) = 0,
\end{align}
where
\begin{align}
A \geq 0, \quad B > 0, \quad 2B + A(t_3 - t_1)(t_3 - 2t_2 + t_1) > 0.
\end{align}
The goal of this paper is to study the existence of at least one, two, three and infinitely many monotone positive solutions for the third order two-point generalized right focal boundary value problem (1.1) and (1.2) by utilizing the Krasnosel’skii and Leggett–Williams fixed-point theorems. It is easy to see that these third order three-point boundary value problems (1.4)–(1.10) do not include the third order two-point generalized right focal boundary value problem (1.1) and (1.2) as special cases.

The paper is divided into four sections. In Section 2, we provide some notation and lemmas, which play key roles in this paper. In Section 3, we use the Krasnosel’skii and Leggett–Williams fixed-point theorems to establish several existence results of at least one, two, three, and infinitely many monotone positive solutions to the third order two-point generalized right focal boundary value problem (1.1) and (1.2). In Section 4, we construct four examples to illustrate our results.

2. Preliminaries and lemmas

Let \( X \) be a Banach space and \( Y \) be a cone in \( X \). A mapping \( \alpha \) is said to be a nonnegative continuous concave functional on \( Y \) if \( \alpha : Y \rightarrow [0, +∞) \) is continuous and
\[\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y), \quad x, y \in Y, \quad t \in [0, 1].\]
x is said to be a positive solution of Eqs. (1.1) and (1.2) if \( x \) is a solution of Eqs. (1.1) and (1.2) and \( x(t) > 0 \) for each \( t \in (a, b) \). Throughout this paper, we assume that \( C[a, b] \) denotes the Banach space of all continuous functions on \( [a, b] \) with the supremum norm \( \|u\| := \sup_{t \in [a, b]} |u(t)| \) for each \( u \in C[a, b] \). \( p \) and \( q \) are constants with \( a < p < q < b \),
\[g(t) = (t - a) \left( b - \frac{t + a}{2} \right), \quad h(t) = \frac{1}{2} \left( \frac{t - a}{b - a} \right)^2, \quad t \in [a, b],\]
\[\psi(s) = \min\{f(t, x) : t \in [p, q], x \in [h(p)s, s]\}, \quad s \geq 0,\]
\[\varphi(s) = \max\{f(t, x) : t \in [a, b], x \in [0, s]\}, \quad s \geq 0,\]
Lemma 2.1. Assume that (1.3) holds.

(a) If \( \min\{f_0, f_\infty\} > \frac{m}{h(p)} \), then \( \max\{\psi_0, \psi_\infty\} > m \);
(b) If \( \min\{f_0, f_\infty\} < k \), then \( \min\{\varphi_0, \varphi_\infty\} < k \).

Lemma 2.2. (a) The functions \( h \) and \( g \) are increasing on \([a, b]\), and

\[
0 \leq h(t) \leq \frac{1}{2}, \quad 0 \leq g(t) \leq \frac{1}{2}(b - a)^2, \quad t \in [a, b];
\]

(b) \( h(t)g(s) \leq G(t, s) \leq g(s), \quad t, s \in [a, b]; \)
(c) \( G(t, s) \leq G(x, s), \quad t, x, s \in [a, b] \) and \( t \leq x \).

Proof. It is easy to see that (a) is true. Now we show that (b) holds. For \( a \leq s \leq t \leq b \), by (a) we know that

\[
G(t, s) = (s - a)\left(t - \frac{s + a}{2}\right) \leq (s - a)\left(b - \frac{s + a}{2}\right) = g(s)
\]

and

\[
G(t, s) = (s - a)\left(t - \frac{s + a}{2}\right) \geq \frac{1}{2}(s - a)(t - a)
\]

\[
\geq \frac{1}{2} \left(\frac{t - a}{b - a}\right)^2 g(s) = h(t)g(s).
\]
For $a \leq t < s \leq b$, by (a) we deduce that
\[ G(t, s) = \frac{1}{2}(t - a)^2 \leq \frac{1}{2}(s - a)^2 \leq g(s) \]
and
\[ G(t, s) = \frac{1}{2}(t - a)^2 \geq \frac{1}{2} \left( \frac{t - a}{s - a} \right)^2 (s - a) \left( b - \frac{s + a}{2} \right) = h(t)g(s). \]

Next, we show that (c) holds. Let $t, x, s \in [a, b]$ and $t \leq x$. We have to consider the following cases. Case 1. Suppose that $s \leq t \leq x$. It follows that
\[ G(t, s) = (s - a) \left( t - \frac{s + a}{2} \right) \leq (s - a) \left( x - \frac{s + a}{2} \right) = G(x, s). \]

Case 2. Suppose that $t \leq x \leq s$. Then
\[ G(t, s) = \frac{1}{2}(t - a)^2 \leq \frac{1}{2}(x - a)^2 = G(x, s). \]

Case 3. Suppose that $t < s \leq x$. Set
\[ H(s) = (s - a) \left( x - \frac{s + a}{2} \right) - \frac{1}{2}(t - a)^2, \quad s \in (t, x]. \]

Notice that
\[ H'(s) = x - \frac{s + a}{2} - \frac{s - a}{2} = x - s \geq 0, \quad s \in (t, x] \]
and
\[ H(t) = (t - a) \left[ x - \frac{t + a}{2} - \frac{t - a}{2} \right] = (t - a)(x - t) \geq 0. \]

It follows that $H(s) \geq 0$ for each $s \in (t, x]$; that is,
\[ G(t, s) \leq G(x, s), \quad t, x, s \in [a, b] \text{ and } t \leq x. \]

This completes the proof. \(\square\)

**Lemma 2.3.** Let (1.3) hold. Then
(a) Eqs. (1.1) and (1.2) has a solution $y \in C[a, b]$ if and only if the operator $T : C[a, b] \to C[a, b]$ defined by
\[ Tx(t) = \int_a^b G(t, s) f(s, x(s))ds, \quad t \in [a, b], \, x \in C[a, b] \]
has a fixed point $y \in C[a, b]$;
(b) Each solution $y \in C[a, b]$ of Eqs. (1.1) and (1.2) is nonnegative and nondecreasing.

**Proof.** (a) is clear. We show only that (b) holds. Assume that $y \in C[a, b]$ is a solution of Eqs. (1.1) and (1.2). In view of (a) and Lemma 2.2, we infer that for any $t, x \in [a, b]$ with $t \leq x$,
\[ 0 \leq y(t) = Ty(t) = \int_a^b G(t, s) f(s, y(s))ds \leq \int_a^b G(x, s) f(s, y(s))ds = Ty(x) = y(x). \]

This completes the proof. \(\square\)

**Lemma 2.4.** Let (1.3) hold. If there exist two positive constants $L$ and $r$ satisfying
\[ f(t, s) \leq Ls, \quad t \in [a, b], \, s \in [r, +\infty), \tag{2.1} \]

...
then for each \( l > 0 \), there exists \( M > \max\{l, r\} \) such that
\[
f(t, s) \leq LM, \quad t \in [a, b], \ s \in [0, M].
\]

**Proof.** Suppose that \( f \) is bounded on \([a, b] \times [0, +\infty)\). Then, for each \( l > 0 \), there exists \( M > \max\{l, r\} \) such that
\[
f(t, s) \leq LM, \quad t \in [a, b], \ s \in [0, +\infty).
\]

Suppose that \( f \) is unbounded on \([a, b] \times [0, +\infty)\). It follows from the continuity of \( f \) that for every \( l > 0 \), there exist \( M > \max\{l, r\} \) and \( t_0 \in [a, b] \) satisfying
\[
f(t, s) \leq f(t_0, M) \leq LM, \quad t \in [a, b], \ s \in [0, M].
\]

This completes the proof. \( \square \)

**Remark 2.1.** Obviously, (2.2) is equivalent to \( \varphi(M) \leq LM \).

**Lemma 2.5 (Krasnosel’skii Fixed-Point Theorem).** Let \((X, \| \cdot \|)\) be a Banach space and let \( Y \subset X \) be a cone in \( X \). Assume that \( A \) and \( B \) are open subsets of \( X \) with \( 0 \in A, \ A \subset B \) and \( T : Y \cap (\overline{B} \setminus A) \to Y \) is a completely continuous operator such that: either
\[
\begin{align*}
(a) \ & \|Tu\| \leq \|u\| \text{ for } u \in Y \cap \partial A, \text{ and } \|Tu\| \geq \|u\| \text{ for } u \in Y \cap \partial B, \text{ or} \\
(b) \ & \|Tu\| \geq \|u\| \text{ for } u \in Y \cap \partial A, \text{ and } \|Tu\| \leq \|u\| \text{ for } u \in Y \cap \partial B.
\end{align*}
\]

Then \( T \) has at least one fixed point in \( Y \cap (\overline{B} \setminus A) \).

**Lemma 2.6 (Leggett–Williams Fixed-Point Theorem).** Let \( T : \overline{P}_c \to \overline{P}_c \) be a completely continuous operator and \( \alpha \) be a nonnegative continuous concave functional on \( P \) such that \( \alpha(x) \leq \|x\| \) for all \( x \in \overline{P}_c \). Suppose that there exist \( 0 < d_0 < a_0 < b_0 \leq c \) such that
\[
\begin{align*}
(a) \ & \{x \in P(\alpha, a_0, b_0) : \alpha(x) > a_0\} \neq \emptyset \text{ and } \alpha(Tx) > a_0 \text{ for } x \in P(\alpha, a_0, b_0); \\
(b) \ & \|Tx\| < d_0 \text{ for } \|x\| \leq d_0; \\
(c) \ & \alpha(Tx) > a_0 \text{ for } x \in P(\alpha, a_0, c) \text{ with } \|Tx\| > b_0.
\end{align*}
\]

Then \( T \) has at least three fixed points \( x_1, x_2, x_3 \) in \( \overline{P}_c \) satisfying
\[
\|x_1\| < d_0, \quad a_0 < \alpha(x_2), \quad \|x_3\| > d_0 \text{ and } \alpha(x_3) < a_0.
\]

### 3. Main results

In this section, by using the Krasnosel’skii fixed-point theorem, we establish a few existence results of at least one, two, three, and infinitely many monotone positive solutions for Eqs. (1.1) and (1.2). We give two existence results of at least three monotone positive solutions for Eqs. (1.1) and (1.2) by means of the Leggett–Williams fixed-point theorem.

**Theorem 3.1.** Let (1.3) hold. Suppose that there exist two different positive constants \( c \) and \( j \) satisfying \( \varphi(c) \leq kc \) and \( \psi(j) \geq mj \), respectively. Then Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \( u \in P \) with \( \min\{c, j\} \leq \|u\| \leq \max\{c, j\} \).

**Proof.** Without loss of generality, we may assume that \( c < j \). Define an operator \( T : P \to C[a, b] \) by
\[
Tx(t) = \int_a^b G(t, s)f(s, x(s))ds, \quad t \in [a, b], \ x \in P.
\]

In view of Lemma 2.2, (1.3) and (3.1), we conclude that
\[
\|Tx\| = \sup_{t \in [a, b]} \int_a^b G(t, s)f(s, x(s))ds \\
\leq \int_a^b g(s)f(s, x(s))ds, \quad x \in P
\]
and
\[ Tx(t) = \int_a^b G(t, s) f(s, x(s)) ds \]
\[ \geq h(t) \int_a^b g(s) f(s, x(s)) ds \]
\[ \geq h(t) \| Tx \|, \quad t \in [a, b], \quad x \in P. \]

That is, \( T : P \to P \). Furthermore, \( T \) is completely continuous by an application of the Arzela–Ascoli Theorem. Let \( x \in \partial P_j \). Then \( \| x \| = j \) and \( \| h(t) \| \leq x(t) \leq j, t \in [p, q] \). Note that \( \psi(j) \geq mj \) if and only if \( f(t, s) \geq mj \) for \( t \in [p, q], s \in [jh(p), j] \). In light of (1.3), (3.1) and Lemma 2.2, we deduce that
\[ \| Tx \| = \sup_{t \in [a, b]} \int_a^b G(t, s) f(s, x(s)) ds \]
\[ \geq \sup_{t \in [p, q]} h(t) \int_p^q g(s) f(s, x(s)) ds \]
\[ \geq h(p) \int_p^q g(s) f(s, x(x)) ds \]
\[ \geq mjh(p) \int_p^q g(s) ds = j. \]

That is,
\[ \| Tx \| \geq \| x \|, \quad x \in \partial P_j. \] (3.2)

Suppose that \( x \) is in \( \partial P_c \). Obviously, \( \| x \| = c \) and \( 0 \leq h(t) \| x \| \leq x(t) \leq c, t \in [a, b] \). It is easy to verify that \( \varphi(c) \leq kc \) if and only if \( f(t, s) \leq kc, t \in [a, b], s \in [0, c] \). Therefore,
\[ \| Tx \| = \sup_{t \in [a, b]} \int_a^b G(t, s) f(s, x(s)) ds \]
\[ \leq \int_a^b g(s) f(s, x(s)) ds \]
\[ \leq kc \int_a^b g(s) ds = c, \]
which yields that
\[ \| Tx \| \leq \| x \|, \quad x \in \partial P_c. \] (3.3)

It follows from (3.2), (3.3) and Lemma 2.5 that the operator \( T \) has at least one fixed point \( u \) in \( \overline{P_j} \setminus P_c \). Consequently, Lemma 2.3 ensures that Eqs. (1.1) and (1.2) has at least one nondecreasing solution \( u \) in \( \overline{P_j} \setminus P_c \) with \( 0 < c \leq \| u \| \leq j \). Since \( u(t) \geq h(t) \| u \| \geq ch(t) > 0, t \in (a, b) \), it follows that \( u \) is a nondecreasing positive solution of Eqs. (1.1) and (1.2). This completes the proof.

**Theorem 3.2.** Let (1.3) hold. If there exist two positive numbers \( j \) and \( i \) satisfying, respectively, \( \psi(j) \geq mj \) and \( f(t, s) \leq ks, \quad t \in [a, b], \quad s \in [i, +\infty) \),
\[ f(t, s) \leq ks, \quad t \in [a, b], \quad s \in [i, +\infty), \] (3.4)
then Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \( x \in P \) with \( \| x \| \geq j \).

**Proof.** Lemma 2.4, (3.4) and Remark 2.1 ensure that there exists \( l > \max\{j, i\} \) satisfying \( \varphi(l) \leq kl \). Thus Theorem 3.2 follows from Theorem 3.1. This completes the proof.

From Theorems 3.1 and 3.2, we have the following result.
Theorem 3.3. Let (1.3) hold. If there exist four positive constants \( c, j, i \) and \( w \) with \( w > \max\{c, j\} > \min\{c, j\} \) satisfying \( \varphi(c) \leq kc, \psi(j) \geq mj, \psi(w) \geq mw \) and (3.4), then Eqs. (1.1) and (1.2) possesses at least two nondecreasing positive solutions \( x, y \in P \) with \( \min\{c, j\} \leq \|x\| \leq \max\{c, j\} \) and \( \|y\| \geq w \).

Theorem 3.4. Let (1.3) hold. If there exist four positive constants \( c, j, i, w \) with \( j < c < w \) satisfying \( \varphi(c) < kc, \psi(j) > mj, \psi(w) > mw \) and (3.4), then Eqs. (1.1) and (1.2) possesses at least three nondecreasing positive solutions \( x, y, z \in P \) with \( j < \|x\| < c < \|y\| < w < \|z\| \).

**Proof.** We assert that there exist two constants \( j_1, c_1 \) with \( j < j_1 < c_1 < c \) and
\[
\varphi(c_1) = kc_1 \quad \text{and} \quad \psi(j_1) = mj_1. \tag{3.5}
\]

Notice that (1.3) implies that \( \varphi \) and \( \psi \) are continuous and
\[
\frac{\varphi(c)}{c} < k < m < \frac{\psi(j)}{j} \leq \frac{\psi(j_1)}{j_1}. \tag{3.6}
\]

Since \( f \) is continuous on \([a, b] \times \mathbb{R}\), it follows that \( \varphi \) and \( \psi \) are continuous on \([0, +\infty)\). Using the continuity of \( \varphi \) and (3.6), we deduce that there exists some \( c_1 \in (j, c) \) satisfying \( \frac{\varphi(c)}{c_1} = k \). Since
\[
\frac{\psi(c_1)}{c_1} \leq \varphi(c_1) \leq k < m < \frac{\psi(j)}{j},
\]
it follows that there exists \( j_1 \in (j, c_1) \) satisfying \( \psi(j_1) = mj_1 \). That is, (3.5) is true. Lemma 2.4 yields that there exists \( l > \max\{w, i\} \) with \( f(t, s) \leq kl, \, t \in [a, b], \, s \in [0, l] \). Consequently, \( \varphi(l) \leq l \). Similarly we could infer that there exist three constants \( c_2, w_2 \) and \( w_3 \) with \( c < c_2 < w_2 < w < w_3 < l \) satisfying
\[
\varphi(c_2) = kc_2, \quad \psi(w_2) = mw_2, \quad \psi(w_3) = mw_3. \tag{3.7}
\]

Thus Theorem 3.1, (3.6) and (3.7) ensure that Eqs. (1.1) and (1.2) possesses at least three nondecreasing positive solutions \( x, y, z \in P \) with
\[
0 < j < j_1 \leq \|x\| \leq c_1 < c < c_2 \leq \|y\| \leq w_2 < w < w_3 \leq \|z\| \leq l.
\]

This completes the proof. \( \square \)

Similar to the proof of Theorems 3.1 and 3.4, we have the following.

Theorem 3.5. Let (1.3) hold. Assume that there exist two different positive constants \( c \) and \( j \) satisfying \( \varphi(c) < kc \) and \( \psi(j) > mj \), respectively. Then Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \( u \in P \) with \( \min\{c, j\} < \|u\| < \max\{c, j\} \).

Theorem 3.6. Let (1.3) hold. If one of the following conditions
\[
\min\{\varphi_0, \varphi_\infty\} < k \quad \text{and} \quad \psi(j) > mj \quad \text{for some} \quad j > 0; \tag{3.8}
\]
\[
\max\{\psi_0, \psi_\infty\} > m \quad \text{and} \quad \varphi(c) < kc \quad \text{for some} \quad c > 0 \tag{3.9}
\]
is satisfied, then Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution in \( P \).

**Proof.** Without loss of generality, we assume that (3.8) holds. Since \( \min\{\varphi_0, \varphi_\infty\} < k \), it follows that there exists a positive constant \( l \neq j \) with \( \varphi(l) < kl \). Therefore Theorem 3.5 and \( \psi(j) > mj \) yield that Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \( u \in P \) with \( \min\{l, j\} < \|u\| < \max\{l, j\} \). This completes the proof. \( \square \)

From Theorem 3.6 and Lemma 2.1, we obtain the following result.

Theorem 3.7. Let (1.3) hold. If either
\[
\min\{f_0, f_\infty\} < k \quad \text{and} \quad \psi(j) > mj \quad \text{for some} \quad j > 0
\]
or
\[
\max\{f_0, f_\infty\} > \frac{m}{h(p)} \quad \text{and} \quad \varphi(c) < kc \quad \text{for some} \quad c > 0
\]
is fulfilled, then Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution in \( P \).
Theorem 3.8. Let (1.3) hold. If either \( \varphi_{1} < k \) or \( \varphi_{0} > m \) is satisfied, then Eqs. (1.1) and (1.2) has infinitely many nondecreasing positive solutions \( \{x_{n}\}_{n \geq 1} \subset P \) with \( \lim_{n \to \infty} \|x_{n}\| = 0 \).

Proof. Since \( \varphi_{1} < k \) and \( \varphi_{0} > m \), it follows that there exist two sequences \( \{c_{n}\}_{n \geq 1} \) and \( \{j_{n}\}_{n \geq 1} \) such that

\[
\varphi(c_{n}) < kc_{n}, \quad \psi(j_{n}) > mj_{n}, \quad c_{n} > j_{n} > c_{n+1} \quad \text{for each } n \geq 1 \text{ and } \lim_{n \to \infty} c_{n} = 0. \tag{3.10}
\]

It follows from Theorem 3.5 and (3.10) that for each \( n \geq 1 \), Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \( x_{n} \in P \) with \( j_{n} < \|x_{n}\| < c_{n} \). Obviously, \( \lim_{n \to \infty} \|x_{n}\| = 0 \). This completes the proof. \( \square \)

By the same arguments used in Theorem 3.8, we have the following result.

Theorem 3.9. Let (1.3) hold. If either \( \varphi_{\infty} < k \) or \( \varphi_{\infty} > m \) is satisfied, then Eqs. (1.1) and (1.2) has infinitely many nondecreasing positive solutions \( \{x_{n}\}_{n \geq 1} \subset P \) with \( \lim_{n \to \infty} \|x_{n}\| = +\infty \).

Theorem 3.10. Let (1.3) hold. Suppose that there exist constants \( d_{0}, a_{0}, b_{0} \) and \( c \) with \( 0 < d_{0} < a_{0} < b_{0} \leq c \) such that

\[
\begin{align*}
    f(t, 0) & > 0 \quad \text{for some } t_{0} \in [a, b]; \\
    f(t, s) & < kd_{0}, \quad t \in [a, b], \ s \in [0, d_{0}]; \\
    f(t, s) & > a_{0}m, \quad t \in [p, q], \ s \in [a_{0}, b_{0}]; \\
    f(t, s) & \leq kc, \quad t \in [a, b], \ s \in [0, c].
\end{align*}
\tag{3.11-3.14}
\]

Then Eqs. (1.1) and (1.2) possesses at least three nondecreasing positive solutions \( x_{1}, x_{2}, x_{3} \in \overline{P}_{c} \) satisfying

\[
\|x_{1}\| < d_{0}, \quad a_{0} < x_{2}(p), \quad \|x_{3}\| > d_{0} \quad \text{and} \quad x_{3}(p) < a_{0}.
\]

Proof. Define the operator \( T : P \to C[a, b] \) by

\[
T(x)(t) = \int_{a}^{b} G(t, s)f(s, x(s))ds, \quad t \in [a, b], \ x \in P.
\]

As in the proof of Theorem 3.1, we conclude that \( T \) is a completely continuous operator from \( P \) into itself. Put \( \alpha(x) = \min_{t \in [p, q]} |x(t)|, x \in P. \) It is easy to see that \( \alpha \) is a nonnegative continuous concave functional on \( P \) and \( \alpha(x) \leq \|x\| \) for each \( x \in P. \) In view of Lemma 2.2, (1.3) and (3.12), we obtain that for any \( \|x\| \leq d_{0}, \)

\[
\|T(x)\| = \sup_{t \in [a, b]} \int_{a}^{b} G(t, s)f(s, x(s))ds \\
\leq \int_{a}^{b} g(s)f(s, x(s))ds \\
< kd_{0} \int_{a}^{b} g(s)ds = d_{0},
\]

which yields that condition (b) of Lemma 2.6 is satisfied. Let \( x \) be in \( \overline{P}_{c}. \) By virtue of Lemma 2.2, (1.3) and (3.14), we get that

\[
\|T(x)\| = \sup_{t \in [a, b]} \int_{a}^{b} G(t, s)f(s, x(s))ds \\
\leq \int_{a}^{b} g(s)f(s, x(s))ds \\
< kc \int_{a}^{b} g(s)ds = c
\]
Lemma 2.6, we can easily conclude that Eqs. (1.1) and (1.2) follow from Lemma 2.2, (3.11) and (3.13), we infer that for any \( x \) \( \in P(\alpha, a_0, b_0) \),

\[
\alpha(Tx) = \min_{t \in [p, q]} \int_a^b G(t, s) f(s, x(s))ds \\
\geq \min_{t \in [p, q]} h(t) \int_a^b g(s) f(s, x(s))ds \\
\geq h(p) \int_p^q g(s) f(s, x(s))ds \\
> h(p) a_0 m \int_p^q g(s)ds = a_0.
\]

That is, condition (a) of Lemma 2.6 is fulfilled. Similarly, for any \( x \) \( \in P(\alpha, a_0, c) \) and \( \|Tx\| > b_0 \), we arrive at

\[
\alpha(Tx) = \min_{t \in [p, q]} \int_a^b G(t, s) f(s, x(s))ds \\
\geq h(p) \int_a^b g(s) f(s, x(s))ds \\
\geq h(p) \|Tx\| > h(p) b_0 \geq a_0.
\]

Therefore, condition (c) of Lemma 2.6 is satisfied. It follows from Lemma 2.6 that the operator \( T \) has at least three fixed points \( x_1, x_2, x_3 \in P_c \) satisfying

\[
\|x_1\| < d_0, \quad a_0 < x_2(p), \quad \|x_3\| > d_0 \quad \text{and} \quad x_3(p) < a_0.
\]

Notice that Lemma 2.3, (3.11) and the definition of \( P \) mean that the fixed points \( x_1, x_2, x_3 \) of \( T \) are nondecreasing positive solutions of Eqs. (1.1) and (1.2). This completes the proof. \( \square \)

**Theorem 3.11.** Let (1.3) hold. Suppose that there exist positive constants \( d_0, a_0, b_0 \) and \( c \) with \( d_0 < a_0, \frac{a_0}{h(p)} \leq b_0 \) satisfying (3.11)–(3.13) and

\[
f(t, s) \leq ks, \quad t \in [a, b], s \in [c, +\infty).
\]

Then there exists \( c_0 > \max\{b_0, c\} \) such that Eqs. (1.1) and (1.2) possesses at least three nondecreasing positive solutions \( x_1, x_2, x_3 \in P_{c_0} \) satisfying

\[
\|x_1\| < d_0, \quad a_0 < x_2(p), \quad \|x_3\| > d_0 \quad \text{and} \quad x_3(p) < a_0.
\]

**Proof.** Lemma 2.4 and (3.15) ensure that there exists \( c_0 > \max\{c, b_0\} \) satisfying

\[
f(t, s) \leq kc_0, \quad t \in [a, b], s \in [0, c_0].
\]

Hence Theorem 3.11 follows from Theorem 3.10. This completes the proof. \( \square \)

**Remark 3.1.** According to the proof of Theorem 3.10 (resp. Theorem 3.11), we can easily conclude that Eqs. (1.1) and (1.2) possesses at least two nondecreasing positive solutions and one nondecreasing nonnegative solution in \( P_c \) (resp. \( P_{c_0} \)) provided that condition (3.11) is omitted.
4. Examples

In this section, we construct four examples to illustrate our results.

Example 4.1. Let \(a = 0, b = 3, p = 1, q = 2, c = 1, j = 20\) and
\[
\begin{align*}
f(t, s) &= \begin{cases} 
\frac{3 - ts}{27}, & t \in [0, 3], \ s \in (-\infty, 1], \\
\frac{27 - t}{27} + ts(s - 1), & t \in [0, 3], \ s \in (1, +\infty).
\end{cases}
\end{align*}
\]
It is easy to verify that \(k = \frac{1}{9}, h(1) = \frac{1}{18}, m = \frac{27}{5},\)
\[
\begin{align*}
\psi(c) &= \max \left\{ \frac{3 - ts}{27} : t \in [a, b], \ s \in [0, c] \right\} = \frac{1}{9} = kc, \\
\psi(j) &= \min \left\{ \frac{3 - t}{27} + ts(s - 1) : t \in [p, q], \ s \in \left[ \frac{j}{18}, j \right] \right\} = \frac{10261}{27} > 108 = mj
\end{align*}
\]
and \(f\) satisfies (1.3). It follows from Theorem 3.1 that Eqs. (1.1) and (1.2) has at least one nondecreasing positive solution \(x\) in \(P\) such that \(1 \leq \|x\| \leq 20.\)

Example 4.2. Let \(a = -1, b = 2, p = 0, q = 1\) and
\[
f(t, s) = \frac{1}{75} |s|(1 + t^2) + 1944(2 + t)s^2, \quad t \in [a, b], \ s \in \mathbb{R}.
\]
Obviously, \(k = \frac{1}{3}, h(p) = \frac{1}{18}, m = \frac{27}{5},\)
\[
\begin{align*}
\overline{f}_0 &= \limsup_{s \to 0^+} \frac{1}{s} \max \{f(t, s) : t \in [a, b]\} \\
&\leq \limsup_{s \to 0^+} \frac{1}{s} \left( \frac{|s|}{25} + 7776s^2 \right) = \frac{1}{25} < k
\end{align*}
\]
and (1.3) holds. It follows from \(j = \frac{9}{20}\) that
\[
\psi(j) = \min \left\{ f(t, s) : t \in [p, q], s \in \left[ \frac{j}{h(p)}, j \right] \right\} \geq 1358 j + 12 j^2 > mj.
\]
Hence, Theorem 3.7 ensures that Eqs. (1.1) and (1.2) possesses at least one nondecreasing positive solution in \(P.\)

Example 4.3. Let \(a = 0, b = 2, p = 1, q = 2, j = 1, c = 288, w = 4224, i = 141 510 094 848\) and
\[
f(t, s) = \begin{cases} 
\frac{252(|s| + t)}{21 + 12t(1 - s)}, & t \in [a, b], \ s \in (-\infty, 1], \\
12(1 + t) + \frac{1}{32} t^2(s - 1), & t \in [a, b], \ s \in (1, 288], \\
12(1 + t) + \frac{1}{32} t^2(s - 1) + 1236(s - 288)^2, & t \in [a, b], \ s \in (288, 4224], \\
12(1 + t) + \frac{1}{32} t^2(s - 1) + 17688 761 856 + \frac{t}{16}(s - 4224), & t \in [a, b], \ s \in (4224, +\infty).
\end{cases}
\]
It is easy to see that \(h(p) = \frac{1}{8}, k = \frac{3}{8}, m = \frac{48}{11}\) and that (1.3) holds. Clearly, we have that
\[
\psi(j) = \min \left\{ \frac{252(|s| + t)}{21 + 12t(1 - s)} : t \in [p, q], s \in \left[ \frac{j}{h(p)}, j \right] \right\} > \frac{252(1/8 + 1)}{21 + 12 \times 2 \times 7/8} = \frac{27}{4} > mj.
\]
\[ \varphi(c) = \max \left\{ 12(1 + t) + \frac{1}{32} t^2 (s - 1) : t \in [a, b], s \in [0, c] \right\} \]

\[ \leq 36 + \frac{287}{8} < 108 = kc, \]

\[ \psi(w) = \min \left\{ 12(1 + t) + \frac{1}{32} t^2 (s - 1) + 1236(s - 288)^2 : t \in [a, b], s \in (288, 4224) \right\} \]

\[ > 24 + \frac{527}{32} + 1236 \times 240^2 > 71393600 > mw \]

and

\[ f(t, s) = 12(1 + t) + \frac{1}{32} t^2 (s - 1) + 17688761856 + \frac{t}{16} (s - 4224) \]

\[ \leq 36 + \frac{s - 1}{8} + 17688761856 + \frac{s - 4224}{8} \]

\[ \leq \frac{s}{4} \leq ks, \quad t \in [a, b], s \in [i, +\infty). \]

That is, the conditions of Theorem 3.4 are fulfilled. Consequently, Theorem 3.4 guarantees that Eqs. (1.1) and (1.2) has at least three nondecreasing positive solutions \( x, y, z \in P \) satisfying

\[ 1 < \|x\| < 288 < \|y\| < 4224 < \|z\|. \]

**Example 4.4.** Let \( a = 1, b = 3, p = 2, q = 3, d_0 = 1, a_0 = 2, b_0 = 100, c = \frac{6243}{33} \) and

\[ f(t, s) = \begin{cases} 
3 - t + \frac{t^2}{32} s^2, & t \in [a, b], s \in (-\infty, 1], \\
3 - t + \frac{t^2}{32} s^2 + \frac{768}{11}, & t \in [a, b], s \in (1, 2], \\
3 - t + \frac{t^2}{32} s^2 + \frac{768}{11}, & t \in [a, b], s \in (2, 100], \\
3 - t + \frac{t^2}{32} s^2 + \frac{768}{11} + \left| \sin \left( \frac{s^4 - ts}{t + t^3 s^2 (s^3 - 100^3)} \right) \right|, & t \in [a, b], s \in (100, +\infty). 
\end{cases} \]

Notice that \( k = \frac{3}{8}, h(p) = \frac{1}{8}, m = \frac{48}{11}. \) It is easy to verify that (1.3) holds, \( f(1, 0) = \frac{1}{16}. \)

\[ f(t, s) = \begin{cases} 
3 - t + \frac{t^2}{32} s^2 - \frac{3 - t}{32} s^2 + \frac{t^2}{144}, & t \in [a, b], s \in [0, d_0], \\
3 - t + \frac{t^2}{32} s^2 + \frac{768}{11}, & t \in [a, b], s \in [a_0, b_0], \\
3 - t + \frac{t^2}{32} s^2 + \frac{768}{11} + \left| \sin \left( \frac{s^4 - ts}{t + t^3 s^2 (s^3 - 100^3)} \right) \right|, & t \in [a, b], s \in [0, c]. 
\end{cases} \]

That is, the assumptions of Theorem 3.10 are fulfilled. Consequently, Theorem 3.10 implies that Eqs. (1.1) and (1.2) has at least three nondecreasing positive solutions \( x_1, x_2, x_3 \in P_c \) satisfying

\[ \|x_1\| < 1, \quad \|x_2(p)\| > 2, \quad \|x_3\| > 1, \quad x_3(p) < 2. \]

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**References**