Invariant Imbedding and Order-of-Scattering Theory in Radiation Field

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Abstract—Around fifty years ago, the principles of invariance were introduced in the study of radiative transfer in planetary atmospheres and then extensively extended to astrophysics by Ambarsumian and Chandrasekhar. Furthermore, the invariance concept was pursued and generalized by Bellman, Kalaba and Wing, who gave the term “Invariant Imbedding” for the systematic use of invariance concept and functionals in various problems of mathematical physics and engineering. Then the two-points boundary value problem was transformed into the initial value problem.

In the present paper, starting from the low-order of scattering and transmission functions, the higher order of these global functions is approximated with the aid of invariant imbedding expression and then the initial value solution of the two components radiation field has been also dealt with.

1. INTRODUCTION

The diffuse reflection and transmission of radiation by a finite target slab plays an important role in radiation dosimetry. Whereas the usual method of solving the linearized Boltzmann equation of radiation has been used in the two-points boundary value problem, in recent years an invariant imbedding approach to the initial value problem has been developed in astrophysics, radiation dosimetry and applied mathematics (cf. 1 and 2). It has been thereafter shown that the novel method can be applied to a much broader class of scattering problems with the aid of high-speed computers, including the neutron and gamma rays transport problems encountered in biomathematical effects of radiation (cf. 3).

Taking into account the finite-order scattering processes in turbid slab, in a series of papers Bellman et al. (cf. 4–9) obtained an initial value solution of the energy-dependent transport problems. Furthermore, the theory has been extended to the finite-order scattering processes in the two radiation approximation (cf. 10), the double layer approximation (cf. 11), the inverse problem of determining a plane source and others (cf. 12–18).

In the present paper, with the aid of an invariant imbedding in radiation dosimetry, the energy-dependent equation of transfer in a finite optical thickness of homogeneous, noncoherently and anisotropically scattering turbid slab has been transformed into Cauchy system of scattering and transmission functions. Then, in a manner similar to the above arguments, a set of Riccati-type of the nonlinear integro-differential equations for the finite order scattering and transmission
functions has been obtained as recurrence formulae for the above cumulative functions, which have undergone interactions once, twice and so forth up to any order \( n (n \geq 1) \) during the passage through the slab. In the limit, these cumulative functions tend to these Chandrasekhar’s functions for the standard transfer problem. Furthermore, the invariant imbedding is applied for the initial value solutions of the two-components radiation field, i.e., the Cauchy system governing the scattering and transmission matrices.

2. BASIC EQUATIONS

2.1. Equation of Transfer

Consider the homogeneous, noncoherently and anisotropically scattering turbid slab of finite optical thickness \( x \). Suppose that parallel rays of finite energy \( E_0 \) is uniformly incident on \( z = 0 \) in the direction \( \Omega_0 = (\theta_0 = \cos^{-1} u (0 \leq u \leq 1), 0 \leq \phi_0 \leq 2\pi) \), where \( u \) is the cosine of inclination and \( \phi_0 \) is the azimuth.

Define the effective radiation flux by \( F \). Then, \( I(z, E, \Omega) \) denotes the intensity of radiation at depth \( 0 \leq z \leq x \), energy \( E \), and direction \( \Omega (0 \leq \theta = \cos^{-1} u \leq \pi, 0 \leq \phi \leq 2\pi) \). Furthermore, let the upwards (or downwards) directed intensity of radiation be defined by \( I(z, E, +\Omega) \) (or \( I(z, E, -\Omega) \)), whereas \( +\Omega = (+u, \phi) \). Let \( \mu(E) \) and \( P(E, \Omega; E', \Omega') \) be respectively the total extinction coefficient and the energy-dependent phase function. \( P \)-function corresponds to the scattering probability that a photon transposes from the \( (E', \Omega') \) state to the \( (E, \Omega) \) state.

The equation of radiative transfer in the above field takes the form

\[
\frac{d}{dz} I(z, E, \Omega) - \mu(E) I(z, E, \Omega) = \frac{1}{4\pi} \int_0^\infty \int_{-1}^1 \int_{0}^{2\pi} P(E, \Omega; E', \Omega') I(z, E', \Omega') dE' d\Omega',
\]

(1)

where \( d\Omega' = dv' d\phi' \), together with the boundary condition

\[
I(0, E, -\Omega) = \pi F \delta(E - E_0) \delta(\Omega - \Omega_0), \quad I(x, E, +\Omega) = 0.
\]

(2)

2.2. Scattering and Transmission Functions

Let the scattering and transmission functions be

\[
I(0, E, +\Omega) = \frac{F}{4\pi} S(x; E, \Omega; E_0, \Omega_0), \quad (3)
\]

\[
I(x, E, -\Omega) = \frac{F}{4\pi} T(x; E, \Omega; E_0, \Omega_0). \quad (4)
\]

The initial conditions of the above functions are as below.

\[
S(0; E, \Omega; E_0, \Omega_0) = 0, \quad (5)
\]

\[
T(0; E, \Omega; E_0, \Omega_0) = 0. \quad (6)
\]

To solve the transfer equation (1) together with the boundary conditions (2) is not easy analytically and numerically. Then, based on an invariant imbedding (cf. 3), it is solved as an initial value problem. In other words, the \( S \)- and \( T \)-functions are obtained as an initial value solution, and then the internal radiation field is also obtained in terms of the \( S \)- and \( T \)-functions.

The principle of invariant imbedding in the diffuse radiation field takes the form

\[
I(z; E, \Omega) - \frac{1}{4\pi} \int_0^\infty \int_{0}^{2\pi} S(z; E, \Omega; E', \Omega') I(z, E', \Omega') dE' d\Omega',
\]

(7)
and

\[ I(z; E, -\Omega) = \frac{E}{4\Omega} \left[ T(z; E, \Omega; E_0, \Omega_0) + e^{-2\mu(E)/\nu}4\pi\delta(E - E_0)\delta(\Omega - \Omega_0) \right] + \frac{1}{4\pi} \int_0^\infty \int_0^1 \int_0^{2\pi} \int_0^{2\pi} S(z; E, \Omega; \Omega'; E') I(z, E', -\Omega') dE' d\Omega'. \] (8)

Equations (7) and (8) show analytically the invariant imbedding procedure in the diffuse reflection and transmission processes for the downwards and upwards incoming radiation intensities.

On differentiating respectively equations (7) and (8) with respect to \( z \), and passing to the limit \( z = 0 \), after some minor rearrangement of terms, we get a set of the following Riccati-type of the nonlinear integro-differential equations as below.

\[ \frac{\partial}{\partial x} S(x; E, \Omega; E_0, \Omega_0) + \left( \frac{\mu(E)}{\nu} + \frac{\mu(E_0)}{\nu} \right) S = P(E, \Omega; E_0, -\Omega_0) \]

\[ + \frac{1}{4\pi} \int_0^\infty \int_0^1 \int_0^{2\pi} P(E, \Omega; E'', \Omega'') S(x; E'', \Omega''; E_0, \Omega_0) dE'' d\Omega'' \]

\[ + \frac{1}{4\pi} \int_0^\infty \int_0^1 \int_0^{2\pi} S(x; E, \Omega; E', \Omega') P(E', -\Omega'; E_0, -\Omega_0) dE' d\Omega' \]

\[ + \frac{1}{16\pi^2} \int_0^\infty \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} S(x; E, \Omega; E, \Omega') P(E', -\Omega'; E''', \Omega''') dE' d\Omega' \]

\[ \times S(x; E''', \Omega'''', E_0, \Omega_0) dE''' d\Omega''' \] (9)

and

\[ \frac{\partial}{\partial x} T(x; E, \Omega; E_0, \Omega_0) + \frac{\mu(E)}{\nu} T = P(E, -\Omega; E_0, -\Omega_0)e^{-z\mu(E_0)/\nu} \]

\[ + \frac{1}{4\pi} \int_0^\infty \int_0^1 \int_0^{2\pi} P(E, -\Omega; E'', -\Omega'') T(x; E'', \Omega''; E_0, \Omega_0) dE'' d\Omega'' \]

\[ + \frac{1}{4\pi} e^{-z\mu(E)/\nu} \int_0^\infty \int_0^1 \int_0^{2\pi} S(x; E, \Omega; E', \Omega') P(E', \Omega'; E_0, -\Omega_0) dE' d\Omega' \]

\[ + \frac{1}{16\pi^2} \int_0^\infty \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} S(x; E, \Omega; E', \Omega') P(E', \Omega'; E'', -\Omega'') \]

\[ \times T(x; E'', \Omega''; E_0, \Omega_0) dE'' d\Omega'' \] (10)

The initial conditions of equations (9) and (10) are given by equations (5) and (6).

3. FINITE ORDER SCATTERING AND TRANSMISSION FUNCTIONS

Whereas an initial value solution of the transfer equation (1) can be solved analytically, however, it can be also obtained in terms of the finite-order scattering and transmission functions in the case of appropriate thickness of the slab and low energy photons.

Let \( S(n, x; E, \Omega; E_0, \Omega_0) \) and \( T(n, x; E, \Omega; E_0, \Omega_0) \) be respectively the scattering and transmission functions in the radiation field for the scattering of \( n (\geq 1) \) times.

Then, the \( S- \) and \( T- \)functions in the diffuse radiations field are respectively expressed in terms of the finite-order \( S- \) and \( T- \)functions under consideration.

\[ S(x; E, \Omega; E_0, \Omega_0) = \sum_{n=1}^\infty S(n, x; E, \Omega; E_0, \Omega_0). \] (11)

\[ T(x; E, \Omega; E_0, \Omega_0) = \sum_{n=1}^\infty T(n, x; E, \Omega; E_0, \Omega_0). \] (12)
3.1. Equation of Transfer

The equation of transfer in the finite-order scattering field takes the form

$$v \frac{d}{dz} I(n, z, E, \Omega) - \mu(E) I(n, z, E, \Omega)$$

$$= \frac{[1 - d(n, 0)]}{4\pi} \int_0^\infty \int_0^{2\pi} P(E, \Omega; E', \Omega') I(n - 1, z, E', \Omega') dE' d\Omega',$$

(13)

where \( I(n, z; E, +\Omega) \) and \( I(n, z; E, -\Omega) \) represent respectively the upwards and downwards directed intensities of finite times scattered radiation. Equation (13) should be solved subject to the following boundary conditions.

$$I(n, 0; E, -\Omega) = \pi F \delta(E - E_0) \delta(\Omega - \Omega_0) d(n, 0),$$

(14)

$$I(n, x; E, +\Omega) = 0,$$

(15)

where \( d(n, 0) \) is the delta function of Kronecker.

3.2. Scattering and Transmission Functions

Invariant imbedding in the finite-order scattering of radiation takes the form

$$I(n, z, E, +\Omega) = \frac{1}{4\pi v} \sum_{i=0}^{n-1} \int_0^\infty \int_0^{2\pi} S(n - i, x - z; E, \Omega; E', \Omega')$$

$$\times I(i, z, E', \Omega') dE' d\Omega', \quad n \geq 1$$

(16)

and

$$I(n, z; E, -\Omega) = \frac{F}{4\pi v} [T(n, z; E, \Omega; E_0, \Omega_0) + e^{-z\mu(E)/v} 4\pi v \delta(E - E_0) \delta(\Omega - \Omega_0) d(n, 0)]$$

$$+ \frac{1}{4\pi v} \sum_{i=1}^{n-1} \int_0^\infty \int_0^{2\pi} S(n - i, x; E, \Omega; E', \Omega') I(i, z; E', \Omega') dE' d\Omega', \quad n \geq 0.$$  

(17)

On differentiating respectively equations (16) and (17) with respect to \( z \), passing to the limit \( z = 0 \), after some minor rearrangements of terms, we have

$$\frac{\partial}{\partial z} S(n, x; E, \Omega; E_0, \Omega_0) + \left( \frac{\mu(E)}{v} + \frac{\mu(E_0)}{v} \right) S = P(E, \Omega; E_0, -\Omega_0) d(n - 1, 0)$$

(18)

and

$$\frac{\mu(E)}{v} T(n, x; E, \Omega; E_0, \Omega_0) + \frac{\partial T}{\partial z} = P(E, -\Omega; E_0, -\Omega_0) e^{-z\mu(E)/v} d(n - 1, 0)$$

(19)

$$+ \frac{1}{4\pi} \int_0^\infty \int_0^{2\pi} P(E, -\Omega; E''', -\Omega''') T(n - 1, x; E'', \Omega'''; E_0, \Omega_0) dE'' d\Omega'''$$

$$+ \frac{1}{4\pi} e^{-z\mu(E_0)/v} \int_0^\infty \int_0^{2\pi} S(n - 1, x; E, \Omega; E', \Omega') P(E', \Omega'; E_0, \Omega_0) dE' d\Omega'$$

$$+ \frac{1}{16\pi^2} \sum_{i=1}^{n-2} \int_0^\infty \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} S(n - i - 1, x; E, \Omega; E', \Omega')$$

$$\times P(E', \Omega'; E''', -\Omega''') T(i, x; E'', \Omega'''; E_0, \Omega_0) dE'' d\Omega''' dE' d\Omega'.$$
The initial conditions are as below.

\[ S(n, 0; E, \Omega; E_0, \Omega_0) = 0, \quad (20) \]
\[ S(0, 0; E, \Omega; E_0, \Omega_0) = 0, \quad (21) \]
\[ T(n, 0; E, \Omega; E_0, \Omega_0) = 0, \quad (22) \]
\[ T(0, 0; E, \Omega; E_0, \Omega_0) = 0. \quad (23) \]

Then, the \( S \)- and \( T \)-functions for \( n = 1 \) takes the form

\[ S(1, x; E, \Omega; E_0, \Omega_0) = \frac{v_0 P(E; \Omega; E_0, -\Omega_0)}{u_0(E) + v_0(E_0)} \left\{ 1 - \exp \left[ -x \left( \frac{\mu(E)}{v} + \frac{\mu(E_0)}{u} \right) \right] \right\}, \quad (24) \]
\[ T(1, x; E, \Omega; E_0, \Omega_0) = \frac{v_0 P(E, -\Omega; E_0, -\Omega_0)}{u_0(E) - v_0(E_0)} [\exp(x\mu(E_0)/u - x\mu(E)/v), \quad (25) \]

and

\[ T(1, x; E, \Omega; E, \Omega) = \frac{v_0 P(E, -\Omega; E, -\Omega)}{\mu(E_0)} \exp(-x\mu(E)/v) \quad \text{for } u = v, \mu(E_0) \neq \mu(E). \quad (26) \]

In the case of high-order scattering, starting from equations (24)--(26), and solving successively the integro-differential equations, the required high-order \( S \)- and \( T \)-functions are obtained.

4. TWO-COMPONENTS RADIATION FIELD

In the real radiation field, we treat with the multi-component radiation field. For example, if gamma rays of given energy impinge on the medium, it produces Compton electrons inside the medium. These secondary electrons can also produce gamma rays by Brehmsstrahlung such that the emergent beams will be composed of two components. In the analysis of the passage of the multicomponents radiation through the matter, we deal with the incident beam consisting of the two-components vectors. Then, the scattering and transmission beams will also contain two components. Thus, the scattering and transmission beam functions consist of the two-by-two matrices. Also, the differential cross-sections will be described by two-by-two matrices.

In a manner similar to the former section, an invariant imbedding arguments result in the Cauchy system governing the scattering and transmission matrices.

4.1. The Equation of Transfer

In a manner similar to the preceding section, consider the diffuse reflection and transmission of a two-components radiation field by a homogeneous, noncoherently and anisotropically scattering target slab of the geometrical thickness \( z \), whose top at \( z = 0 \) is illuminated in a direction \(-\Omega_0\). The two-components are denoted by subscript 1 and 2. Now, in a steady state situation, we consider the two-components radiation field for an incident illumination of direction \(-\Omega_0\) at \( z = 0 \). The transfer equation for the two-components radiation field takes the form

\[
v \frac{dI(n; z; E_i, \Omega_i)}{dz} - \mu_i(E_i) I(n; z; E_i, \Omega_i) \]
\[ = -B^*(n; z; E_i, \Omega_i) - \frac{1}{4\pi} \int_0^\infty dE'_i \int_{2\pi} d\Omega'_i \mu^*(E_i, \Omega_i; E'_i, \Omega'_i) \]
\[ \times I((n-1); z; E'_i, \Omega'_i) - \frac{1}{4\pi} \int_0^\infty dE'_j \int_{2\pi} d\Omega'_j \mu^*(E_i, \Omega_i; E'_j, \Omega'_j) \]
\[ \times I((n-1); z; E'_j, \Omega'_j), \quad i, j = 1, 2 \text{ and } i \neq j, \quad (27)\]
where $\mu_i(E_i)$ is expressed in terms of the differential phase function $\mu'(E_j', \Omega_j'; E_i, \Omega_i)$, $i = 1, 2$,

$$\mu_i(E_i) = \sum_j \int_0^{\infty} dE_j' \int d\Omega_j' \mu'(E_j', \Omega_j'; E_i, \Omega_i), \quad i, j = 1, 2. \quad (28)$$

In equation (27), the source function $B^*$ at level $z$ is given by

$$B^*(n; z; E_i, \Omega_i) = \frac{1}{4\pi} \mu^*(E_i, \Omega_i; E_i, \Omega_i) \int_0^{\infty} d(E_i, \Omega_i) d(n, 1) + \frac{1}{4\pi} \mu^*(E_i, \Omega_i; E_j, \Omega_j) \int_0^{\infty} d(E_j, \Omega_j) d(n, 1), \quad i \neq j. \quad (29)$$

In equation (29), $I^*(0; E_{i0}; \Omega_{i0})$ is the $i$th component of the incident radiation,

$$I^*(0; E_{i0}; \Omega_{i0}) = \pi F_i \delta(E - E_{i0}) \delta(\Omega - \Omega_{i0}), \quad i = 1, 2. \quad (30)$$

$$\mu^*(E_i, \Omega_i; E_j', \Omega_j') = 4\pi \mu'(E_i, \Omega_i; E_j', \Omega_j') \frac{E_j'}{E_j}. \quad (31)$$

### 4.2. Scattering and Transmission Matrices

The scattering and transmission matrices are defined as below.

$$S(n; x, E; \Omega; E_0, \Omega_0) = \begin{pmatrix} S(n; x, E_1; \Omega_1; E_0, \Omega_0) & S(n; x, E_1; \Omega_1; E_2, \Omega_2) \\ S(n; x, E_2; \Omega_2; E_0, \Omega_0) & S(n; x, E_2; \Omega_2; E_2, \Omega_2) \end{pmatrix}, \quad (32)$$

$$T(n; x, E; \Omega) = \begin{pmatrix} T(n; x, E_1; \Omega_1; E_0, \Omega_0) & T(n; x, E_1; \Omega_1; E_2, \Omega_2) \\ T(n; x, E_2; \Omega_2; E_0, \Omega_0) & T(n; x, E_2; \Omega_2; E_2, \Omega_2) \end{pmatrix}. \quad (33)$$

Then, the scattered and transmitted intensities are expressed as below.

$$I(n; 0; E, \Omega) = \frac{1}{4\pi} S(n; x; E, \Omega; E_0, \Omega_0) F, \quad (34)$$

$$I(n; x; E, -\Omega) = \frac{1}{4\pi} T(n; x; E, \Omega; E_0, \Omega_0) F. \quad (35)$$

Furthermore, the invariant imbedding for the internal radiation field takes the form as follows:

$$I(n; z; E, \Omega) = \frac{1}{4\pi} S(n; x - z; E, \Omega; E_0, \Omega_0) F + \frac{1}{4\pi} \sum_{m=1}^{n-1} \int_0^{\infty} \int_{2\pi} S(n - m; x - z; E, \Omega; E', \Omega') \times I(m; z; E', -\Omega') dE' d\Omega', \quad (36)$$

where

$$F = \begin{bmatrix} F_1 e^{-\mu(E_{i0}) z / u_{i0}} \\ F_2 e^{-\mu(E_{20}) z / u_{20}} \end{bmatrix},$$

and

$$I(n; z; E, -\Omega) = \frac{1}{4\pi} T(n; z; E, \Omega; E_0, \Omega_0) F + \frac{1}{4\pi} \sum_{m=1}^{n-1} \int_0^{\infty} \int_{2\pi} S(n - m; z; E, \Omega; E', \Omega') \times I(m; z; E', -\Omega') dE' d\Omega', \quad (37)$$

where $n \geq 1$. 

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On making use of the differential and limiting procedures in a similar manner to the Section 3, we have

\[
\left( \frac{\mu(E)}{v} + \frac{\mu(E_0)}{u} \right) S(n; x; E, \Omega; E_0, \Omega_0) + \frac{\partial}{\partial x} S(n; x; E, \Omega; E_0, \Omega_0) \\
= \mu^*(E, \Omega; E_0, \Omega_0) d(n, 1) \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi}^\infty \mu^*(E, \Omega; E', \Omega') S(n - 1; x; E', \Omega'; E_0, \Omega_0) dE' \frac{d\Omega'}{v'} \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi}^\infty S(n - 1; x; E, \Omega; E'', \Omega'') \mu^*(E'', -\Omega'', E_0, \Omega_0) dE'' \frac{d\Omega''}{v''} \\
+ \left( \frac{1}{4\pi} \right)^2 \sum_{m=1}^{n-2} \int_0^\infty \int_0^\infty \int_{2\pi}^\infty S(n - m - 1; x; E, \Omega; E', \Omega') \\
\times \mu^*(E', -\Omega', E''', \Omega''') S(m; x; E''', \Omega'''; E_0, \Omega_0) dE' \frac{d\Omega'}{v'} dE'' \frac{d\Omega''}{v''} .
\]

and

\[
\frac{\mu(E)}{v} + \frac{\partial}{\partial x} T(n; x; E, \Omega; E_0, \Omega_0) \\
= \mu^*(E, -\Omega; E_0, -\Omega_0) e^{-\mu(E_0)\pi(1/u)} d(n, 1) \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi}^\infty \mu^*(E, -\Omega; E'', -\Omega'') T(n - 1; x; E'', \Omega'', E_0, \Omega_0) dE'' \frac{d\Omega''}{v''} \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi}^\infty S(n - 1; x; E, \Omega; E', \Omega') \mu^*(E', \Omega', E_0, -\Omega_0) e^{-\mu(E_0)\pi(1/u)} dE' \frac{d\Omega'}{v'} \\
+ \left( \frac{1}{4\pi} \right)^2 \sum_{m=1}^{n-2} \int_0^\infty \int_0^\infty \int_{2\pi}^\infty S(n - m - 1; x; E, \Omega; E', \Omega') \\
\times \mu^*(E', \Omega'; E''', \Omega''') T(m; x; E''', \Omega'''; E_0, \Omega_0) dE' \frac{d\Omega'}{v'} dE'' \frac{d\Omega''}{v''} .
\]

where \( \mu^* \)-matrix is given by

\[
\mu^*(E, \Omega; E_0, \Omega_0) = \left( \begin{array}{ccc}
\mu^*(E_1, \Omega_1; E_{10}, \Omega_{10}) & 0 \\
0 & \mu^*(E_2, \Omega_2; E_{20}, \Omega_{20})
\end{array} \right) .
\]

The initial conditions for equations (38) and (39) are as below.

\[
S(n; 0; E, \Omega; E_0, \Omega_0) = 0, \quad n = 1, 2, \ldots,
\]

\[
S(0; x; E, \Omega; E_0, \Omega_0) = 0,
\]

\[
T(n; 0; E, \Omega; E_0, \Omega_0) = 0, \quad n = 1, 2, \ldots,
\]

\[
T(0; x; E, \Omega; E_0, \Omega_0) = 0.
\]

Equations (38) and (39) correspond to \( S \)- and \( T \)-matrices, respectively. On the other hand, the incident beam in some dosimetry problems may involve only single component, say the gamma rays alone. Since these give rise to electrons inside the media, we shall have both the components in the emergent radiation. Therefore, we will be interested in writing down equations governing \( S_{11}, S_{21}, T_{11}, \) and \( T_{21} \) components of the scattering and transmission matrices.
\[
\begin{align*}
1 \mu(E_2) S(n; x; E_2, \Omega_2; E_{10}, \Omega_{10}) + \frac{\partial}{\partial x} S(n; x; E_2, \Omega_2; E_{10}, \Omega_{10}) \\
= \mu^*(E_2, \Omega_2; E_{10}, -\Omega_{10}) d(n, 1) \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi} S(n - 1; x; E_2, \Omega_2; E_{10}, \Omega_{10}) dE' \frac{d\Omega''}{\nu''} \\
+ \mu^*(E_2, \Omega_2; E''_2, \Omega''_2) S(n - 1; x; E''_2, \Omega''_2; E_{10}, \Omega_{10}) dE'' \frac{d\Omega''}{\nu''} \\
+ \frac{1}{4\pi} \int_0^\infty \int_{2\pi} [S(n - 1; x; E_2, \Omega_2; E'_1, \Omega'_1) \mu^*(E'_1, -\Omega'_1; E_{10}, -\Omega_{10}) \\
+ S(n - 1; x; E_2, \Omega_2; E'_2, \Omega'_2) \mu^*(E'_2, -\Omega'_2; E_{10}, -\Omega_{10})] dE' \frac{d\Omega'}{\nu'}.
\end{align*}
\]

The \( S_{11}\)- and \( S_{21}\)-functions of order \( n=1 \) are provided by

\[
S(1; x; E_1, \Omega_1; E_{10}, \Omega_{10}) = \frac{\nu \mu(E_1, \Omega_1; E_{10}, \Omega_{10})}{\nu \mu(E_{10}) + \mu(E_1)} \\
\times \left\{ 1 - \exp\left[ - \left( \frac{\mu(E_1)}{v} + \frac{\mu(E_{10})}{u_{10}} \right) x \right]\right\}.
\]
and
\[
S(1; x; E_2, \Omega_2; E_{10}, \Omega_{10}) = \frac{v}{\mu(E_2)} \mu^*(E_2, \Omega_2; E_{10}, \Omega_{10}) 
\times \left\{ \exp \left[ -\frac{\mu(E_{10})}{u_{10}} x \right] - \exp \left[ -\frac{\mu(E_1)}{v} x \right] \right\}. \tag{48}
\]

5. DISCUSSION

In the present paper, it is shown how to solve effectively the transport problems in radiation dosimetry via invariant imbedding. In other words, an invariant imbedding transformed the two-points boundary value problem into the initial value problem. Furthermore, it is extended to the finite-order scattering and transmission functions in the case of finite appropriate thickness and low energy photons. Then, we dealt with the diffuse reflection and transmission of two-components radiation by a homogeneous, noncoherently and anisotropically scattering slab. Thus, the scattering and transmission functions consist of the two-by-two matrices.

REFERENCES