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# Viscosity Approximating Solutions to ODE Systems That Admit Shocks, and Their Limits

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We study nonlinear systems of ordinary differential equations that arise when considering stationary one-dimensional systems of conservation laws with forcing terms defined in a bounded interval. We construct weak entropy solutions of bounded variation which are pointwise and $\mathscr{L}^1$ limits of solutions of regularized, i.e., viscous, systems, where the limit is taken in the viscosity parameter. In particular, no oscillations occur either for the viscous solutions or for the inviscid one. We also discuss the possible formation of boundary layers when boundary values are prescribed for the viscous regularized equations. As applications, first we show the existence of transonic solutions of bounded variation with strong shocks for the equation of stationary gas flow in a duct of variable area as a pointwise limit of artificial viscosity solutions. We analyze their properties depending on the kind of duct as well as on the boundary data of the regularized problem. Second we show that the model applies to the hydrodynamic modeling for semiconductor devices for some particular heat conduction terms and added diffusion to the energy equation. In particular, we show that under the assumption of bounds for the state variables, there exists a regular solution for that particular viscous heat conducting model. Also, if the bounds for the state variables are uniform in the vanishing parameters, we obtain the existence of an inviscid weak entropy solution of bounded variation as a pointwise limit of the regular ones.


## 1. INTRODUCTION

We consider a system of stationary conservation laws with forcing terms

$$A(x)nv = j \tag{1.1.1}$$

$$F_1(n,T)_x = \left(\frac{mj^2}{n} + knT\right)_x = S_1(n,T,x) \tag{1.1.2}$$

$$F_2(n,T)_x = \left(\frac{mj^2}{n^2} + \alpha T\right)_x = S_2(n,T,x),\ x \text{ in the bounded interval } [0,1], \tag{1.1.3}$$



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where $j$ is a positive constant; $n, v, T$ usually represent density, velocity, and temperature, respectively, and $m$, $k$, and $\alpha$ are dimensionless physical parameters depending on the fluxes of the model and their constitutive relationships. We may allow $m$ to depend on $x$; this is the case in nozzle flow in a duct of variable area (See Section 5).

These are the steady state model of the equations of motion for a one-dimensional fluid flow modified by the presence of external forces or sources, arising, for example, from a collision term in statistical mechanical theory, and/or coupling with external forces, or geometrical effects produced by sources or drains.

Appendix 1 is dedicated to a simple introduction to gas dynamics and their thermodynamic relations. Still, we mention here that the above systems of equations model a compressible flow where the pressure $P = knT$ represents the perfect gas law and the temperature $T$ is defined through a differential relation between the internal energy and the entropy state variable $S$.

Thus, system (1.1) would model stationary inviscid flows which may be nonisentropic and admit moderate shocks so that the energy equation cannot be "replaced" by a pressure–density relationship.

The aim of this paper is to construct admissible solutions of the system (1.1) ("inviscid" solutions) as pointwise limits of solutions of the boundary value problem associated with higher order regularized systems for (1.1) (by adding viscosity or heat conduction) that are of uniform bounded variation. In particular, the limiting solution has $x$-sided limits at any point.

As an application, we prove existence of weak solutions of the inviscid transonic stationary gas nozzle flow equations (5.1), as limits of solutions to regularized ones.

It is well known that solutions of the system (1.1) will admit discontinuities, but require an extra condition for uniqueness of the initial value problem in the corresponding transient model. Here, we take the classical entropy conditions; i.e., density increases across a discontinuity along the particle path (see [CF, MP, ZR]). Indeed, we shall see, as expected, that for polytropic gases the entropy state variable $S$ will have a discontinuity at the same value where the density $n$ that solves system (1.1) is discontinuous.

We say that $(n, T)$ is an admissible inviscid solution of (1.1) if it is a generalized solution of the system of Eqs. (1.1) and satisfies the entropy condition.

We construct an inviscid solution of bounded total variation of system (1.1) that arises as a pointwise limit of solutions to "viscous" systems related to (1.1), with total variation bounded uniformly in viscosity. (By viscosity we mean adding a "small" second-order factor.)

For many examples it is an open question if the viscous systems we introduce are physical ones. However, we may obtain an inviscid solution as a pointwise limit of an "artificial viscosity" solution of the higher order system of equations.

We shall present some of the conditions for admissible second-order viscous terms such that the solutions of the viscous systems are uniformly bounded in viscosity measure. These bounds also depend very strongly on the behavior of the right-hand terms $S_1$ and $S_2$.

In order to carry out our program, we present in Section 2 a change of the state variables, the viscous system, and the related boundary value problem. The new system in the new state variable will have decoupled the flux functions and will be of a more manageable form

$$u_x = \tilde{S}_1(u, w, x) \tag{1.2.1}$$

$$F(w)_x = \tilde{S}_2(u, w, x), \qquad x \in I, \tag{1.2.2}$$

where $F'(w) = 0$ for exactly one value of $w$. These new variables appear to be the natural ones as the density $n$, expressed in terms of $u$ and $w$, is monotone in $w$ and the value $w = 0$ is the equation of the sonic line in terms of the original variables $n$ and $T$. Consequently, $w$ will admit discontinuities as a function of $x$. That means, as we shall see later that the viscosity term that yields the increasing density condition *must* be the one that makes $w$ increase across a discontinuity, i.e., must jump from a supersonic region ($w < 0$) to a subsonic region ($w > 0$) keeping $F(w)$ continuous as a function of $x$.

We construct a full solution of the viscous-regularized boundary value problem associated with (1.2) for a mathematical model, where the source-forcing terms $S_1$ and $S_2$ satisfy certain growth conditions.

In Section 3, we follow [G1] to show that we have existence and uniform BV estimates for the viscous solution of the regularized equation corresponding to (1.2.2) and so we obtain an inviscid entropy solution of bounded variation as pointwise and $L^1$ limit of the viscous ones, as the lateral limit exists at every point $x$.

In Section 4, a brief discussion of the possible boundary layer related to a boundary value problem is included that is associated with system (1.1). Boundary layers are admissible at both end points.

Assuming that the current flow constant $j$ is positive and following the characterization presented in the work of S. B. Hsu and T. L. Liu [HL], solutions of the boundary value problem of the regularized system associated with (1.2) tend to inviscid solutions satisfying the corresponding system for $\epsilon = 0$, except for possible discontinuities at the boundary. For the stationary models, [HL] have analyzed the boundary layer for the case of singular nonlinear Sturm–Liouville problems with coupling effects as the corresponding ones to transonic flow through a nozzle. There they indicate the form of the boundary layer based on classical analysis of the layer. However, for first-order quasilinear scalar equations with boundary conditions, Bardoes *et al.* [BLN] give the correct way to write boundary conditions for the evolution problem.

Thus, the conditions of the boundary layer are the following:

At the upstream boundary $x = 0$, $(w_0^+ - w)(F(w(x)) - F(w_0^+)) > 0$ for all $w$ between $w_0^+$ and $w_0$ continuous in $x$.

At the downstream boundary $x = 1$, $(w_1^- - w)(F(w(x)) - F(w_1^-)) < 0$ for all $w$ between $w_1^-$ and $w_1$.

Such an inviscid solution is called an asymptotic state because it represents the large-time state of transient solutions of the time dependent system with given end states at $x = \pm\infty$

A proof of the condition on the boundary layer is presented in [G1, G2] for a general forcing-source term, with different arguments of those of classical layer analysis.

In Section 5 we show applications of this model to gas flow through a nozzle duct of variable cross section and to hydrodynamic or energy transport modeling of semiconductor devices.

In particular, we work out the corresponding regularized system associated with the stationary equation for gas dynamics in a nozzle duct with varying area. It is remarkable that for the gas nozzle flow equations (see (1.5)) the second equation (1.2.2) decouples completely from the first one (1.2.1), so that $S_2 = S_2(w, x)$ and thus, system (1.2) is easily solvable, allowing us to show the existence of a weak entropy solution as a limit of solutions to viscous regularized systems.

## 2. Change of State Variables and Regularization

In order to carry out our program, we first rewrite system (1.1) in a new set of state variables. So we consider the system

$$F_1(n, T, j)_x = S_1(n, T, x, j)$$
$$F_2(n, T, j)_x = S_2(n T, x, j), \tag{2.1}$$

where $F_1$ and $F_2$ are defined as in (1.1).

The jacobian $\partial(F_1, F_2)/\partial(n, T)$ vanishes along the curve

$$0 = \frac{mj^2}{n^2}\left[(2k - \alpha) + \frac{k}{mj^2}\alpha n^2 T\right].$$

The resulting curve $\alpha n^2 T = k_j$, with $k_j = (\alpha - 2k)mk^{-1}j^2$, is referred as the "sonic line," according to the interpretation given below, and will be used to define one of the new variables as follows.

We take the following change of variables: Set

$$\begin{aligned} u &= \frac{mj^2}{n} + knT \qquad &&\text{(momentum flux)} \\ w &= \alpha n^2 T - k_j \qquad &&\text{(sonic line equation)}. \end{aligned} \tag{2.2}$$

Indeed the curve $\alpha n^2 T = k_j$ is referred to as the sonic line in the following sense. Since the pressure $P = knT$, the curve $w = \alpha n^2 T - k_j = 0$ can be recast as

$$w = \frac{\alpha}{k}\frac{j^2}{v^2}\frac{P}{n} - k_j = 0.$$

Thus, if we define the thermodynamic variable $c$ as

$$c^2 = \frac{\alpha j^2}{kk_j}\frac{P}{n} = \frac{\alpha}{m(\alpha - 2k)}\frac{P}{n},$$

then the curve $w = 0$ can be written as

$$w = k_j\left(\frac{c^2}{v^2} - 1\right) = 0.$$

Therefore, we call the values of $x$, where $w(x) = 0$ or, equivalently, $(c^2/v^2)(x) = 1$, the sonic points. Regions where $w < 0$ are supersonic ones and regions where $w > 0$ are subsonic ones.

In order to set up the equations in the variables defined by (2.2), we write the energy flux in the new variables. From (1.1.3)

$$\frac{1}{n^2}(mj^2 + \alpha n^2 T) = \frac{1}{n^2}(mj^2 + k_j + w) = \frac{1}{n^2}\left(\frac{(\alpha - k)}{k}mj^2 + w\right). \tag{2.3}$$

From the momentum flux (1.1.2) we have that

$$u = \frac{mj^2}{n} + knT = \frac{1}{n}(mj^2 + kn^2T) = \frac{1}{n}\frac{k}{\alpha}\left(2\frac{(\alpha - k)}{k}mj^2 + w\right). \quad (2.4)$$

So replacing $1/n^2$ from (2.4) into (2.3), we obtain that

$$F_2(n,T) = \frac{(w + a_j)\alpha^2 u^2}{(w + 2a_j)^2 k^2}, \quad (2.5)$$

where we call

$$a_j = \frac{\alpha - k}{k}mj^2 = mj^2 + k_j. \quad (2.6)$$

Therefore Eqs. (1.1) in the new variables, as defined in (2.2), become

$$u_x = S_1(n(u,w), T(u,w), x) \quad (2.7.1)$$

$$(F(w)u^2)_x = S_2(n(u,w), T(u,w), x)(k^2/\alpha^2), \quad (2.7.2)$$

where

$$F(w) = \frac{w + a_j}{(w + 2a_j)^2}$$

with density

$$n = \frac{k(w + 2a_j)}{\alpha u} \quad (2.8.1)$$

and temperature

$$T = \frac{(w + k_j)}{(w + 2a_j)^2}\frac{\alpha}{k^2}u^2. \quad (2.8.2)$$

We note that the change of variables $(n, T){:} \to (u, w)$ is an admissible one as

$$\frac{\partial(u,w)}{\partial(n,T)} = \begin{vmatrix} -\frac{mj^2}{n^2} + kT & kn \\ 2\alpha nT & \alpha n^2 \end{vmatrix} = -\alpha mj^2 - k\alpha n^2 T \neq 0,$$

provided $T \geq 0$.

We anticipate that $u$ and $F(w)u^2$ will be Lipschitz functions of $x$ in $\bar{I}$ and, hence, will also be $F(w)$. Therefore the jump condition $[F(w)u^2] = 0$ is the same as $[F(w)] = 0$. That means that the change of variables conserves momentum and energy.

Also, we remark here that $n$ is monotone in $w$. This will be fundamental in order to construct entropy solutions using the new system.

Using (2.7.1) to eliminate $u_x$ in (2.7.2) we obtain that the system (2.7) is reduced to

$$u_x = \tilde{S}_1(u,w,x) \tag{2.9.1}$$

$$F(w)_x = \left(\frac{w + a_j}{(w + 2a_j)^2}\right)_x = \frac{k^2\tilde{S}_2(u,w,x)}{\alpha^2 u^2} - \frac{2}{u}\frac{w + a_j}{(w + 2a_j)^2}\tilde{S}_1(u,w,x). \tag{2.9.2}$$

Renaming the right-hand sides $S_1$ and $S_2$, respectively, we have reduced the system (2.1) to

$$\left.\begin{aligned} u_x &= S_1(u,w,x) \\ F(w)_x &= S_2(u,w,x) \end{aligned}\right\}, \quad 0 \le x \le 1. \tag{2.10}$$

Since the physical problem does not allow negative temperatures, we are interested in $n^2T \ge 0$ and we want $w = \alpha n^2 T - k_j \ge -k_j$; that is, from (2.6), $w \ge -k_j > -a_j$. Therefore we redefine $F(w)$ to be zero if $w \le -a_j$, as well as $S_1$ and $S_2$.

The new system (2.10) has a sonic line equation,

$$\frac{\partial(u, F(w))}{\partial(u,w)} = F_w = 0,$$

where

$$F_w = \frac{-w}{(w + 2a_j)^3}.$$

We show the graph of $F(w)$ in Fig. 1.

*Remark.* In the case of gas through a nozzle duct (i.e., Eqs. (5.14)) the right-hand side $S_2(u,w,x)$ does not depend on $u$. So the second equation uncouples from the first one.

*A Viscous-Heat Conducting Boundary Value Problem; Existence and Uniform Bounds*

We regularize system (2.10) by adding a higher order term with a small coefficient. The *a priori* estimates to the new problem are uniformly

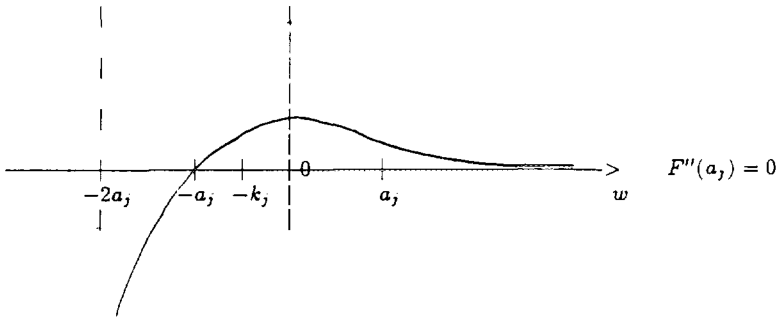


FIG. 1. Graph of $F(w)$.

independent of this coefficient provided the right-hand sides have a particular growth condition.

The method uses the concept of finding an "invariant region" in the domain of the state variables, which is independent of the viscosity parameter. Then, by classical comparison theorems of ordinary differential equations, the solution cannot leave the invariant region. This method is in the spirit of the work on invariant regions developed by Chueh, Conley, and Smoller [CCS] and Weinberger [W] in order to find bounds for solutions of systems of partial differential equations. Because of various technical differences between our problem and the ones treated in [W, CCS] (in particular, a survey on invariant regions can be found in Smoller [S, Chap. 14] and references therein), we present the proofs, which, as in the work of the above references, are dependent on the higher order regularization, as well as on the form of the force-source terms $S_1$ and $S_2$.

Once the conditions on the right-hand sides $S_1$ and $S_2$ are imposed and the uniform invariant region in the $u, w$ plane is found, the remaining estimates and convergence results are basically independent of the form of the right-hand side, provided the graph of the new flux function $F(w)$ has the concavity near a single sonic value $w = w_S$ as the one shown in Fig. 1 near the sonic value $w = 0$.

The regularized boundary value problem we consider is

$$u_x = S_1(u, w, x) \tag{2.11.1}$$

$$E_\epsilon(u, w) = -F(w)_x + S_2(u, w, x) + \epsilon(B(w)w_x)_x = 0,$$

$$x \in I = (0, 1), \tag{2.11.2}$$

$$w(0) = w_0, \qquad w(1) = w_1, \qquad u(0) = u_0 > 0.$$

We assume $w_0, w_1 > -k_j$; and $B$ a differentiable function of $w$, defined for $w > -a_j$ satisfying $B(w) > 0$, $B'(w) \le 0$, for $w > -a_j$.

*Remark.* System (2.11) might also be viewed as the transformed one, corresponding to the boundary value problem

$$\left(m\frac{j^2}{n} + knT\right)_x = S_1(n,T,x)$$

$$\left(\frac{mj^2}{n^2} + \alpha T\right)_x = S_2(n,T,x) + \epsilon\left(\frac{\alpha}{k}\right)^2\left(\frac{mj^2}{n} + knT\right)^2 \cdot \left(B\left(\alpha n^2 T - k_j\right)\left(\alpha n^2 T - k_j\right)_x\right)_x,$$

where $n$ and $T$ are prescribed at $x = 0$ and $x = 1$.

In this section, we present a proof of how to obtain a priori uniform bounds for solutions of (2.11) in the case where the growth on the forcing term $S_1$ is prescribed, and $w^\epsilon$ has $\epsilon$-uniform upper and lower barrier functions with respect to the operator $E_\epsilon$ in [0, 1]. We set up Assumption $\mathscr{A}$ in order to provide for both of these conditions. First, let us define the family of sets $R_A$ depending on one parameter by

$$R_A = \left\{(u,w) \colon P_1^A \le u \le P_3^A, P_2^A \le w \le P_4^A\right\}. \tag{2.12}$$

$R_A$ form a continuous family in the parameter $A$ of smooth rectangles where $\partial R_A$ is given by functions $P_i^A(x)$, $i = 1,\ldots,4$, that vary continuously with respect to $A$, uniformly in $x$, satisfying

$$\begin{matrix} P_1^A \to k_1 \ge 0, & P_3^A \to \infty \\ P_2^A \to k_2 \ge -a_j, & P_4^A \to \infty \end{matrix} \quad \text{as } A \to \infty,\ x \in [0,1]. \tag{2.12.1}$$

Hence, $\partial R_A = \bigcup_{i+1}^4 \partial_i R_A$, where $\partial_i R_A = P_i^A(x)$, $x \in [0,1]$, $i = 1,\ldots,4$.

*Assumption* $\mathscr{A}$. Let $\mathscr{M}_A$ be a nonempty continuous one-parameter family of regions given by smooth rectangles $R_A$ defined as in (2.12) satisfying the condition (2.12.1) such that the "edges" $P_i^A$ of the rectangle $R_A$ are given by functions independent of $\epsilon$ and satisfy that $P_1^A$ and $P_3^A$ are "contracting" curves for $S_1(u^\epsilon, w^\epsilon, x)$, respectively, and such that $P_2^A$ and $P_4^A$ are sub and super solutions of $w^\epsilon$ with respect to $E_\epsilon$ in [0, 1], respectively, for $A$ sufficiently large. That is, dropping $\epsilon$ for the rest of the assumption, there is an $A^*$ such that for all $A \ge A^*$, the evaluation of the terms $S_1$ and $E_\epsilon$ on the boundaries given by $P_i^A(x)$, $i = 2, \dots, 4$, become

$$\begin{aligned} &S_1(P_1^A, w, x) > 0 \\ &S_1(P_3^A, w, x) < 0, \qquad \text{for any } w \text{ if } P_2^A \le w \le P_4^A; \\ &E_\epsilon(u, P_2^A) > 0 \\ &E_\epsilon(u, P_4^A) < 0, \qquad \text{for any } u \text{ if } P_1^A < u < P_3^A. \end{aligned} \tag{2.13}$$

Using the standard notation for invariant regions, condition (2.13) is equivalent to

$$\begin{gathered} (S_1(u,w), E_\epsilon(u,w)) \cdot \boldsymbol{\nu} < 0 \qquad \text{on } \partial R_A; \\ S_1(P_1^A, P_i^A) > 0, \qquad S_1(P_3^A, P_i^A) < 0, \qquad \text{for } i = 2, 4, \end{gathered} \tag{2.14}$$

where $\boldsymbol{\nu}$ denotes the outer normal direction to $\partial R_A$ at $(u, w)$, excluding the "corner" values of the region $R_A$.

The next theorem will show that Assumption $\mathscr{A}$ provides a sufficient condition to find the desired uniform bounds.

THEOREM 2.1 (A priori uniform bounds. Comparison theorem). *Under the assumption $\mathscr{A}$, there exists an $\infty > A^* > 0$ such that if $u^\epsilon$ and $w^\epsilon$ solve the boundary value problem* (2.11) *then $(u^\epsilon, w^\epsilon) \in R_{A^*}$ with $A^*$ independent of $\epsilon$.*

*Proof.* First we recall that solutions $w^\epsilon$ of the second-order operator $E_\epsilon(u, w) = 0$ defined by (2.11.2), with prescribed boundary data are bounded by a constant $M^\epsilon$, where $-a_j < M^\epsilon < \infty$. Next, Assumption $\mathscr{A}$ sets conditions on the equation so that all the functions $P_i^A(x)$ with $A \ge A^*$ are a continuous family in the parameter $A$ of upper and lower barriers of the boundary value problem at any $x \in I$, provided that $P_i^A$ controls the boundary data for $A \ge A^*$. The continuity in the parameter $A$ is needed in order to find functions $P_i^{A^\epsilon}(x)$, $i = 2, 4$, that have first-order contact with $w^\epsilon$ solution of $E_\epsilon(u, w) = 0$ as well as to obtain the $\epsilon$ uniformity of the bounds. By first-order contact we mean that these functions and their tangent planes must coincide at the contact point.

Indeed take $A_1$ as the first parameter that makes $P_1^A(0) < u(0) < P_3^A(0)$ and $P_2^A(r) < w(r) < P_4^A(r)$, $r = 0, 1$, for all $A \ge A_1$. Rename $A^* = \max\{A_1, A^*\}$.

Now let us see that if $u^\epsilon$ and $w^\epsilon$ are solutions (2.11) then they cannot have a contact point with $\partial R_{A^*}$.

Since $A^* \ge A_1$, then $(u^\epsilon(0), w^\epsilon(0)) = (u_0, w_0)$ and $(u^\epsilon(0), w^\epsilon(1)) = (u_0, w_1)$ are in the interior of $R_{A^*}$.

Now, let $x_0 \in I$ be the value of $x$ that makes the first contact point with $\partial R_{A^*}$ by increasing $x$. This first contact could take place at any one of the four edges of the rectangle. Namely, either

(i) $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \partial_1 R_{A^*}$ or (ii) $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \partial_3 R_{A^*}$

or

(iii) $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \partial_2 R_{A^*}$ or (iv) $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \partial_4 R_{A^*}$.

We consider the four cases, one at a time.

(i) If $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \overline{\partial_1 R_{A^*}}$; that is, $u^\epsilon(x_0) = P_1^{A^*}(x_0)$ for the first value of $x$ (i.e., $u^\epsilon(x) \neq P_1^{A^*}(x_0)$ for $x$ in $[0, x_0)$ since $u^\epsilon(0) > P_1^{A^*}(0)$. Then $u$ decreases toward $P_1^{A^*}$ at $x_0$, so $u_x^\epsilon(x_0) \leq 0$. On the other hand, $u_x^\epsilon(x_0) = S_1(P_1^A, w^\epsilon, x_0) > 0$ which contradicts the previous statement. So, $u^\epsilon > P_1^{A^*}$.

(ii) Similarly if $(u^\epsilon(x_0), w^\epsilon(x_0)) \in \overline{\partial_3 R_{A^*}}$ then $u^\epsilon(x_0) = P_3^{A^*}$ for the first value of $x$, so $u_x^\epsilon(x_0) \geq 0$ which contradicts condition (2.14), $u_x^\epsilon(x_0) = S_1(P_3^{A^*}, w^\epsilon, x_0) < 0$. So $u^\epsilon < P_3^{A^*}$.

(iii) and (iv) If either $w^\epsilon(x_0) = P_2^{A^*}(x_0)$ or $w^\epsilon(x_0) = P_4^{A^*}(x_0)$, on one hand we have condition (2.14) which says

$$\begin{aligned} E_\epsilon(u, P_i^A) &= -F'(P_i^A(x_0))P_{i_x}^A(x_0) + S_2(u, P_i^A, x_0) \\ &\quad + \epsilon\big(B(P_i^A(x_0))P_{i_x}^A(x_0)\big)_x \begin{cases} > 0, & i = 2 \\ < 0, & i = 4, \end{cases} \end{aligned} \tag{2.16}$$

On the other hand, if the first contact is produced at $x_0$, then $x_0$ is in the interior of $I$, but it might not be a first-order contact, so we must look for a parameter $A^\epsilon = A(\epsilon)$ such that the contact between $w^\epsilon$ and $P_i^{A^\epsilon}$ will be of first order. Indeed, since $-a_j \leq w^\epsilon \leq \infty$, so by condition (2.12.1), we slide $P_i^A$ continuously in $A$, $i = 2, 4$, from $P_i^{A^*}$ toward $-a_j$ and $\infty$, respectively, so that we can choose the first $A^\epsilon$ such that $P_i^{A\epsilon}$ and $w^\epsilon$ coincide in first order at a point $x_{i,1}$, so that, if $x_0$ is still not a point on which $(w^\epsilon - P_i^{A^*})_x(x_0) = 0$, $i = 2, 4$, we can take an $A^\epsilon \geq A^*$ such that,

$$\begin{aligned} &\text{(iii) } w^\epsilon - P_2^{A\epsilon} \text{ has a minimum at } x_{2,1}, \quad (w^\epsilon - P_2^{A^\epsilon})(x_{2,1}) = 0; \\ &\text{(iv) } w^\epsilon - P_4^{A^\epsilon} \text{ has a maximum at } x_{4,1} \quad (w^\epsilon - P_2^{A^\epsilon})(x_{4,1}) = 0. \end{aligned} \tag{2.17}$$

Therefore, from (2.16) with $x_0$ replaced by $x_{2,1}$ and (2.17) we obtain that in (iii), dropping $\epsilon$ from $u$ and $w$ momentarily,

$$\begin{aligned} 0 &< E_\epsilon(u,w) - E_\epsilon(u, P_2^{A^\epsilon})(x_{2,1}) \\ &= -F'(w(x_{2,1}))(w - P_2^{A^\epsilon})_x(x_{2,1}) \\ &\quad - S_2(u,w,x_{2,1}) + S_2(u, P_2^{A^\epsilon}, x_{2,1}) \\ &\quad - \epsilon B'(w(x_{2,1}))\left(w_x^2 - (P_2^{A^\epsilon})_x^2\right)(x_{2,1}) \\ &\quad - \epsilon B(w(x_{2,1}))(w - P_2^{A^\epsilon})_{xx}(x_{2,1}) \\ &= -\epsilon B(w(x_{2,1}))(w - P^{A^\epsilon})_{xx}(x_{2,1}). \end{aligned}$$

As $B(w)$ is a positive function, then $(w - P_2^{A^\epsilon})_{xx}(x_{2,1}) < 0$, $A^\epsilon \geq A^*$, which contradicts (2.17) (iii).

Similarly, for (iv), one can obtain that $(w - P_4^{A\epsilon})_{xx}(x_{2,1}) > 0$, $A^\epsilon \geq A^*$ which contradicts (2.17) (iv).

Hence, no such contact can take place. Therefore,

$$(u^\epsilon, w^\epsilon) \in \operatorname{int} R_{A\epsilon}, \qquad A^\epsilon \geq A^*.$$

Finally, we need a result that gives the $\epsilon$ uniformity of the bound. Here, we also need that the family $R_A$ is continuous in $A$. Let $w^A = P_2^A$, $A^* \leq A \leq A^\epsilon$, then $w^A$ is a continuous family of subsolutions of $E_\epsilon(w^\epsilon) = 0$ in $I$ such that $w^{A^\epsilon} < w^\epsilon$ in $I$ and $w^A|_{\partial I} < w^\epsilon|_{\partial I}$ for all $A^* \leq A \leq A^\epsilon$. Then $w^A < w^\epsilon$ for all $A^* \leq A \leq A^\epsilon$. (This is a classical result for operators satisfying a comparison principle as the one described before. A proof of this result was written in [G1].) An equivalent result holds for $w = P_4^A$ as a continuous family of supersolutions of $E_\epsilon = 0$ and the corresponding reversed inequalities. Hence, $(u^\epsilon, w^\epsilon) \in \operatorname{int} R_{A^*}$, that is,

$$P_2^{A^*} < w^\epsilon < P_4^{A^*}, \qquad P_1^{A^*} < u^\epsilon < P_3^{A^*} \qquad \text{in } I \text{ uniformly in } \epsilon. \quad (2.18)$$

Next, we need to find a bound for the derivative of $w^\epsilon(x)$.

LEMMA 2.2 (A priori bound for the first-order derivative of $w$). *If $w^\epsilon$ is a differentiable solution of $E_\epsilon(u^\epsilon, w^\epsilon) = 0$ in $I = (0,1)$ and we let $K$ and $C$ be the constants such that $K < w^\epsilon$ and $\|w^\epsilon\|_{L^\infty} \leq C$ uniformly in $\epsilon$, then $\epsilon w_x^\epsilon$ is uniformly bounded in $\bar{I}$ by $\lambda(K,C)$.*

*Proof.* We follow the same idea as in Lemma 4 in [G1]. The only difference in this case with respect to the one in that paper is the viscous term $(B(w)w_x)_x$. Indeed, taking the rescaling $\phi(x) = w^\epsilon(\epsilon x)$ the equation

$E_\epsilon(u, w^\epsilon) = 0$ is reduced to

$$0 = F'(\phi)\phi_x - \epsilon S_{21}(u, \phi, x) - (\beta(\phi))_{xx},$$

where $\beta' = B > 0$, and $\min_{\phi > K}\beta(\phi > B^* > 0$, for $\beta(K)$ chosen positive.

Set $\sigma = \beta(\phi)$. Since $\beta$ is a monotone increasing positive function, then $\beta^{-1}$ is well defined, and $\beta$ and $\beta^{-1}$ preserve Lipschitz functions; that is, $\phi$ has a bounded derivative at a point iff $\sigma$ has a bounded derivative at the same point. Thus, we transform the above equation into an equation for $\sigma$, namely,

$$0 = F'(\beta^{-1}(\sigma))(B(\sigma))^{-1}\sigma_x - \epsilon S_2(u, \beta^{-1}(\sigma), x) - \sigma_{xx}.$$

Since $\phi$ is uniformly bounded and $\beta(\phi)$ is positive then $\sigma$ is positive and uniformly bounded in $\bar{I}$.

In addition, using the uniform bounds on $\phi(x)$ we have that solutions of the above equation are solutions of the elliptic differential inequality

$$\sigma_{xx} \le D(|\sigma_x| + 1) \tag{2.19}$$

in $I' = (0, \epsilon^{-1})$ and $D = D(K, C, B, \beta)$.

We take the following differentiable barrier functions for $\sigma_x(x)$ at a point $x_0 \in \bar{I}'$, where $\sigma(x)$ is the solution of the differential inequality (2.19). Let

$$b^+(x - x_0) = -\frac{C}{\delta^2}((x - x_0) - \delta)^2 + C + \sigma(x_0)$$

for an arbitrarily small $\delta$, defined in the interval $[x_0, x_0 + \delta]$. If $x_0 + \delta > \epsilon^{-1}$, we do a symmetric construction in $[x_0 - \delta, x_0]$.

Also define

$$b^-(x - x_0) = -b^+(x - x_0) \qquad \text{in } [x_0, x_0 + \delta].$$

First, we note that, since $\sigma > 0$.

$$\begin{aligned} b^-(0) &= -\sigma(x_0) < \sigma(x_0) = b^+(0), \\ b^-(\delta) &= -C - \sigma(x_0) < \sigma(x_0 + \delta) < C + \sigma(x_0) = b^+(\delta). \end{aligned} \tag{2.20}$$

Then, we need to check that $b^+$ and $b^-$ are super and subsolutions of the elliptic inequality (2.19) in $[x_0, x_0 + \delta]$, respectively. Indeed,

$$b^+_{xx} = -\frac{2C}{\delta^2} \qquad |b^+_x| \le \frac{4C}{\delta}|x - x_0| < \delta.$$

Hence,

$$b_{xx}^{+} - D(|b_x^{+}| + 1) = -\frac{2C}{\delta^2} - D\left(\mathscr{O}\left(\frac{4C}{\delta}\right) + 1\right) < 0 \quad (2.21.1)$$

if $\delta$ is chosen small enough.

Then, $b^{+}$ is a strict supersolution of (2.19) in $[x_0, x_0 + \delta]$, for a value of $\delta(K, C)$. Analogously, $b^{-}$ is a strict subsolution in the same interval as

$$b_{xx}^{-} - D(|b_{+}^{-}| + 1) = \frac{2C}{\delta^2} - D\left(\mathscr{O}\left(\frac{4C}{\delta}\right) + 1\right) > 0 \quad (2.21.2)$$

if $\delta$ is small enough, where $\delta$ depends on $D$, independent of $\epsilon$.

From (2.20) and (2.21), by the standard maximum principle applied to the inequality (2.19),

$$b^{-} \leq \sigma \leq b^{+} \qquad \text{in } [x_0, x_0 + \delta],$$

where $\delta$ is independent of $x_0$ and $\epsilon$. Assuming that $\sigma$ is differentiable, approaching the first derivatives at the point $x_0$, we obtain

$$-\frac{2C}{\delta} \leq \sigma_x(x_0) \leq \frac{2C}{\delta}.$$

Thus, we have obtained that $|B(\phi)\phi_x| \leq 2C/\delta$. Hence

$$|\epsilon w_x^{\epsilon}| \leq \frac{MC}{\delta} \qquad \text{independent of } \epsilon, \text{ uniformly in } \bar{I}, \quad (2.22)$$

where $M$ depends on the function $B$ and $\delta = \delta(K, C, B, \beta)$.

We finish this section by showing that the problem (2.11) is solvable under Assumption $\mathscr{A}$. We make use of a special case of the Leray–Schauder fixed point theorem combined with classical theorems of ordinary differential equations.

THEOREM 2. *Let* $\mathbf{T}$ *be a compact mapping of a Banach space* $\mathscr{B}$ *into itself and suppose there exists a constant* $M$ *such that* $\|x\|_{\mathscr{B}} \leq M$, *for all* $x \in \mathscr{B}$ *and* $\sigma \in [0, 1]$, *satisfying* $x = \sigma \mathbf{T} x$; *then* $\mathbf{T}$ *has a fixed point.*

A proof of this theorem can be found in [GT, Sec. 10.2].

In order to apply this theorem we construct the operator $\mathbf{T}_\delta(v)$: $C^{0,1}(I) \to C^{0,1}(I)$ for $\delta \ll w_0, w_1$, in the following way.

Given $v \in C^{0,1}(I)$, for $\tilde{v} = (v + a_j)^{+}$, we solve first

$$u_x = S_1(u, \tilde{v}, x), \qquad u(0) = u_0. \quad (2.23)$$

We assume $S_1(u, \tilde{v}(x), x)$ to be a continuous function in the variables $(\tilde{v}(x), x)$ over the region $R = [P_2^{A^*}, P_4^{A^*}] \times \bar{I}$ and uniformly Lipschitz continuous with respect to $u$ over $[P_1^{A^*}, P_3^{A^*}]$. Let $M = \sup_R S_1$ ($M$ depends on $P_i^{A^*}$, $i = 1, \ldots, 4$, from Theorem 1, and on $j$). Then (2.23) has a unique solution $u = u(x)$ on $[0, x + \alpha]$, where $\alpha = \min(1, P_3^{A^*}/M)$.

This is the classical Picard–Lindelöf theorem and if $\alpha < 1$, one can use extension theorems to have the solution defined over a maximal interval of existence which will contain $\bar{I}$. For details we refer to Hartman [H].

Having solved (2.23), we solve

$$\epsilon B(w) w_{xx} = F^{\delta}(\tilde{v})_x - S_2(u, \tilde{v}, x) - \epsilon B'(\tilde{v})(\tilde{v}_x)^2$$
$$w(0) = w_0, \qquad w(1) = w_1, \tag{2.24}$$

where

$$F^{\delta}(v) = \begin{cases} F(v), & v > \delta \\ F(\delta), & v \leq \delta, \end{cases}$$

and we define $w = \mathbf{T}_{\delta}(v)$. Then the following theorem holds.

THEOREM 3 (Existence of a viscous regular solution). *$\mathbf{T}_{\delta}$ defined as above has a fixed point.*

*Proof.* $\mathbf{T}_{\delta}$ is compact mapping of $\mathscr{B} = C^{0,1}(I)$. Indeed, from Eq. (2.24)

$$\|w\|_{C^{1,1}(I)} \leq K(\epsilon, \delta, M), \qquad \text{whenever } \|v\|_{C^{0,1}(\bar{I})} \leq M,$$

where $K(\epsilon, \delta, M)$ is a constant depending on $\epsilon, \delta, M$ and on the form of Eq. (2.24).

Next, let $w^* = \sigma \mathbf{T}_{\delta}(w^*), \sigma \in [0, 1]$. It follows from Theorem 2.1 and Lemma 2.2 that

$$P_2^{A^*} \leq w^* \leq P_4^{A^*}$$

and

$$|w_x^*| \leq K\left(\frac{\sigma}{\epsilon}, \delta\right).$$

Then $\|\tilde{w}\|_{C^{0,1}(I)} \leq M$. The proof of continuity of $T_{\delta}$ is a standard one and can be found in [GT], so the Leray–Schauder theorem says that there is a $w \in C^{0,1}(I)$ such that

$$u_x = S_1(u, \tilde{w}, x)$$
$$\epsilon B(w) w_{xx} = F^{\delta}(\tilde{w})_x - S_2(u, \tilde{w}, x) - \epsilon B'(\tilde{w})(\tilde{w}_x)^2$$

in $I, w(0) = w_0$, and $w(1) = w_1$.

Now, taking $-a_j < \delta < P_2^{A^*}$, where $P_2^{A^*}$ is the uniform lower bound for $w$ (from (2.18)), we have that $F^\delta(\tilde{w}) = F(w)$. Therefore $w$ is a solution of

$$\begin{aligned} u_x &= S_1(u,w,x) \\ F(w)_x &= S_2(u,w,x) + \epsilon(B(w)w_x)_x, \qquad x \in I = (0,1), \\ u(0) &= u_0, \qquad w(0), \qquad w(1) = w_1, \end{aligned} \tag{2.25}$$

which is the original system.

Before getting into the limiting process in Section 3 we present two examples where there exists an $\epsilon$-uniformly bounded regular solution for the regularized boundary value problem.

The first one is a mathematical model with an appropriate growth condition on $S_1$ and $S_2$ which satisfies Assumption $\mathscr{A}$.

The second and more interesting one is the existence of an $\epsilon$-uniformly bounded regular solution of the boundary value problem for the stationary $\epsilon$-(artificial) viscous model of gas nozzle flow corresponding to the inviscid model described by Eq. (5.1) or, equivalently, (5.3).

Although Assumption $\mathscr{A}$ will not be satisfied for the nozzle flow model, still we have a comparison theorem and, consequently, the desired $\epsilon$-uniform bounds, due to the very useful fact that the nozzle flow equations in the new state variables $(u,w)$ decouple the nonlinear flux equation in (2.10) from the variable $u$.

This will result in a transformation from the nozzle flow viscous boundary value problem associated with system (5.3) to a boundary value problem (2.25), where $S_2 = S_2(w,x)$, so that the $\epsilon$-uniform bounds will follows from a simple modification of the comparison Theorem 2.1. We devote Section 5 to working out the details of the nozzle flow equations.

EXAMPLE 1. Consider a mathematical model where the transformed source-force terms $\mathscr{S}_1 = \mathscr{S}_1(u,w)$ and $\mathscr{S}_2 = \mathscr{S}_2(u,w)$ satisfy a condition that the family of "rectangles" defined by (2.12) and (2.12.1) becomes a family of "straight edge contracting" rectangles for the vector field $(\mathscr{S}_1, \mathscr{S}_2)$. That is, Assumption $\mathscr{A}$ is satisfied by a family of $\partial R_A$, where $P_i^A(x) = P_i^A$, $i = 1,\dots,4$, are constant functions satisfying condition (2.12.1) and

$$\begin{gathered} (\mathscr{S}_1, \mathscr{S}_2) \cdot \boldsymbol{\nu}(u,v) < 0 \qquad \text{on } \partial R_A, \\ \mathscr{S}_1(P_1^A, P_i^A) > 0, \qquad \mathscr{S}_1(P_3^A, P_i^A) < 0, \qquad i = 2 \text{ or } 4, \end{gathered} \tag{2.26}$$

holds for all $A \geq A^*$.

Indeed, condition (2.26) means that each rectangle $\partial R_A$ is an "attractor" for the vector field $(\mathscr{S}_1, \mathscr{S}_2)$. In particular, it implies that condition (2.13) holds (as each $P_i^A$ is constant). Therefore, Assumption $\mathscr{A}$ holds independently of the form of the "flux" function $F(w)$, and consequently, the comparison Theorem 2.1, the estimate of the derivatives from Lemma 2.2, hold as well as the existence theorem 3. Hence, a boundary value problem given by (2.11), where the terms $\mathscr{S}_1$ and $\mathscr{S}_2$ satisfy condition (2.26), where $R_A$ is defined as in (2.12) with the family $P_i^A$, $i = 1, \dots, 4$, satisfy (2.12.1), has a solution $(u^\epsilon, w^\epsilon)$ in $C^{1,1}(I)$, where $\|u, w, u_x, \epsilon w_x\|_{L^\infty} \leq \kappa$, $\kappa$ independent of $\epsilon$, but depending on the boundary data and the physical parameters involved in Eqs. (2.11).

## 3. The Existence of a Weak Entropy Solution of the Limiting Problem

In this section we use the vanishing viscosity method to prove the existence of a solution $(u, w)$ of problem (2.10), where $u$ is a strong classical solution and $w$ is a weak solution in the sense of the integral identity,

$$\int_I (F(w)\psi_x + S(u, w, x)\psi)\, dx = 0 \tag{3.1}$$

which is valid for any $\psi \in C_0^2(I)$.

Also we show that the function

$$\mathscr{H}(w) = (F(0) - F(w)) \operatorname{sign} w$$

satisfies the condition that

$$\mathscr{H}(w)(x) + Cx \text{ is monotone increasing,} \tag{3.2}$$

where $C = \sup_I \mathscr{S}_2(u^\epsilon, w^\epsilon, x)$. That is, $(\mathscr{H}(w))_x$ is a measure bounded below by $-C$. Actually, condition (3.2) or, equivalently,

$$\int_I (\mathscr{H}(w) + C)\phi_x \geq 0$$

for all $\phi \in C_0^2(I)$ a positive test function, represents the classical "entropy condition" for the "transonic case" in the sense of Olĭenik [O], Vol'pert [V], and Kružkov [K] for first-order quasilinear equations. Also see Bardos, Leroux, and Nedelec [BLN] for the problem with boundary conditions.

In particular, condition (3.2) will imply that the weak solution $w$ of (3.1) can be written as a sum of a Hölder-$\frac{1}{2}$ continuous function plus a monotone increasing function.

We stress here that the sign of the added $\epsilon$ viscosity in Eq. (2.11) is fundamental to obtain condition (3.2). If $\epsilon$ were chosen negative, the monotonicity property of (3.2) would reverse.

In order to obtain a convergence result for the $\epsilon$-viscosity solutions we proceed very similarly as in [G1] due to the similarity of the system under consideration. Indeed, the "flux" function $F(w)$ is not one to one and has just one extreme point in the domain where $w^\epsilon$, so, solution of (2.25) is uniformly bounded. Therefore, we define

$$\begin{aligned}\mathscr{H}(w^\epsilon) &= (F(w^\epsilon) - F(0))\operatorname{sign}(-w)\\ &= \left(\frac{1}{4a_j} - F(w^\epsilon)\right)\operatorname{sign} w. \qquad (3.3)\end{aligned}$$

Now $\mathscr{H}(w^\epsilon)$ is a monotone increasing function of its argument well defined in $\{w \geq -a_j\}$.

We show that $\mathscr{H}(w^\epsilon)(x)$ is a function of bounded variation in $I$, with a TV-norm bounded independently of $\epsilon$. In fact, multiplying (2.25) by $H_\delta(w) = -\operatorname{sign}_\delta(w)$, a $\delta$-regularization of sign$(-w)$ and integrating with respect to $x$, $\mathscr{H}(w^\epsilon)(x)$ satisfies the equation

$$\begin{aligned}(\mathscr{H}_\delta(w^\epsilon))_x = O(\delta)w_x^\epsilon + (H_\delta S_2)(w^\epsilon) + \epsilon\left(\int H_\delta B(w^\epsilon)\right)_{xx}\\ - \epsilon(H_\delta' B(w^\epsilon))(w_x^\epsilon)^2 \qquad (3.4)\end{aligned}$$

with $\mathscr{H}_\delta(w^\epsilon)(x) = (F(0) - F(w^\epsilon))H_\delta(w)(x)$.

Hence, since the TV-norm is defined as

$$\mathrm{TV}_I(\mathscr{H}_\delta(w^\epsilon)) = \int_0^1 |\mathscr{H}_\delta(w^\epsilon)_x|\, dx,$$

we need to see that the integral on the right-hand side of (3.2) is uniformly bounded in $\epsilon$.

Now, in order to estimate this integral, we use the following lemma which gives an estimate of the integral in $I$ of the last term in (3.4), which is nonnegative as $B(w^\epsilon) > 0$ and $H_\delta'(w^\epsilon) \leq 0$.

LEMMA 3.1. *For each fixed $\epsilon$, there exists a $\delta_0 = \delta_0(\epsilon)$ such that*

$$\int_0^1 -\epsilon(H_\delta' B(w^\epsilon))(w_x^\epsilon)^2 \leq K,$$

*where $K$ is a constant independent of $\epsilon$ and $\delta$ for $\delta < \delta_0$.*

A proof for this lemma can be found in [GL, Lemma 5]. The independence of $\epsilon$ for the bound $K$ relies on the $\epsilon$-independent bounds obtained for $u^\epsilon, w^\epsilon$, and $\epsilon w_x^\epsilon$ obtained in Theorem 2.1 and Lemma 2.2 from Section 2.

The next "key" lemma gives an $\epsilon$-uniform estimate for (3.4). We sketch a proof for it, although it is proven in detail in [G1, Lemma 6].

LEMMA 3.2. *For each fixed $\epsilon$ there exists a $\delta_0 = \delta_0(\epsilon)$ such that*

$$\mathrm{TV}_I(\mathscr{H}_\delta(w^\epsilon)) \leq K$$

*for $\delta < \delta_0$ and $K$ a constant independent of $\epsilon$ and $\delta$.*

*Proof.* By the existence theorem 2.3 we know that $w^\epsilon$ is regular so $\mathscr{H}_\delta(w^\epsilon)$ is of bounded variation. We want to compute the variation and see that it is bounded independently of $\epsilon$ and $\delta$.

In order to analyze this estimate, we compute

$$(F_{\delta,\epsilon})_k = \sum_{0 \leq n \leq k} \int_{I_n} |(\mathscr{H}_\delta(w^\epsilon))_x| \, dx$$

in any number of intervals $I_n, 0 \leq u \leq k$, where $w_x^\epsilon$ does not change sign (i.e., the end points of $I_n$ are local extreme points of $w^\epsilon$). Since $\mathscr{H}_\delta(w^\epsilon)$ is a monotone function of $\omega^\epsilon$ then $\mathscr{H}'_\delta(w^\epsilon)$ does not change sign in $I_n$ either, and then

$$(F_{\delta,\epsilon})_k = \sum_{0 \leq n \leq k} \left| \int_{I_n} (\mathscr{H}_\delta(w^\epsilon))_x \, dx \right|.$$

Then replacing the integrand in the above expression by Equation (3.3), $(F_{\delta,\epsilon})_k$ is estimated by the sum over $n$ of the absolute value of the integral of each term over $I_n$.

The three first resulting sums are $\epsilon, \delta$-uniformly controlled by using the $\epsilon$-uniform estimates over $w^\epsilon, u^\epsilon$, and $\epsilon w_x^\epsilon$ and the quadratic behavior of $F(w)$ about $w = 0$. The last resulting term is given by

$$\sum_{0 \leq n \leq k} \left| \int_{I_n} A(w^\epsilon)(x) \, dx \right|, \qquad \text{where } A(w^\epsilon) = -\epsilon(H'_\delta B)(w^\epsilon)(w_k^\epsilon)^2 \geq 0.$$

Therefore, by Lemma 3.1,

$$\sum_{0 \leq n \leq k} \left| \int_{I_n} A(w^\epsilon) \, dx \right| = \sum_{0 \leq n \leq k} \int_{I_n} A(w^\epsilon) \, dx \leq \int_0^1 A(w^\epsilon) \, dx \leq K$$

with $K$ independent of $\delta$ and $\epsilon$.

THEOREM 3.3 (Passing to the limits). (i) *The functions* $\mathscr{H}(w^\epsilon)(x)$ *as defined in* (3.3) *are of bounded variation in* $I$ *and their total variation norm is independent of* $\epsilon$.

(ii) *The family* $\{\mathscr{H}(w^\epsilon)\}$ *has a sequence* $\{\mathscr{H}(w^{\epsilon_n})\}$, $\epsilon_n \to 0$ *as* $n \to \infty$, *that converges pointwise and in every* $L_p(I)$, $1 \le p < \infty$, *to a function* $\mathscr{H}_0(w)$ *of bounded variation.*

The proof of Theorem 3.3 is very much like the proofs of Theorems 2 and 3 in [G1]. Part (i) follows by taking the limit as $\delta \to 0$ to the family $\{\mathscr{H}_\delta(w^\epsilon)\}_{\delta \le \delta_0}$. This limit exists as the family is of $\delta$-uniform bounded variation and $\mathscr{H}(w^\epsilon)$ is continuous in $x$. Thus, $\mathscr{H}(w^\epsilon) = \lim_{\delta \to 0} \mathscr{H}_\delta(w^\epsilon)$ uniformly in $I$, and $\mathrm{TV}_I(\mathscr{H}(w^\epsilon)) \le K$, $K$ independent of $\epsilon$.

Part (ii) follows from applying Helly's theorem and Kolmogorov's compactness condition (see [N, XVII, Sect. 3]) to the family of $\epsilon$-uniform bound variation function $\{\mathscr{H}(w^\epsilon)\}$.

Therefore, it follows that

$$\mathrm{TV}(\mathscr{H}(w^\epsilon)(x)) \le K \qquad \text{uniformly in } \epsilon \tag{3.5}$$

and there exists a function $\mathscr{H}_0(x)$ such that

$$\lim_{\epsilon \to 0} \mathscr{H}(w^\epsilon)(x) = \mathscr{H}_0(x) \qquad \text{pointwise and in } L_p(I),\ 1 \le p < \infty, \tag{3.6}$$

and also that

$$\mathrm{TV}(\mathscr{H}_0(x)) \le K.$$

Thus, $\mathscr{H}(w^\epsilon)(x)$ is a uniformly bounded family of $L^1(I)$ functions converging to $\mathscr{H}_0(x)$ in $L^1(I)$. Since $\mathscr{H}^{-1}(w)$ is a continuous function of $w$, then we define

$$w(x) = \lim_{\epsilon_n \to 0} \mathscr{H}^{-1}(\mathscr{H}(w_n^\epsilon))(x) = \mathscr{H}^{-1}(\mathscr{H}_0(x)), \tag{3.7}$$

and so $w^{\epsilon_n} \to w$ pointwise and in $L^1(I)$. Also, we must take care of the equation $u_x^\epsilon = S_1(u^\epsilon, w^\epsilon, x)$. Here it is easy to see that $\mathrm{TV}_I(u^\epsilon) = \int_0^1 |u_x^\epsilon|\, dx = \int_0^1 |S_2(u^\epsilon, w^\epsilon, x)| \le K$, where $K$ is a constant depending on the bounds for $u^\epsilon$ and $w^\epsilon$, which are $\epsilon$-independent.

Again, making use of Helly's theorems, there exists a function $u(x)$ such that

$$u(x) = \lim_{\epsilon_m \to 0} u^{\epsilon_m}(x), \qquad \text{as } m \to \infty. \tag{3.8}$$

Next, we see that these $u$ and $w$ are inviscid weak solutions of problem (2.10) in $I$.

THEOREM 3.4. *The pair* $(u, w)$ *defined by* (3.7) *and* (3.8), *respectively, are a solution of problem* (2.10), *where* $u$ *is a classical solution and* $w$ *is a weak solution. Also,* $\mathscr{H}(w)(x) = (F(0) - F(w))\operatorname{sign} w$ *is a function of bounded variation, with* $\mathrm{TV}_I(\mathscr{H}(w)(\epsilon)) \leq K$, $K$ *independent of* $\epsilon$ *and therefore* $w$ *has bounded variation independent of* $\epsilon$.

*Proof.* Let $\phi$ and $\psi$ be a pair of test functions in $C_0^2(I)$, multiplying equations (2.11) and integrating by parts,

$$\int_I u^{\epsilon}\phi_x + \int_I S_1(u^{\epsilon_m}, w^{\epsilon_m}, x)\phi = 0 \tag{3.9.1}$$

and

$$\int_I F(w^{\epsilon_n})\psi_x + \int_I S_2(u^{\epsilon_n}, w^{\epsilon_n}, x)\psi - \epsilon_n \int_I \beta(w^{\epsilon_n})\psi_{xx} = 0, \tag{3.9.2}$$

where $\beta'(w^{\epsilon}) = B(w^{\epsilon})$. Using that $S_1$ and $S_2$ are continuous functions of $u^{\epsilon}$ and $w^{\epsilon}$ in $I$, as well as that $F$ is a continuous function of $w^{\epsilon}$ in $I$, then $w^{\epsilon_n} \to w$, $u^{\epsilon_m} \to u$, $S_1(u^{\epsilon_m}, w^{\epsilon_m}, x) \to S_1(u, w, x)$, and $S_2(u^{\epsilon_n}, w^{\epsilon_n}, x) \to S_2(u, w, x)$, all limits as $\epsilon_n, \epsilon_m \to 0$, converge in $L^1(I)$.

Finally, it can also be shown that

$$\mathscr{H}(w)(x) = (F(0) - F(w))\operatorname{sign} w,$$

where $w$ is the weak solution (2.10) defined above, is also a function of bounded variation with $\mathrm{TV}(\mathscr{H}(w)(x)) \leq K$ uniformly in $I$ and $K$ independent of $\epsilon$, and therefore, $w$ is a function of bounded variation independent of $\epsilon$. Also, from (3.10.2) it can be shown that $F(w)(x)$ is a Lipschitz function of $x$ in $I$.

*The Entropy Condition*

We prove in detail the following theorem. This result assures us that the weak solution we have found in the limiting process satisfies the compatible entropy condition for the density $n$ as $n = n(u, w)$ is monotone in $w$.

THEOREM 3.5. *Let* $w$ *be the weak solution defined in* (3.7) *of problem* (2.11.2) *for* $\epsilon = 0$; *then there is a constant* $C > 0$ *such that the function* $\mathscr{H}(w)(x)$ *defined above satisfies*

$$(\mathscr{H}(w))_x + C > 0 \tag{3.10}$$

*in the sense of the distributions.*

*Proof.* Following [G1], let $\phi \in C_0^2(I)$ be any positive test function. Multiply (3.4) by $\phi$ and integrate; then

$$\begin{aligned} -\int_I \mathscr{H}_\delta(w^\epsilon)\phi_x + \int_I O(\delta) w_x^\epsilon \phi \\ = \int_I (H_\delta S_2)(w^\epsilon)\phi + \epsilon \int_I \left( \int H_\delta(w^\epsilon) B(w^\epsilon) \right) \phi_{xx} \\ - \epsilon \int H_\delta'(w^\epsilon) B(w^\epsilon)(w_x^\epsilon)^2 \phi. \end{aligned} \tag{3.11}$$

Now let $C = \sup S_2(w^\epsilon)$; $C$ is independent of $\epsilon$. Then

$$0 = \int_I (Cx\phi)_x = \int_I Cx\phi_x + \int_I C\phi. \tag{3.12}$$

Subtracting (3.11) from (3.12) we obtain

$$\begin{aligned} -\int_I (\mathscr{H}_\delta(w^\epsilon) + Cx)\phi_x \\ = \int_I (H_\delta S_2(w^\epsilon) + C)\phi + \int_I O(\delta) w_x^\epsilon \phi \\ + \epsilon \int_I \left( \int H_\delta(w^\epsilon) B(w^\epsilon) \right) \phi_{xx} - \epsilon \int_I H_\delta'(w^\epsilon) B(w^\epsilon)(w_x^\epsilon)^2 \phi \\ = A_1 + A_2 + A_3 + A_4. \end{aligned}$$

First, $A_1 = \int_I ((H_\delta S_2)(w^\epsilon) + C)\phi \geq 0$ as $|H_\delta| \leq 1$. Next, $A_4 = -\epsilon \int H_\delta'(w^\epsilon) B(w^\epsilon)(w_x^\epsilon)^2 \phi \geq 0$, as $H_\delta' = -\operatorname{sign}_\delta(w^\epsilon) \leq 0$ and $B(w^\epsilon) > 0$. Next, $|A_2| \leq O(\delta)\epsilon^{-1}K$ where $K$ depends only on the bound of the family $\{\epsilon w_x^\epsilon\}$ and the function $\phi$. Then $A_2 \to 0$ uniformly in $I$ as $\delta \to 0$ for each fixed $\epsilon$.

Finally,

$$A_3 = \epsilon \int_I \left( \int H_\delta(w^\epsilon) B(w^\epsilon) \right) \phi_{xx} \to 0 \qquad \text{as } \epsilon \to 0,$$

since $H_\delta(w^\epsilon) B(w^\epsilon)$ are uniformly bounded in $I$ independently of $\delta$ and $\epsilon$. Therefore, one obtains

$$-\int_I (\mathscr{H}_\delta(w^\epsilon) + Cx)\phi_x \geq A_2(\delta) + A_3(\epsilon);$$

thus taking the limit first as $\delta \to 0$ and then as $\epsilon \to 0$, we obtain that

$$\int -(\mathscr{H}(w) + Cx)\phi_x \, dx \geq 0 \tag{3.13}$$

for any positive test function $\phi$, where $w$ is the weak solution defined in (3.7).

Hence

$$(\mathscr{H}(w) + Cx)_x = \mathscr{H}(w)_x + C \geq 0 \tag{3.14}$$

in the sense of the distributions. In particular, $\mathscr{H}(w(x))$ is a Lipschitz function of $x$ plus a monotone increasing function. Since $\mathscr{H}(w)$ is a monotone increasing function of $w$. Then we obtain that $w(x) = \mathscr{H}^{-1}(\mathscr{H}(w(x)))$ is the sum of a monotone increasing function plus a $\frac{1}{2}$-Hölder continuous function, as the behavior of $F(w)$ is quadratic near $w$ equals zero. Thus, we set the regularity of the weak solution $w$ in the following lemma.

LEMMA 3.6. (a) *Let $w$ be the weak solution defined in* (3.7) *and let $u$ defined in* (3.8) *be the solution of the problem* (2.11) *for $\epsilon = 0$. Then,*

(a) *$w(x) = G(x) + \alpha(x)$, where $G(x) \in C^{1/2}(I)$ and $\alpha(x)$ is a monotone increasing function in $I$ with at most a countable number of discontinuities.*

(b) *$u(x)$ is a $C^{0,1}(I)$ function (Lipschitz).*

(c) *$F(w)(x)$ is a Lipschitz function in $I$.*

(c) *$w(x)$ is a Lipschitz function in $I'$, a closed subinterval of $I$, where $w(x) < 0$ (or $w(x) > 0$) for every $x \in I'$. Thus, $w(x)$ is continuous in the supersonic (subsonic) region. Here $w_{\text{sonic}} = 0$. $w(x)$ can have discontinuities only at subsets $I' \subset I$ such that $w$ takes values above and below $w_{\text{sonic}}$ in subsets of $I'$ with positive measure. This indicates that $I'$ is a transonic region of Eq.* (2.10).

(e) *No oscillations are admissible, as the lateral limit exists at every point of the interval $I$. That is, solution $w$ cannot have oscillations with wave length going to zero as $\epsilon$ goes to zero. In other words, $w$ has a lateral derivative almost everywhere.*

The proof of (a) was developed prior to the statement of the lemma. Statement (b) is a classical result of regularity of ordinary differential equations as $S_1(u, w, x)$ is a locally bounded function of its arguments. Statement (c) is a classical observation of the differentiation theorem in $L^1(I)$ and therefore (d) and (e) come as a trivial conclusion of the form of $F(w)$ and the Lipschitz continuity of $F(w)$ with respect to $x$.

We conclude by pointing out that $(n, T)$ is an admissible inviscid solution of (1.1) given by the change of variables (2.8) and that it satisfies the expected regularity:

(a) the momentum flux, conserved as

$$\left(\frac{mj^2}{n} + knT\right)(x) = u(x), \tag{3.15}$$

is a Lipschitz continuous function, so that the Rankine–Hugoniot jump condition is satisfied for the momentum equation.

(b) From (1.1) and (2.7.2), the energy flux, conserved as

$$\left(\frac{mj^2}{n^2} + \alpha T\right)(x) = \left(\frac{\alpha}{k}\right)^2 u^2 F(w), \tag{3.16}$$

is a Lipschitz continuous function, so the Rankine–Hugoniot jump condition is also satisfied for the energy equation.

(c) The entropy condition on

$$w = \alpha n^2 T - k_j$$

indicates that $n^2 T$ is the sum of a $C^{1/2}(I)$ plus a monotone increasing function, where the discontinuities are from values $(n^-, T^-)$ to $(n^+, T^+)$, where $(n^-)^2 T^-(x) \le k_j \le (n^+)^2 T^+(x)$. This last inequality, along with the jump condition for the momentum flux,

$$\left(\frac{mj^2}{n^-} + kn^- T^-\right) = \left(\frac{mj^2}{n^+} + kn^+ T^+\right),$$

imply that

$$n^- < n^+ \qquad \text{and} \qquad kn^- T^- < kn^+ T^+ \tag{3.17}$$

across a discontinuity. Hence the density $n(x)$ and the pressure $p(x) = knT(x)$ satisfy the classical entropy condition.

It is worth mentioning that another way to see that the classical entropy condition is satisfied is by looking at the density $n$ as a function of $u$ and $w$. Indeed, as $u(x)$ is continuous throughout the domain and $n = n(u, w)$ is monotone in $w$, then $n$ increases across a discontinuity whenever $w$ jumps from $w^- < 0 < w^+$ satisfying the jump condition for the momentum and energy flux.

Also, from (2.8) $T = (\alpha/k^2)u^2(w + k_j)/(w + 2a_j)^2$; then it can be shown that $T$ increases across a discontinuity whenever $w$ becomes discontinuous. An argument similar to the one in Lemma 3.7 below will prove this result.

Finally, we want to remark that the same analysis done to the system of Eqs. (1.1) could have been done to system (A.6) which corresponds to the equations of motion for an ideal gas with constant specific heat expressed in terms of the density $n(x)$ and entropy $S(x)$.

Here the change of state variables that lead us to the same analysis is

$$u = \frac{j^2}{n} + (\gamma - 1)n^\gamma \exp\{S/c_v\}\left(= \frac{j^2}{n} + knT\right)$$

$$w = \gamma n^{\gamma+1} \exp\{S/c_n\} - k_j\left(= \gamma n^2 T - k_j\right), \tag{3.18}$$

where $k_j = j^2((2 - \gamma)/(\gamma - 1))$. (Here the temperature becomes $T = n^{\gamma-1}\exp\{S/c_v\}$.)

*Remark*. Here, for simplicity of notation, $m = m(x)$ is assumed to be the constant value 1. For $m$ nonconstant the change of state variable is slightly modified to one very similar to the one corresponding to gas flow in a nozzle with variable cross section $A(x)$, as will be done in Section 5.

Consequently, Eq. (A.6) in the new state variables becomes

$$u_x = \mathscr{S}_1(u, w, x)$$

$$\left(j^2 u^2 \frac{\gamma^2}{(\gamma - 1)^2} F(w)\right)_x = \mathscr{S}_2(u, w, x) \tag{3.19}$$

with $F(w) = (w + \alpha_j)/(w + 2\alpha_j)^2$, $\alpha_j = k_j + j^2$.

These equations differ from Eqs. (2.7) just by a constant. Therefore, all the analyses and results obtained in the last two sections would apply accordingly to system (3.16).

Hence, expressing the density and entropy in the new state variables,

$$n = \frac{1}{u}\frac{(\gamma - 1)}{\gamma}(2\alpha_j + w)$$

$$S = c_v \log\left\{\frac{u^{\gamma+1}}{(\gamma - 1)}\left(\frac{\gamma}{(\gamma - 1)}\right)^\gamma \frac{w + k_j}{(2\alpha_j + w)^{\gamma+1}}\right\}, \tag{3.20}$$

then if $w$ has a discontinuity at a value of $x_0$, from values $w^-$ to $w^+$ such that $F(w^-)(x_0) = F(w^+)(x_0)$ and $w^- < 0 < w^+$, as $u$ is continuous everywhere, we immediately obtain that the density $n$ becomes discontinuous and that $n^-(x) < (\gamma - 1)\alpha_j/u(x_0)\gamma < n^+(x_0)$. The entropy $S$ also is discontinuous, as $(w + k_j)/(2\alpha_j + w)^{\gamma+1}$ is discontinuous, satisfying that $S^-(x_0) < S^+(x_0)$.

This last statement is proved in the following lemma.

LEMMA 3.7. *Let $S$ be the entropy state variable defined by* (3.20). *Let $x_0$ be a point of discontinuity in $I$ for $w$. Then $S$ discontinuous at $x_0$ and $S^-(x_0) < S^+(x_0)$.*

*Proof.* $-k_j < w^-(x_0) < 0 < w^+(x_0)$, with $F(w^-)(x_0) = F(w)(x_0)$. $S$ is discontinuous at $x_0$, if and only if $\mathscr{S}(w) = (w + k_j)/(w + 2\alpha_j)^{\gamma+1}$ is discontinuous at $x_0$ and $S(w^-)(x_0) < S(w^+)(x_0)$ if and only if $\mathscr{S}(w^-)(x_0) < \mathscr{S}(w^+)(x_0)$. So it is enough to show this last inequality. In fact, it can be shown that $R(w) = \mathscr{S}(w)/F^{\gamma}(w)$ is monotone in $w$, so that, since $w^- < w^+$ with $F(w^-) = F(w^+)$, then $\mathscr{S}(w^-)(x_0) < \mathscr{S}(w^+)(x_0)$. Therefore the proof is completed by showing that

$$\frac{d}{dw}\log R(w) > 0,$$

as the log is a monotone function. Indeed,

$$\frac{d}{dw}\log R(w) = \frac{\mathscr{S}'}{\mathscr{S}}(w) - \gamma\frac{F'}{F}(w). \tag{3.21}$$

Using that $a_j = k_j + j^2 = j^2/(\gamma - 1)$ and that $a_j + j^2 = \gamma a_j$, we compute the right-hand side of (3.21) and obtain

$$\begin{aligned} -&\frac{\gamma}{(w + 2a_j)} + \frac{a_j + j^2}{(w + 2a_j)\left(w + a_j - j^2\right)} \\ &- \gamma\left[-\frac{1}{(w + 2a_j)} + \frac{a_j}{(w + 2a_j)(w + a_j)}\right] \\ &= \frac{a_j + j^2}{(w + 2a_j)\left(w + a_j - j^2\right)} - \frac{\gamma a_j}{(w + 2a_j)(w + a_j)} > 0. \end{aligned}$$

*Remark.* This theory does not predict the location of the discontinuities of the state variables, but finds inviscid solutions as the limit of some (real or artificial) viscous-heat conducting ones.

## 4. Boundary Layer Analysis

In this section we investigate what the boundary layer looks like near the ends for the viscous boundary value problem,

$$u_x^\epsilon = \mathscr{S}_1(u^\epsilon, w^\epsilon, x), \tag{4.1.1}$$

$$-F(w^\epsilon)_x + \mathscr{S}_2(u^\epsilon, w^\epsilon, x) + \epsilon(B(w^\epsilon)w_x^\epsilon)_x = 0, \qquad x \text{ in } (0,1)$$

$$w^\epsilon(0) = w_0, \qquad w^\epsilon(1) = w_1, \qquad u^\epsilon(0) = u_0 > 0. \tag{4.1.2}$$

A boundary layer may develop at both endpoints of the boundary, only for the $w^\epsilon$ solution of (4.1). This suggests that the inviscid limiting solution of (4.1) for $\epsilon = 0$ might not solve the initial value problem for system (2.10) corresponding to the inviscid transonic model, i.e., when boundary data is prescribed at the inflow (or upstream) boundary.

For first-order quasilinear scalar equations with boundary conditions, [BLN] give the correct way to write the boundary conditions for the evolution problem.

For the stationary models, [HL] have analyzed the boundary layer for the case of singular nonlinear Sturm–Liouville problems with coupling effects as the corresponding ones to transonic flow through a nozzle. There, they also indicate the form of the boundary layer based on classical analysis of the layer.

Recalling that the current flow constant $j$ is positive and following the characterization presented in [HL], the solutions of system (4.1) tend to inviscid solutions satisfying the corresponding system for $\epsilon = 0$, except for possible discontinuities.

The interior discontinuities must keep the function $F(w(x))$. At the upstream boundary $x = 0$, $(w_0^+ - w)(F(w(x)) - F(w_0^+)) > 0$ for all $w$ between $w_0^+$ and $w_0$ continuous in $x$. At the downstream boundary $x = 1$, $(w_1^- - w)(F(w(x)) - F(w_1^-)) < 0$ for all $w$ between $w_1^-$ and $w_1$. Such an inviscid solution is called an asymptotic state because it represents the large-time state of transient solutions of the time-dependent system with given end states at $x = \pm\infty$ (see Liu [L2]).

These boundary conditions can also be expressed in the following terms: if the prescribed values of $w_0$ and $w_1$ are subsonic, i.e., $w_0$ and $w_1 > 0$, a "subsonic" boundary layer may be formed in the upstream boundary, which in our problem is at $x = 0$, and a "shock boundary layer" may be formed in the downstream boundary at $x = 1$. That is, let $w$ be the limiting inviscid solution defined by (3.7); then the upstream boundary limiting value

$$w_0^+ = \lim_{\substack{x \to 0 \\ x > 0}} w(x) \text{ must stay subsonic}, \qquad \text{i.e., } w_0^+ \geq 0. \tag{4.2}$$

On the other hand, the downstream limiting value $w_1^- = \lim_{x \to 1, x<1} w(x)$ satisfies either

$$w_1^- = w_1$$

or

$$w_1^- < w_1^* < 0 \qquad \text{with } F(w_1^-) < F(w_1^*) = F(w_1). \tag{4.3}$$

That is, $w_1^-$, if different from $w_1$, not only must become supersonic but, further, stay below the conjugate value of $w_1$ by $F(w)$.

A proof of this statement is given in [G1] with different arguments from classical analysis of the layer. Indeed, as discussed in [G2], a discontinuity which does not necessarily form at a distance $\epsilon$ from the boundary is undetectable with standard boundary layer analyses.

Similarly, for supersonic data, i.e., $w_0$ or $w_1 < 0$, we can state that at the upstream boundary either

$$w_0^+ = \lim_{\substack{x \to 0 \\ x>0}} w(x) = w_0$$

or

$$w_0^+ > w_0^* > 0 \qquad \text{with } F(w_0^+) < F(w_0^*) = F(w_0), \tag{4.4}$$

and at the downstream boundary

$$w_1^- = \lim_{\substack{x \to 1 \\ x<1}} w(x) \le 0. \tag{4.5}$$

Finally, we see that $u^\epsilon$ will have no boundary layer as

$$u_0^- = \lim_{\substack{x \to 0 \\ x>0}} u(x) = u(0) = u_0, \tag{4.6}$$

where $u$ is the limiting solution corresponding to the inviscid problem to (4.1).

Next, we shall present a sketch of the proof of the above statement. For complete details see [G1]. We use the formula obtained by integrating Eq. (4.1.2),

$$\begin{aligned} &F(w^\epsilon(b^\epsilon)) - F(w^\epsilon(a^\epsilon)) \\ &\quad = \int_{a^\epsilon}^{b^\epsilon} \mathscr{S}_2(u^\epsilon, w^\epsilon, x)\, dx + \epsilon\big[(B(w^\epsilon)w_x^\epsilon)(b^\epsilon) - (B(w^\epsilon)w_x^\epsilon)a^\epsilon)\big], \end{aligned}$$

for points $b^\epsilon$ and $a^\epsilon$ at "both sides" of the forming layer, so that the difference $F(w_1^-) - F(w_1)$ is of order $F(w^\epsilon(b^\epsilon)) - F(w^\epsilon(a^\epsilon))$, whose sign can be controlled by choosing $a^\epsilon$ and $b^\epsilon$ carefully, depending on the difference $w_1^- - w_1$ (resp. $w_0^+ - w_0$), so we can make the sign of the right-hand side of

$$\begin{aligned} F(w_1^-) - F(w_1) &= \int_{a^\epsilon}^{b^\epsilon} \mathscr{S}_2(u^\epsilon, w^\epsilon, x)\,dx + \epsilon(B(w^\epsilon)w_x^\epsilon)(b^\epsilon) - \epsilon(B(w^\epsilon)w_x^\epsilon)(a^\epsilon) \\ &\quad + [F(w_1^-) - F(w^\epsilon(w^\epsilon))] + [F(w^\epsilon(a^\epsilon)) - F(w^\epsilon)] \end{aligned} \tag{4.7}$$

either positive or negative.

The solution of the boundary layer points $a^\epsilon$ and $b^\epsilon$ is done closer to $w_1$ and $w_1^-$, respectively, in the layer neighborhood of $x = 1$ (resp. with $w_0$ and $w_0^+$ at $x = 0$), keeping in mind that we want to estimate the sign of $F(w_1^-) - F(w_1)$ when $w_1^- - w_1$ is assumed positive, or $w_1^- - w_1$ is assumed negative.

In fact, the factor $\int_{a^\epsilon}^{b^\epsilon} \mathscr{S}_2(u^\epsilon, w^\epsilon, x)\,dx$ is not essential since $\mathscr{S}_2$ is uniformly bounded in $\epsilon$. The last two terms can be estimated by $|\mathscr{H}(w_1^-) - \mathscr{H}(w^\epsilon(b^\epsilon))|$ and $|\mathscr{H}(w^\epsilon(a^\epsilon) - \mathscr{H}(w^1)|$, so we must carefully see how to control the sign of $w_x^\epsilon$ at the selected points $a^\epsilon$ and $b^\epsilon$. Thus, the selection of the point $a^\epsilon$ near $w_1^-$ is done after using the following lemma that indicates how to control $|\mathscr{H}(w^\epsilon(a^\epsilon)) - \mathscr{H}(w_1^-)|$ and $\epsilon(B(w^\epsilon)w_x^\epsilon)(a^\epsilon)$ jointly in a boundary layer region controlling how close $a^\epsilon$ can be to the boundary value $x = 1$ so that, we still can choose $b^\epsilon$ to the right of $a^\epsilon$ (i.e., close to $x = 1$) such that we can estimate the two remaining terms at $x = b^\epsilon$.

We state the following lemma, whose proof can be found in [G1].

LEMMA 4.1. *Let $w$ be the weak solution of problem* (4.1) *for $\epsilon = 0$ defined as in* (3.5) *and let $w_1^- = \lim_{x\to 1^-} w(x)$ (resp. $w_0^+ = \lim_{x\to 0^+} w(x)$). Then for every $\delta > 0$ there exists a positive number $\sigma_\delta = \sigma(\delta)$ such that, for any $\sigma < \sigma_j$ fixed there is $\epsilon_0 = \epsilon(\sigma)$, where the following holds: for each $\epsilon < \epsilon_0$ there is $x_1^\epsilon = x_1(\epsilon) \in (1 - \sigma, 1 - \sigma/2)$ (resp. $x_0^\epsilon = x_0(\epsilon) \in (\sigma/2, \sigma)$) such that the pair $(\epsilon, x_1^\epsilon)$ (resp. $(\epsilon, x_0^\epsilon)$) satisfies*

(i) $|\mathscr{H}(w^\epsilon(x_1^\epsilon) - \mathscr{H}(w_1^-)| < \delta/2$ (resp. $|\mathscr{H}(w^\epsilon(x_0^\epsilon)) - \mathscr{H}(w_0^+)| < \delta/2$) and

(ii) either $|(B(w^\epsilon)w_x^\epsilon)(x_1^\epsilon)| \le c(\delta^{1/2}/\sigma)$ (resp. $|(B(w^\epsilon)w_x^\epsilon)(x_0^\epsilon)| \le c(\delta^{1/2}/\sigma)$) or we can choose the sign of $w_x^\epsilon$ at $x_1^\epsilon$ (resp. at $x_0^\epsilon$); i.e.,

$$(B(w^\epsilon)w_x^\epsilon)(x_1^\epsilon) > 0 \qquad \text{or} \qquad (B(w^\epsilon)w_x^\epsilon)(x_1^\epsilon) < 0 \qquad (\text{resp. at } x_0^\epsilon).$$

*Remark*. The role of $\sigma_\delta$ in this lemma is crucial, as $a^\epsilon$ is chosen $x_1^\epsilon$ for $\epsilon \le \epsilon_0(\sigma)$, $\sigma < \sigma_\delta$ means that each $a^\epsilon$ is in the boundary layer, but at a distance not less than $\sigma/2$ from the boundary, so that the estimate of $w_x^\epsilon(a^\epsilon)$ is kept under control uniformly in $\epsilon \le \epsilon_0(\sigma)$, and then, either the term $\epsilon(B(w)w_x^\epsilon)(a^\epsilon)$ converges to zero as $\epsilon \to 0$ or we can choose its sign.

In addition to choosing $a^\epsilon = x_1^\epsilon$, $\epsilon \le \epsilon_0(\sigma, \delta)$ from the above lemma, $b^\epsilon$ is chosen in $(1 - \sigma/2, 1]$ satisfying that $w^\epsilon(b^\epsilon) = w_1$ (so it takes care of the term $F(w^\epsilon(b^\epsilon)) - F(w_1)$ and depending on the initial assumption of the difference $w_1 - w_1^-$ and on the fact that $\mathscr{H}(w^\epsilon)$ is monotone in $w^\epsilon$ and has a limit at any point $x$ in $[0, 1]$, we than have a given sign for $w_x^\epsilon(b^\epsilon)$. Therefore we are able to estimate the sign of the difference $F(w_1^-) - F(w_1)$ by estimating the equality (4.7).

Therefore (4.3) and (4.5) hold. With similar arguments it can be shown (4.2) and (4.4) hold. Finally, it is easy to see by simple integration near the boundary that (4.6) holds, as $\mathscr{S}_1(w^\epsilon, w^\epsilon, x)$ is uniformly bounded in $\epsilon$.

We conclude this section by recalling that

$$w = n^2T - k_j$$

and

$$u = \frac{mj^2}{n} + knT,$$

so if $w^\epsilon$ is producing a boundary layer downstream then $w_1^- < 0$ with $F(w_1^-) < F(w_1) = F((n_1, T_1))$, where $n_1^2T_1 - k_j > 0$, then the resulting inviscid solution $(n, T)$ will have the property that

$$(n_1^-)^2T_1^- - k_j < (n_1^*)^2\, T_1^* - k_j < 0, \tag{4.8}$$

where $(n_1^*, T_1^*)$ are the supersonic conjugate values of $(n_1, T_1)$ by the momentum and energy fluxes satisfying

$$F((n_1^-, T_1^-)) < F((n_1^*, T_1^*)) = F((n_1, T_1)) \tag{4.9}$$

and

$$\frac{mj^2}{n_1^-} + kn_1^-T_1^- = \frac{mj^2}{n_1} + kn_1T_1. \tag{4.10}$$

That is, (4.8) says $n_1^-$ corresponds to a supersonic density and the shock layer is characterized by (4.8)–(4.10). Similar results may be obtained at the upstream boundary, and also the corresponding ones for supersonic prescribed data.

## 5. Applications

Examples of systems of this kind of systems are

*(i) Gas Flow Through a Nozzle Duct of Variable Cross Sections*

This is a very well analyzed classical model in gas dynamics in a duct of slow varying area that can be found in Courant and Friedrichs [CF] and Liepmann and Roshko [LR]. The stationary model can be written as

$$(nv)_x = -\frac{A'(x)}{A(x)}nv \tag{5.1.1}$$

$$(nv^2 + P)_x = -\frac{A'(x)}{A(x)}nv^2 \tag{5.1.2}$$

$$\left(n\left(\frac{1}{2}v^2 + e\right)v + Pv\right)_x = -\frac{A'(x)}{A(x)}\left(n\left(\frac{1}{2}v^2 + e\right)v + Pv\right), \tag{5.1.3}$$

where $A(x)$ is the duct cross section and $n, v, P$, and $e$ denote respectively the density, velocity, pressure, and internal energy per unit mass of the gas. The gas is assumed to be polytropic, i.e., $P = (\gamma - 1)ne$.

It is easy to see that system (5.1) is a special case of system (1.1) when the temperature $T$ state variable is replaced by the internal energy $e$ state. In this case, $m = (A(x))^{-2}$; $k = \gamma - 1$ and $\alpha = \gamma$, where $\gamma$ is the adiabatic constant.

The transport problem for which these equations represent its stationary state have been extensively studied by Liu [L1], where he analyzes nonlinear stability and instability, and Liu and Glaz [LG], where local interactions of nonlinear waves are resolved through asymptotic analysis and it is then used to construct a numerical calculation of the transonic flow in the nozzle. Other numerical simulations and discussions have been done by Glimm, Marshall, and Plohr [GMP] and references therein.

Since Eqs. (5.1) imply that smooth flow remains isentropic along the particle path (see Appendix), the weak shock (or relatively small sized discontinuities for the pressure) model is given by a solution of a scalar equation depending on the current $j$ parameter, where then the pressure is a given power of the density ($P = (\gamma - 1)n^{\gamma}$). This case has been treated analytically by Hsu and Liu [HL], where they study properties of the solutions for the corresponding viscous regularized equation of the form

$$\begin{aligned} \epsilon u'' &= f(u)' - c(x)h(u), \qquad x \in [0,1] \\ u(0) &= u_l, \qquad u(1) = u_r, \end{aligned} \tag{5.2}$$

where $f$ is convex, $f(0) = f'(0) = 0$. The function $h(u)$ represents the coupling of the source due to the geometry and gas flow, and $c(x)$ represents the strength of the source. There, they analyze the number of solutions, their asymptotic shape for small $\epsilon$, and their stability and instability when viewed as stationary solutions of the corresponding time evolution equation. A small variation in the arguments used by the author [G1, G2] shows that a solution $u^\epsilon$ of (5.2) converges pointwise and in $L^1(I)$ to an admissible entropy solution with bounded variation of the inviscid problem associated with (5.2) that arises at $\epsilon = 0$. It is also proven that the solution (5.2) might form an upstream boundary layer and a downstream shock layer, depending on the boundary data.

*Strong Shock Stationary Transonic Solutions*

In this section we prove the existence of solutions for the stationary nozzle flow equations, associated with system (5.1), as a pointwise and $L^1(I)$ limit of (artificial) viscous-heat conducting regular solutions of a higher order system with appropriate boundary data.

Thus from (5.1) we obtain the system of equations,

$$A(x)nv = j = \text{const} \tag{5.3.1}$$

$$F_1(n,e)_x = \left(\frac{j^2}{A^2(x)n} + (\gamma - 1)ne\right)_x$$

$$= \frac{-A(x)}{A(x)}\frac{j^2}{A^2(x)n} = S_1(n,e,x), \tag{5.3.2}$$

$$F_2(n,e)_x = \left(\frac{j^2}{A^2(x)n^2} + \gamma e\right)_x = 0 = S_2(n,e,x), \tag{5.3.3}$$

where $n$ and $e$ denote density and internal energy per unit mass, respectively, $A(x)$ is the duct cross section, $1 < \gamma \le \frac{5}{3}$ is the adiabatic constant and $j$ is a constant parameter for the current flow.

We follow the steps developed in Section 2. First, we compute the equation of the line where the jacobian $\partial(F_1, F_2)/\partial(n, e)$ vanishes,

$$0 = \frac{(\gamma - 1)}{A^2(x)n^2}\left[-\frac{j^2(2-\gamma)}{\gamma - 1} + \gamma A^2 n^2 e\right]. \tag{5.4}$$

As $1 < \gamma < \frac{5}{3}$, denoted by

$$k_j = j^2\frac{(2-\gamma)}{\gamma - 1} > 0, \tag{5.5}$$

the set of new state variables is

$$u = \frac{j^2}{A^2(x)n} + (\gamma - 1)ne \qquad \text{(momentum flux)},$$

$$w = \gamma A^2(x)n^2 e - k_j \qquad \text{(sonic line equation)}. \qquad (5.6)$$

Therefore, proceeding as in Section 2, we express Eqs. (5.3) in the new state variables, for which we need to express the energy flux in terms of $u$ and $w$, that is, from

$$A^2(x)nu = j^2 + \frac{(\gamma-1)}{\gamma}(w + k_j) = \left(\frac{\gamma-1}{\gamma}\right)\left(2\frac{j^2}{\gamma-1} + w\right).$$

So

$$n = \frac{1}{A^2(x)u}\left(\frac{\gamma-1}{\gamma}\right)[2\alpha_j + w] \qquad (5.7)$$

with

$$\alpha_j = \frac{j^2}{\gamma - 1} = k_j + j^2 \qquad (5.8)$$

and

$$F_2(n,e) = A^2(x)u^2\frac{\gamma^2}{(\gamma-1)^2}(2\alpha_j + w)^2(\alpha_j + w). \qquad (5.9)$$

Hence, denoted by

$$F(w) = \frac{w + \alpha_j}{(w + 2\alpha_j)^2}, \qquad (5.10)$$

system (5.3) becomes

$$u_x = -(\ln A)'\gamma\alpha_j\frac{u}{(2\alpha_j + w)} \qquad (5.11.1)$$

$$\left\{\left(\frac{\gamma}{\gamma-1}A(x)\right)^2 u^2 F(w)\right\}_x = 0. \qquad (5.11.2)$$

Next, we see that the system uncouples as the equation of conservation of energy becomes independent of the momentum flux variable $u$. Since

(5.11.2) can be written as

$$(F(w))_x u^2 A(x) = -2F(w)\left[u^2 A(x) A'(x) - u u_x\right], \qquad (5.12)$$

then, replacing the value of $u_x$ in (5.12) by Eq. (5.11.1) and dividing by $u^2 A^2(x)$, (5.12) becomes

$$\begin{aligned}(F(w))_x &= -2(\ln A)' F(w)\left[1 - \frac{\gamma \alpha_j}{2\alpha_j + w}\right] \\ &= -2(\ln A)' F(w)\left(\frac{w + k_j}{2\alpha_j + w}\right).\end{aligned}$$

Finally, in order to obtain a physically consistent model we define

$$S_2(w, x) = \begin{cases} -2(\ln A)' F(w)\left(\dfrac{w + k_j}{2\alpha_j - + w}\right), & w \geq -k_j \\ 0, & w < -k_j. \end{cases} \qquad (5.13)$$

The reason behind this cutoff term comes from the consideration that the equations of motion are defined for physical state variables $\rho$ and $e$ nonnegative, so that $\rho^2 e \geq 0$. In particular the relationships given for the source terms, as well as the flux $F(w)$, written in the new variables (5.6), are not defined for $w < -k_j$.

Therefore, $S_2(w, x)$, defined in (5.13), continuous for all $w$, is consistent with the physical problem under consideration. Thus, system (5.3) becomes

$$u_x = -(\ln A)' \frac{\gamma \alpha_j}{(2\alpha_j + w)} u = S_1(u, w, x) \qquad (5.14.1)$$

$$F(w)_x = S_2(w, x) \qquad (5.14.2)$$

through the change of state variables defined by (5.6), with $F(w)$ given by (5.10), $S_2(w, x)$ by (5.3), and $k_j = j^2((2 - \gamma)/(\gamma - 1))$ and $\alpha_2 = k_j + j^2 = j^2/(\gamma - 1)$.

Note that system (5.14) has exactly the same fluxes as the one used throughout Sections 2 and 3, as well as the property that the density $n$, given in terms of the new variable $u$ and $w$ by (5.7), is monotone in $w$. See that the graph of $F(w)$ is the same as the one pictured in Fig. 1. Hence, we have a system with the same characteristics and properties as the one we worked out through Sections 2 and 3. This motivates us to choose a class of viscosity-heat conducting terms of the form $\epsilon(B(w)w_x)_x$ satisfying $B(w) > 0$, $B'(w) \leq 0$. Indeed we shall see immediately that the boundary value problem

$$u_x = -(\ln A)' \frac{\gamma \alpha_j}{(2\alpha_j + w)} u \tag{5.15.1}$$

$$E^\epsilon(w) = -F(w)_x + S_2(w,x) + \epsilon(B(w)w_x)_x = 0 \tag{5.15.2}$$

$$u(0) = u_0, \qquad w(0) = w_0, \qquad w(1) = w_1, \qquad \epsilon > 0, \tag{5.15.3}$$

with the particular choice $B(w) \asymp w^{-\beta}$, $\beta \geq , 1$, as $w$ becomes large, has a solution $w^\epsilon, u^\epsilon$ such that

$$\|u^\epsilon, u_x^\epsilon, w^\epsilon, \epsilon w_x^\epsilon\|_\infty \leq \kappa$$

and

$$0 < U_0 \leq u^\epsilon, \qquad -k_j < w^\epsilon, \qquad \text{uniformly in } I,$$

where $\kappa$ and $U_0$ depend on $u_0, w_0, w_1$, $A(x)$, and $\gamma$, independently of the viscosity measure $\epsilon$.

Once this bound is obtained, we can use the existence Leray–Schauder-type theorem 3 and, consequently, the limiting process developed in Section 2 to find an "inviscid" solution pair $(u, w)$ of problem (5.15), where $u$ is a classical regular solution and $w$ is a function with bounded variation and a generalized solution satisfying the entropy condition that makes $w$ increase across a discontinuity in the open interval (0, 1) so $w$ is a weak entropy solution. However, $w$ might cavitate; i.e., there might exist an $x_0 \in I$, where $w(x_0) = k_j$ (note that the value $-k_j$ is a stationary point for the first-order ordinary differential equation (5.14.2)). Also, the limiting process is expected to develop boundary layers when solving a boundary value problem, as discussed in Section 4.

### *$\epsilon$-Uniform Bounds for the Gas Nozzle Flow Equations*

Thus, all we need to show is that we can find a $\epsilon$-uniform invariant region by proving a result equivalent to the comparison theorem 1. Actually, due to the decoupling of the system in $(u, w)$ variables Assumption $\mathscr{A}$ can be replaced easily by having a priori bounds on $u$ which will depend on $w$ and on finding $\epsilon$-uniform super- and sub-solutions for the scalar equation $E^\epsilon(w) = 0$, so that a comparison theorem will follow immediately. We shall restrict ourselves to duct models where either $A'(x) > 0$ or $A'(x) < 0$ throughout the interval [0, 1].

LEMMA 5.1 (A priori bound for $u^\epsilon$). *Let $(u^\epsilon, w^\epsilon)$ be a solution of the boundary value problem* (5.15), *where* $-\alpha_j < m \le w^\epsilon \le M < \infty$ *in* $\bar{I}$ *for all* $\epsilon$, *then*

*If* $A'(x) > 0$, *then* $u^\epsilon$ *is monotone decreasing and*

$$0 < u_0(A(0)/A(1))^{\gamma/2} \le u^\epsilon \le u_0 \text{ in } \bar{I}. \tag{5.16.1}$$

*If* $A'(x) < 0$, *then* $u^\epsilon$ *is monotone*

*increasing and* $0 < u_0 \le u^\epsilon \le u_0(A(0)/A(T))^{\gamma/2}$ *in* $\bar{I}$. (5.16.2)

*Proof.* From (4.13), $u^\epsilon$ is a solution of

$$(\ln u^\epsilon)_x = -\frac{A'(x)}{A(x)}\frac{\gamma\alpha_j}{2\alpha_j + w^\epsilon}, \qquad u(0) = u_0.$$

Since $-\alpha_j < m < w^\epsilon \le M$ and $\gamma\alpha_i\{A(x)(2\alpha_j + w^\epsilon)\} > 0$ in $\bar{I}$, the monotonicity of $u$ as a function of $x$ depends on the sign of $A'(x)$. Therefore, if $A'(x) > 0$ in $\bar{I}$, $u$ is decreasing and

$$-\frac{A'(x)}{A(x)}\frac{\gamma}{2} \le -\frac{A'(x)}{A(x)}\frac{\gamma\alpha_j}{2\alpha_j + m} \le (\ln u^\epsilon)_x \le \frac{A'(x)}{A(x)}\frac{\gamma\alpha_j}{2\alpha_j + M}.$$

Integrating in $x$ and taking exponentials, the bound

$$U_0\left(\frac{A(0)}{A(1)}\right)^{\gamma/2} < u_0\left(\frac{AQ(0)}{A(x)}\right)^{\gamma/2} \le u^\epsilon(x) \le u_0\left(\frac{A(0)}{A(x)}\right)^{\gamma\alpha_j/(2\alpha_j+M)} \le u_0 \tag{5.17}$$

holds uniformly in $\bar{I}$ and in $\epsilon$ as $A(0) \le A(x)$,

Similarly, if $A'(x) < 0$ in $I$, $u$ is increasing and $u$ has the bound

$$u_0 \le u^\epsilon(x) \le u_0(A(0)/A(1))^{\gamma/2} \tag{5.18}$$

as $A(x) \le A(0)$, uniformly in $I$ and in $\epsilon$.

Next, we want to find $\epsilon$-uniform upper and lower bounds for the $w^\epsilon$ solution of the boundary value problem, taken from (5.15.2), rewritten as

$$E_\epsilon(w) = \frac{w}{(2\alpha_j + w)^3}w_x - S_2(w, x) + \epsilon(B(w)w_x)_x = 0, \qquad x \in I,$$

$$-k_j < w(0) = w_0, \qquad w(1) = w_1 < B \tag{5.19}$$

with $S_2(w, x)$ given by (5.13).

In fact, due to the form of $S_2(w, x)$, we find that a lower bound is given by the value $-k_j$, which is the first stationary point of $S_2(w, x)$ to the left of the $\min\{w_0, w_1\}$. The lower bound is $\epsilon$-independent and does not depend on the sign of $A'(x)$ (that is, if the duct is convergent or divergent).

LEMMA 5.2 ($\epsilon$-uniform lower bound for $w^\epsilon$). *Let $w^\epsilon$ be a regular solution of $E_\epsilon(w) = 0$ in $I$, with boundary data satisfying $-k_j < w(0) = w_0, w(1) = w_1$. Then*

$$w^\epsilon > -k_j. \tag{5.20}$$

*Proof.* Assume that $\min_I w^\epsilon = w^\epsilon(x_0) \leq -k_j$. First, $x_0$ cannot be on $\partial I$, as the prescribed data is bigger than $-k_j$. Next, if $x_0$ is an interior point, then $w_x^\epsilon(x_0) = 0$. So, let us denote $m = w^\epsilon(x_0)$ and $v^\epsilon = w_x^\epsilon$, and rewrite the second-order equation $E_\epsilon(w^\epsilon) = 0$ as an initial value problem at $x_0$ for the ODE system

$$\begin{aligned} w_x &= v \\ v_x &= \frac{1}{\epsilon B(w)}\{F'(w)v - B'(w)v^2 - S_2(w, x)\} \end{aligned} \tag{5.21}$$

with initial data $w(x_0) = m$ and $v(x_0) = 0$.

Then, by the classical unicontinuation ODE theorems (5.21) has a unique solution in $[x_0, 1]$. Since $w^\epsilon \equiv m$ and $v^\epsilon \equiv 0$ solve (5.21), they must be the unique solution, in particular, $w^\epsilon(1) = m \leq -k_j$, which contradicts that $w^\epsilon$ is a solution of the boundary value problem $E_\epsilon(w) = 0$, $w^\epsilon(1), w^\epsilon(0) > -k_j$.

Next we look for $\epsilon$-uniform upper bounds. First, we need to find a one-parameter family of upper barrier functions to the solution $w^\epsilon$ of the boundary value problem (5.19). They will depend on the sign of $A'(x)$.

Afterwards, we prove a comparison theorem, equivalent to Theorem 1, from Section 2, so that we obtain the desired uniform bounds for the solution pair $(u^\epsilon, w^\epsilon)$ of problem (5.15).

LEMMA 5.3 (Divergent duct, $A'(x) > 0$). *Let $w^\epsilon$ be a solution of the boundary value problem* (5.19). *Any constant function $P_M(x) = M > \max\{w_1, w_0\} > -k_j$ is an upper barrier function to $w^\epsilon$.*

*Proof.* The constant function $P_M(x) = M$ in $\bar{I}$ is a supersolution of the operator $E_\epsilon(w) = 0$, since $A'(x) > 0$ and $M > -k_j$ imply $S_2(P_M(x), x) < 0$ throughout the interval $I$. In particular, $E_\epsilon(P_M) < 0$ for all $M > -k_j$, so that $P_M(x)$ is an upper barrier function to $w^\epsilon$ if $M \geq \max\{w_0, w_1\}$.

Next, we analyze for $A'(x) < 0$. In this case the constant functions are not supersolutions any longer; nevertheless we find an $\epsilon$-uniform one-parameter family of variable dependent supersolutions for the equations $E_\epsilon(w) = 0$, where the family of viscosity terms $\epsilon(B(w)w_x)_x$ satisfies $B(w) \asymp w^{-\beta}, \beta \geq 1$, as $w$ became large ($S_2(w, x)$ as defined in (4.11)).

This particular choice is related to the asympototic behavior of the flux function $F(w)$ and the source term $S_2(w, \cdot\,)$ at infinity.

LEMMA 5.4 (Convergent duct, $A'(x) < 0$). *Let $w^\epsilon$ be a solution of the boundary value problem* (5.17), *where $B(w) \asymp w^{-\beta}, \beta \geq 1$, as $w$ becomes large. There is a positive constant $M^*$ such that $P^M(x) = Me^{-3M(x-1)}$ is an upper barrier function for $w^\epsilon$ in* [0, 1], *for all $M \geq M^*$.*

*Proof.* Since we need to control the solution $w^\epsilon$ from above, we look at the asymptotic order at infinity of each term of the operator $E_\epsilon(w)$. Indeed, $F(w) = (w + \alpha_j)/(w + 2\alpha_j)^2 \asymp 1/w$, $F'(w) \asymp -1/w^2$, and $S_2(w, x) \asymp -2(A'(x)/A(x))(1/w)$, as $w$ becomes large. ($S_2(w, x)$ is as defined in (4.11).) Therefore, we shall see that, for a sufficiently large $M$,

$$E_\epsilon(P^M(x)) \leq E_\epsilon^a(P^M(x)) = \frac{P_x^M}{(P^M)^2} - 2\frac{A'(x)}{A(x)}\frac{1}{P^M} + \epsilon\left(\frac{P_x^M}{(P^M)^\beta}\right)_x < 0 \tag{5.22}$$

and $P^M(0) > w_0$, $P^M(1) > w_1$.

*Construction of $P^M(x)$.* First we need to find a supersolution $\tilde{w}$ to $E_\epsilon^a(w)$ defined in (5.22). We try with

$$\tilde{w} = Me^{-B(x-1)} > M \qquad \text{in } [0, 1],$$

where $M$ and $B$ positive are to be chosen appropriately. Since $\tilde{w}_x = -MBe^{-B(x-1)}$ and $\tilde{w}_{xx} = MB^2e^{-B(x-1)}$, evaluating on $E_\epsilon^a(w)$, we obtain

$$\begin{aligned} E_\epsilon^a(\tilde{w}) = &- \frac{MBe^{-B(x-1)}}{M^2e^{-2B(x-1)}} + 2\left(\frac{-A'(x)}{A(x)}\right)Me^{B(x-1)} \\ &- \epsilon\beta(Me^{-B(x-1)})^{-\beta-1}(MBe^{-B(x-1)})^2 \\ &+ \epsilon(Me^{-B(x-1)})^{-\beta}MB^2e^{-B(x-1)}. \end{aligned}$$

Then $E_\epsilon^a(\tilde{w}) < 0$ if and only if $\tilde{w}^\beta E_\epsilon^a(\tilde{w}) < 0$, so that

$$\begin{aligned}\tilde{w}^\beta E_\epsilon^a(\tilde{w}) &= -(Me^{-B(x-1)})^{\beta-1}B + 2\left(\frac{-A'(x)}{A(x)}\right)(Me^{-B(x-1)})^{\beta-1}\\ &\quad - \epsilon\beta B^2 Me^{-B(x-1)} + \epsilon MB^2 e^{-B(x-1)}\\ &= (Me^{-B(x-1)})^{\beta-1}\left[-B + 2\left(\frac{-A'(x)}{A(x)}\right)\right]\\ &\quad + \epsilon[1-\beta]MB^2 Me^{-B(x-1)}.\end{aligned}$$

Now, since $A'(x) < 0$, we choose

$$M \le \max\left\{\sup_I\{-(\ln A(x))_x\}; w(0); w(1)\right\}. \tag{5.23}$$

Thus, since $1 - \beta \le 0$,

$$E_\epsilon^a(\tilde{w}) \le (Me^{-B(x-1)})^{\beta-1}[-B + 2M] < 0$$

if $B > 2M$. In particular,

$$\tilde{w} = Me^{-3M(x-1)} = P^M(x)$$

is a supersolution of $E_\epsilon^a$.

Now, we take $M^*$ large enough such that $E_\epsilon(P^{M^*}) \le E_\epsilon^a(P^{M^*}) < 0$. Since $P^M(x) > M$, by (5.23) we have that $P^M(x)$, with $M > \max\{M^*, \max\{\sup_I\{-(\ln A)_x\}, w(0), w(1)\}\}$ is a one-parameter $\epsilon$-uniform upper barrier family of functions of $E_\epsilon(w)$, and, in addition,

$$P^M(x) \to \infty \qquad \text{as } M \to \infty \text{ uniformly in } [0,1]. \tag{5.24}$$

Property (5.24) (similar to condition (2.12.1) for $P_4^*(x)$) is fundamental to deriving the comparison result.

Finally, we present the comparison theorem in order to obtain the uniform bounds for the boundary value problem (5.15), the viscous regular approximation to the gas nozzle flow equations.

THEOREM 5.5 ($\epsilon$-uniform bounds). *Let $(u^\epsilon, w^\epsilon)$ be a solution of the boundary value problem* (5.15); *then*

$$\begin{aligned}&0 < U_0 \le u^\epsilon < \kappa\\ &-k_j < w^\epsilon < \kappa \qquad \textit{uniformly in } [0,1],\end{aligned} \tag{5.25}$$

*where $U_0$ and $\kappa$ depend on $u_0, w_0, w_1, A(x)$, and $\gamma$, but are independent on $\epsilon$.*

*Proof.* Due to the nice decoupling of the nozzle duct equations, first we show that $w^\epsilon$ is uniformly bounded. The argument to find the upper bound for $w^\epsilon$ is exactly the same as that used in Theorem 1, case (iv), in Section 2, since by Lemmas 5.3 and 5.4 we have a one-parameter family of functions $P^M(x)$ supersolutions of $E^\epsilon(w)$ for either divergent or convergent ducts, such that $P^M(x) \to \infty$ as $M$ becomes large uniformly in [0, 1]. Therefore, there exists an $A^*$ large enough with

$$w^\epsilon < P^{A^*} \qquad \text{uniformly in } [0,1],$$

and $P^{A^*}(x)$ depends on $w_0, w_1, A(x)$.

Adding the result obtained in Lemma 5.2,

$$-k_j < w^\epsilon < P^{A^*} \qquad \text{uniformly in } [0,1], \tag{5.26}$$

independently of $\epsilon$.

Now, we are in condition to apply Lemma 5.1, the a priori bound for $u^\epsilon$, so that

$$0 < U_0 \le u^\epsilon \le U_1 \qquad \text{uniformly in } [0,1] \tag{5.27}$$

where $U_0$ and $U_1$ depend on $u_0$, $A(x), \gamma$, independently of $\epsilon$. In particular, (5.25) holds, taking $\kappa = \max\{P^{A^*}, U_1\}$.

*Remark.* Neither the upper uniform bounds nor the lower bound of $u^\epsilon$ depend on the current parameter j. This observation would facilitate showing the expected behavior of the flow; as the current parameter goes to zero the flow would remain subsonic (as $w^\epsilon$ would remain bounded by $0 < w^\epsilon < \kappa$), so that the inviscid limiting solution would not admit discontinuities.

*Inherited Properties for the Inviscid Solution*

First, as stated before, using the existence results of Section 2 and the limiting process of Section 3 we can have an admissible solution $(u, w)$ to the inviscid (5.14) in (0, 1); that is, $w$ is a weak solution of (5.14.2) that satisfies the entropy condition, as stated in Section 3, which is a pointwise and $L^1(I)$ limit of $w^\epsilon$, solution of the boundary value problem (5.15), where $w^\epsilon$ is a family of $\epsilon$-uniform bounded variation; that is, $w^\epsilon$ has derivatives $\epsilon$-uniformly bounded in $L^1(I)$ and in particular, the limit function $w$ has derivatives bounded in $L^\infty$ and $L^1$-norm. In particular, $u$ and $w$ inherit the bounds of $u^\epsilon$ and $w^\epsilon$.

Therefore, using the change of variables given by (5.6), we obtain that the pair $(n, e)$ is a weak solution of the stationary inviscid nozzle gas flow

(5.3), where $n$ and $e$ are given by

$$n = \frac{1}{A^2(x)u}\left(\frac{\gamma - 1}{\gamma}\right)\left(w + 2\frac{j^2}{\gamma - 1}\right) \tag{5.28.1}$$

and

$$e = \frac{A^2(x)\gamma}{(\gamma - 1)^2}u^2\left(\left(w + \frac{j^2(2-\gamma)}{\gamma - 1}\right)\Big/\left(w + \frac{2j^2}{\gamma - 1}\right)^2\right). \tag{5.28.2}$$

In particular, $(n, e)$, defined as above, satisfies the entropy and jump conditions across the discontinuities, as is shown in (3.12), (3.13), and (3.14). Therefore, $(n, e)$ is an admissible inviscid stationary solution of nozzle gas flow system (5.3).

Finally, we state the following theorem regarding the regularity of the inviscid solution $(n, e)$, defined as above, and the formation of boundary layers, depending on the kind of boundary data.

THEOREM 5.6 (Divergent duct; $A'(x) > 0$). *Let $n$ and $e$ be the inviscid density and internal energy (resp. solutions) of* (5.3) *given by the transformation* (5.28), *where $(n, w)$ is the limit solution of problem* (5.15) *for $\epsilon = 0$, as defined in Section* 3 *and the theorems therein*:

(i) If the boundary data $n_0, n_1$ and $e_0, e_1$ is such that $w(0), w(1) < 0$, with $w(x) = \gamma A^2(x)n^2e - j^2(2-\gamma)/\gamma - 1$ (i.e., supersonic data). Then $n$ and $e$ are continuous functions of $x$ and $0 \le n \le N$ and $0 \le e \le E$, with $\gamma A^2(x)n^2e < j^2(2-\gamma)/(\gamma - 1)$ in $I$. So, that $n$ is a supersonic continuous solution and no shocks are admissible, there is no upstream boundary layer for the viscous solution, but an downstream supersonic layer might develop.

(ii) if either boundary conditions on $\eta$ and $e$ at $x = 0$ or 1 is such that $w(0)$ or $w(1) > 0$, then $n$ and $e$ might be a shock solution, and $0 \le n \le N$ and $0 \le e \le E$, and a boundary layer may develop at the upstream boundary and a shock layer, at the downstream one.

*Proof.* Let $(u^\epsilon, w^\epsilon)$ be the solution of the boundary value problem (5.15), where the data is given by $u(0) = j^2/A^2(0)n_0 + (\gamma - 1)n_0e_0 = u_0$, and

$$w^\epsilon(0) = \gamma A^2(0)n_0^2e_0 - j^2\frac{2-\gamma}{\gamma - 1} = w_0,$$

$$w^\epsilon(1) = \gamma A^2(1)n_1^2e_1 - j^2\frac{2-\gamma}{\gamma - 1} = w_1;$$

then $(u^\epsilon, v^\epsilon)$ converges pointwise and in $L^1(I)$ to $(u, w)$ there is an admissible solution of (5.14), so that $(n, e)$, as defined by (5.28) is an admissible solution of (5.3).

First, using Theorem 5.5 we have that $(u^\epsilon, w^\epsilon)$ is $\varepsilon$-uniformly bounded as in (5.25), where the form of the bounds are given in Lemmas 5.1, 5.2, 5.3, and 5.4, respectively. These bounds are inherited to $(u, w)$ straightforward and to $(n, e)$ through the relation (5.28).

Since $A'(x) > 0$ (divergent duct model), by inequality (5.16.1) on Lemma 5.1, $u$ monotone decreasing and controlled by

$$0 < u_0 \left( \frac{A(0)}{A(1)} \right)^{\gamma/2} < u(x) < u_0 \quad \text{in } p[0,1]. \tag{5.29}$$

By Lemma 5.2, we know that $w^\epsilon > -k_j$ for all $\epsilon$, so at most we can state that $w \geq -k_j$. This particular, lower bound says that $n^2 e \geq 0$, so that the inviscid solution might have a point $x_0$ in $(0, 1)$ on which it cavitates by either $n(x_0) = 0$ or $e(x_0) = 0$. Actually, this effect can happen in either form of duct.

Next, from Lemma 5.3 we have that $M = \max\{w_1, w_0\}$ gives an upper bound for $w^\epsilon$ and also for $w(x)$.

At this point we break the analysis into case (i) with supersonic data and case (ii) with subsonic data at least in one of the end points.

(i) If $w_1$ and $w_0$ are negative then $w^\epsilon$ and $w$ satisfy

$$-k_j \leq w^\epsilon, \qquad w < M < 0 \quad \text{in} \quad [0,1] \text{ uniformly in } \epsilon. \tag{5.30}$$

In particular, $F(w)$ is monotone increasing as a function of $w$ in $[-k_j, M]$, and a Lipschitz function of $x$ on $[0, 1]$, therefore, is invertible and the inverse is also a continuous function of $x$. So $w(x)$ must coincide with $F^{-1}(F(w))(x)$; hence, $w(x)$ is a continuous function of $x$. As an immediate consequence, by relationship (5.28) we obtain that the density $n$ and internal energy $e$ are a continuous function of the space variable in $[0, 1]$, and the bounds

$$0 \leq n \leq \frac{(A(1))^{\gamma/2}(\gamma - 1)}{\gamma (A(0))^{2+\gamma/2} u_0} \left( M + \frac{2j^2}{\gamma - 1} \right) = N \tag{5.30.1}$$

and

$$0 \leq e \leq \frac{A^2(1)}{j^2 \gamma} u_0 \left( M + \frac{j^2(2 - \gamma)}{\gamma - 1} \right) = E \tag{5.30.2}$$

hold, where $M = \max\{w_0, w_1\} < 0$.

Also, from (5.26) $n$ and $e$ satisfy

$$0 \le \gamma(n^2 e)(x) A^2(x) < \frac{j^2(2-\gamma)}{\gamma - 1} \tag{5.31}$$

for $x \in [0,1]$, so $(n, e)$ is a supersonic continuous inviscid solution. In particular, applying the results from Section 4, this solution cannot have an upstream shock boundary layer, but it might develop a downstream boundary layer that would remain always supersonic.

(ii) If either $w_0$ or $w_1$ is bigger than zero (i.e., are subsonic values) then we obtain that

$$-k_j \le w \le M \qquad \text{in } [0,1],$$

but $M$ is not necessarily negative, then $F(w)$ loses monotonicity on $[-k_j, M]$. Hence $w$ might admit a jump discontinuity, making the density jump, increasing its value across the jump. Therefore bounds from (5.30) for $n$ and $e$ still hold, but no longer (5.31). In particular, the boundary layer is admissible at both boundary points. At the upstream boundary the layer is supersonic or subsonic depending on the sign of $w_0$, and at the downstream boundary a shock layer might develop, as described in Section 4, as classified from (4.2) through (4.6).

THEOREM 5.7 (Convergent duct, $A'(x) < 0$). *Let $(n, e)$ be given as in Theorem 5.6. Then for any kind of boundary data $n_0$, $n_1$, $e_0$, and $e_1$ given, $(n, e)$ might admit shock discontinuities (satisfying the entropy conditions) and are bounded by*

$$0 \le n \le \frac{\gamma - 1}{\gamma A^2(1) u_0}\left(P^M(x) + 2\frac{j^2}{\gamma - 1}\right) \tag{5.31.1}$$

*and*

$$0 \le e \le \frac{A^2(0)}{j^2 \gamma} u_0 \left(\frac{A(0)}{A(1)}\right)^{\gamma/2}\left(P^M(x) + \frac{j^2(2-\gamma)}{\gamma - 1}\right). \tag{5.31.2}$$

*As in Theorem 5.6(ii), boundary layers might develop at both endpoints, as clarified therein.*

*Proof.* The proof differs from that in Theorem 5.6 only in the upper bound for $w$ and the formula for $n$. We use here Lemma 5.1 and inequality (5.18), so that $u$ is bounded by

$$0 < u_0 \le u(x) \le u_0 \left(\frac{A(0)}{A(1)}\right)^{\gamma/2}, \qquad \text{in } [0,1].$$

Now, from Lemma 5.4, if $M \geq M^*$ ($M^*$ from that lemma), then

$$-k_j \leq w(x) \leq P^M(x) = Me^{-3M(x-1)} \leq Me^{3M}.$$

Therefore, using relationship (5.28) with the above bounds for $u$ and $w$, (5.31) holds, and $u$ and $e$ might become transonic.

We pass to the next application.

*(ii) Energy Transport or Hydrodynamic Models for Semiconductor Devices*

These models treat the propagation of electrons in a semiconductor as the flow of a charged, heat conducting gas in an electric field. This electric field $E$ is given by the Poisson equation

$$(\varepsilon\phi_x)_x = -q(C(x) - n), \qquad E = -\phi_x, \tag{5.32}$$

where $\phi$ is the electrostatic potential, $\varepsilon$ the permitivity of the medium, $q$ the space charge, and $C(x)$ represents the given "doping profile" function (usually the model as a piecewise constant time independent function).

System (1.1) would model a stationary electron gas flow in a device where the constitutive relations are characterized by the isotropic/parabolic energy band assumption and the heat conduction and viscosity/diffusion effects are neglected, coupled with the Poisson equation through an external force term contained in $S_1(n,T,x)$ and $S_2(n,T,x)$ given by

$$qnv = j$$

$$F_1(n,T)_x = S_1(n,T,x) = -qnE - \frac{m}{q}\left(\frac{\partial n}{\partial t}\right)_c \tag{5.33.1}$$

$$F_2(n,T)_x = S_2(n,T,x) = -Ej - \left(\frac{\partial w}{\partial t}\right)_c, \tag{5.33.2}$$

where $(\partial n/\partial t)_c$ and $(\partial w/\partial t)_c$ represent the contributions of the collision terms which are usually approximated in terms of momentum and energy relaxation times, and $A(x) = q$, the space charge constant in Eq. (1.1.1).

The hydrodynamic model (5.33) for semiconductor devices was first proposed by Blotekjaer [B], where he obtains the transport equations as the first three moments of the Boltzmann equation. There, the moment expansion is closed at three moments by assuming the Fourier law for heat conduction $\mathbf{q} = -\kappa \nabla T$, where $\kappa$ depends on the state variables and physical constants. Actually, in semiconductor device modeling there is no agreement on how the term $\mathbf{q}$, which corresponds to a nonconvective energyflow coming from the third moment expansion of the Boltzmann equation, should be modeled. For some references regarding work done on this model see Baccarani and Woderman [BW], Fatemi, Jerome, and

Osher [FJO], Gnudi, Odeh, and Rudan [GOR], Gardner [Gr1, Gr2], Jerome and Shu [JS], Odeh and Rudan [OR], where several numerical simulations and discussions have been carried out. These numerical simulations show that the model produces undesirable velocity "overshoots" (i.e., peaks in the velocity state variable $v$) which are very sensitive to the change of some constants on the factor $\kappa$ (see [GOR, JS] for a detailed discussion on the matter.)

Energy transport models that do not neglect convection terms differ from the hydrodynamic one in the formulation of the constitutive relationships for the state variables. They have the fluxes $F_1(n, T)$ and $F_2(n, T)$ defined differently from (1.1), but they might be treated in a similar way as a nonlinear system of ordinary differential equations that admit shock solutions of the same kind of system (5.32), (5.33) (see [SOTG]).

However, no analytical results over existence and well posedness on any of these models have been presented so far regarding the full system of Eqs. (5.32)–(5.33), with a heat conduction term modifying (5.33.2) and appropriate boundary data.

In the case of autonomous systems (i.e., no dependence on $x$), for instance, a case of constant doping profile and collisionless model without heat conduction, a phase portrait analysis was carried out by Markowich and Pietra [MrPr]. However, elementary examples show that these solutions may differ from those obtained by a vanishing viscosity method.

We show here that for the system (1.1)–(5.32)–(5.33) with an appropriate term **B** added to (5.33.2), along with the appropriate boundary data, the resulting boundary value problem will have a regular solution provided $L^\infty$ bounds are found for all state variables. Such a **B** will depend on $n$ and $T$ and its first and second derivatives, so the term **B** will be a diffusion–heat conducting second-order term.

The $L^\infty$ bounds of the state variables will depend very strongly on the form of this second-order term as well as how the force-collision terms $S_1$ and $S_2$ are modeled.

Nevertheless, these models have been treated analytically in the case when the collision terms satisfy that

$$\left(\frac{\partial w}{\partial t}\right)_c - v\left(\frac{\partial n}{\partial t}\right)_c \equiv 0 \text{ in } [0,1]. \tag{5.34}$$

Relationship (5.34) is a sufficient condition to have the property that smooth flow becomes isentropic along a particle path. In particular, collisionless models would correspond to smooth isentropic gas flow models reducing the system (5.33) to a scalar equation for weak shock models (see Appendix 1). Again, as in the previous example of gas flow in a nozzle duct, a reduced system from (5.32)–(5.33) to an inviscid scalar equation for the density $n$, with the pressure modeled as a function of the density only, including collision effects and neglecting heat conduction, coupled with the Poisson equation, would be an appropriate model for solutions that admit weak shocks, that is, discontinuities in pressure and density with a relatively very small jump in size. Such a system has been treated analytically by Degond and Markowich [DM] and the author [G1, G2], for the prescribed boundary data. In [DM] an existence and uniqueness result is obtained under the assumption of smallness in size of the data which must result in a small current flow $j/q$ and low velocities that keep the flow subsonic in the whole interval. In [G1] an existence result is obtained without any assumption on the size of the data, hence, allowing the flow to become transonic in different regions of the interval $[0,1]$. There, the solution pair $(n,\phi)$, depending on the parameter $j$, is found as a pointwise limit of the diffusion-regularized boundary value problem when $\varepsilon n_{xx}$ is added to the corresponding scalar equation ($\varepsilon$ being the "diffusion" measure). The limiting "inviscid" solution is an admissible entropy solution in the classical sense.

In particular, the density $n$ is found to be bounded away from cavitation and away from infinity, and hence, the corresponding velocity is also bounded away from zero and from infinity.

In [G1] and [G2] the boundary layer is analyzed when the prescribed data is given by the value of the density $n$ at both end points and the electric field $E = -\phi_x$ is prescribed inflow. It is shown that the corresponding boundary value problem might not be well posed, if the flow is allowed to become transonic (i.e., if the parameter $j$ is not sufficiently small).

*Assumption of an $L^\infty$-Bound; existence Result of a Regular "Viscous" Solution*

We consider the system given by the Possion equation (5.32) and the motion equations in steady state (5.33), where a second-order term has been added to (5.33.2).

We transform this system into a new one by the transformation given by the change of state variables (2.2), so that the associated boundary value problem becomes

$$(\varepsilon\phi_x)_2 = -q(c(x) - n(u,w)) \tag{5.35.1}$$

$$u_x = S_1(u,w,E,x) \tag{5.35.2}$$

$$F(w)_x - S_2(u,w,E,x) - \varepsilon(B(w)w_x)_x = 0, \qquad x \in I = (0,1), \tag{5.35.3}$$

$$\phi(0) = \phi_0, \quad \phi(1) = \phi_1, \quad u(0) = u_0, \quad w(0) = w_0, \quad w(1) = w_0.$$

where the term $(\varepsilon\mathscr{B}(w)w_x)_x$ is the one corresponding to the one transformed from **B** and which accounts for the diffusion and heat conduction effects, so that $\varepsilon$ is the measure of diffusivity and heat conductivity. The constant current flow $j$ is a constant parameter involved in this system.

Here the argument developed in Section 2 as Assumption $\mathscr{A}$ and the comparison theorem 2.1 must be modified. Still we anticipate that once a result on how to bound the solution $E^\varepsilon$, $u^\varepsilon$, and $w^\varepsilon$ of the above boundary value problem is achieved, then Lemma 2.2 and the existence theorem follow. Therefore, we assume that the boundary value problem (5.35) has $L^\infty$-bounds.

*Assumption* B (A priori bounds; existence of an invariant region.). Let $Q_i$, $i = 1, \ldots, 6$, be differentiable functions of $x$ defined in $I$. Let $E^\varepsilon$, $u^\varepsilon$, and $w^\varepsilon$ be solutions of problem (5.35). Then

$$-\infty < Q_1 \leq E^\varepsilon \leq Q_2 < \infty \tag{5.36.1}$$

$$0 < Q_3 \leq u^\varepsilon \leq Q_4 < \infty \tag{5.36.2}$$

$$-k_j < Q_5 \leq w^\varepsilon \leq Q_6 < \infty. \tag{5.36.3}$$

THEOREM 5.8. *Under Assumption B, there exists a solution $E^\varepsilon$, $u^\varepsilon$, and $w^\varepsilon$ in $C^{0,1}(I)$ of Eqs.* (5.35) *with prescribed boundary data $E^\varepsilon(0) = E_0$, $u^\varepsilon(0) = u_0$, $w^\varepsilon(0) = w_0$, and $w^\varepsilon(1) = w_1$ with $E_0, u_0, w_0$ in the range of the bounds from* (5.36), *such that $E^\varepsilon, u^\varepsilon, w^\varepsilon$ satisfy* (5.36) *and $\|\varepsilon w_x^\varepsilon\|_{L^\infty} < Q_7$, $Q_7$ a positive finite constant.*

*Proof.* Assumption B provides the a priori $L^\infty$-bound to use Lemma 2.2 and a fixed point theorem argument.

Indeed, Lemma 2.2, Theorems 2.2 and 2.3 are valid, with a simple modification that consists in adding Eq. (3.5.1) in the construction of the operator $\mathbf{T}_\delta(v)$ in Theorem 3.2, so that for each $\varepsilon$, there exist functions $E^\varepsilon$, $u^\varepsilon$, and $w^\varepsilon$ in $C^{0,1}(I)$ (i.e., with at least bounded first derivatives) such that they solve system (5.35) and have bounds (5.36), which do not need to be $\varepsilon$-independent in order to grant the existence of a regular diffusion heat conducting solution.

As an immediate consequence of Assumption B and Theorem 5.8, there is an existence result for solutions of a regularized system associated to (5.32)–(5.33). We state this result as a theorem.

THEOREM 5.9. *The system* (5.32)–(5.33), *where* (5.33.2) *is replaced by*

$$F_2(n,T)_x = S_2(n,T,E,x) + \varepsilon(B(w(n,T)w(n,t)_x)_x \tag{5.37}$$

*with boundary data prescribed for $\phi_x(0) = E_0$, $\phi(1) = \phi_1$, $n(0) = n_0$, $n(1) = n_1$, $T(0) = T_0$ and $T(1) = T_1$ has a solution $\phi$, $n$, $T$ in $C^{0,1}(I)$ (i.e., with* *at least bounded first derivatives), with* $\phi(x) = \int_0^{1-x} E(x)\,dx + \phi_1$, *and the bounds for* $\phi_x$, $n$, *and* $T$ *will depend on the bounds* $Q_i, i = 1, \ldots, 6$, *and the transformation of the state variables* (2.2) *satisfying*

$$\kappa_1 \leq -\phi_1 \leq \kappa_2$$
$$0 < K_3 \leq n \leq K_4 < \infty$$
$$0 < K_5 \leq T \leq K_6 < \infty.$$

*Remark.* The bounds solution $\phi, n, T$ depend on the measure of the diffusion–heat conducting parameter $\varepsilon$ of the higher order term in (5.37).

It is not clear, from the theory of semiconductor device modeling, that there is an interest in the behavior of inviscid not heat conducting solutions. However, it might be not very difficult to find bounds for the solution of the regularized problem $Q_i, i = 1, \ldots, 6$, from (5.36) which are independent on $\varepsilon$ and ultimately if the interest in inviscid solutions becomes relevant, then it is necessary that the bounds in Assumption B are uniform in the measure of the diffusion–heat conduction parameter $\varepsilon$.

We stress that achieving a physical meaningful bound for $E^\varepsilon$, $u^\varepsilon$, and $w^\varepsilon$, either uniform or not, in $\varepsilon$, would result in the impossibility of velocity "overshoots" coming from the model, with a viscous-heat conducting term that satisfies the above equations after the change of state variables has been performed, as the bounds for $u^\varepsilon$ and $w^\varepsilon$ should lead to a bound for $n^\varepsilon$ away from cavitation.

Still, if the inviscid limit is taken, the weak admissible solution could produce shocks in density and temperature, but a velocity overshoot is not admissible (by velocity overshoot we mean that the velocity becomes unbounded at a value $x_0$ in the interval of definition $I$).

Finally, under Assumption B we set the conditions for existence of an inviscid solution to problem (5.32), (5.33).

THEOREM 5.10 (Existence of an inviscid admissible solution). *In addition to Assumption B, let* $Q_i, i = 1, \ldots, 6$, *be* $\varepsilon$*-independent as well as* $Q_7$ *from Theorem* 5.8. *Then there exists an "inviscid" solution* $\phi$, $n$, *and* $T$ *to system* (5.32)–(5.33) *which can admit shocks in* $n$ *and* $T$ *satisfying the condition that* $n$ *and* $T$ *increase across the discontinuity in the direction of the particle path* (*i.e.*, $n$ *and* $T$ *are admissible entropy solutions*).

*Proof.* Under Assumption B, Theorem 5.8 and the $\varepsilon$-uniformity of the bounds, all theorems and results of Section 3 can apply, with the minor correction of adding the transformed equation from (5.32), or equivalently $E_x = S_0(u, w, x)$ to problem (2.10). Then, the regularity of $E$ is the same as the regularity of $u$ in Theorem 3.4, the existence theorem of an inviscid solution. Theorem 3.5 (the entropy condition) also follows immediately, as well as Lemma 3.6 and the conclusion (3.15), (3.16), and (3.17).

## APPENDIX

The equations of motion for a one-dimensional fluid flow modified by the presence of external forces or sources arising, for example, from a collision term in statistical mechanical theory, read

$$\rho_t + (\rho v)_x = \mathscr{S}_1 \tag{A.1.1}$$

$$(\rho v)_t + (\rho v^2 + P)_x = \mathscr{S}_2 \tag{A.1.2}$$

$$\left(\tfrac{1}{2}\rho v^2 + \rho e\right)_t + \left\{\left(\tfrac{1}{2}\rho v^2 + \rho e + P\right)v\right\}_x = \mathscr{S}_3, \tag{A.1.3}$$

where $\rho$ denotes density, $v$ is velocity, $P$ is the scalar pressure, and $e = e(P, \rho)$ in the internal energy. The first equation is conservation of mass. The second and third represent the conservation of momentum and energy, respectively.

The force-source terms $\mathscr{S}_1$, $\mathscr{S}_2$, and $\mathscr{S}_3$ are functions depending on the state variables $\rho$, $v$, $P$, and $e$, on the space variable $x$ and on the physical parameters involved in the model of flow equations (A.1) represent, but do not depend on any gradients.

Taking into consideration the thermodynamic relations, we assume that, for smooth changes of pressure, the absolute temperature $T = T(P, \rho)$, and the entropy $S = S(P, \rho)$ are states defined by the differential relation

$$T\,dS = de + P\,d\tau \tag{A.2}$$

with $\tau = 1/\rho$ the specific volume. In particular, for smooth flows the system of equations (A.1) and relation (A.2) lead to a differential equation for the entropy state variable $S$ that reads

$$\rho T \frac{DS}{Dt} = -\mathscr{S}_1\left(-\frac{v^2}{2} + e + \frac{P}{\rho}\right) - v\mathscr{S}_2 + \mathscr{S}_3. \tag{A.3}$$

Hence, whenever the right-hand side of (A.3) vanishes along the interval of definition (in particular, in the absence of force-source terms) or is a function of $S$ with at least one zero, Eq. (A.3) says that the entropy remains constant along the particle path whenever the state variables are smooth functions of the space variable $x$, if the initial data is a stationary point of (A.3).

*Remark.* (i) We note here that the transport equations of nozzle gas flow associated with (5.1), as given in [CF, LP], have source effects modeled such that they make the right-hand side of (A.3) vanish. So, smooth gas flow in nozzles is isentropic along the particle path. In particular, stationary nozzle flow, modeled as in (5.1), must satisfy that the entropy state variable is a piecewise constant function of the space variable $x$.

(ii) In the semiconductor device model that arises from taking moments of the Boltzmann equations (i.e., Eqs. (1.1), (5.32), (5.33), the term $\mathcal{S}_1 = 0$ and the terms $\mathcal{S}_2$ and $\mathcal{S}_3$ include the external force induced by the electric field and the effects of collisions (see (5.33)).

It is easy to verify the collisionless models of the above type have smooth isentropic flow as the force exerted by the contribution of the electric field in the momentum and energy equation always satisfy that $v\mathcal{S}_2 = \mathcal{S}_3$.

For those points $(x, t)$ where the flow will not be smooth, the second law of thermodynamics states that the entropy (or, equivalently, the pressure or density) must increase along the particle path. Hence, the system under consideration involves $\rho$, $P$, $e$, $T$, and $S$ state variables of which only two of them are independent and the rest can be expressed as functions of these two.

It can be shown, from Eqs. (A.1) and (A.2), that the changes of entropy at dicontinuities are of third order with respect to changes in pressure or density. This result is due to Lax [Lx] for any general system of conservation laws. Hence, a change in pressure relatively small in size at a discontinuity from values $P^-$ to $P^+$ would make the entropy almost constant throughout the flow.

Therefore, the pressure $P$ would be modeled as a nonlinear function of the density only, reducing the one space dimensional system of equations of motion (A.1) to a $2 \times 2$ system model for velocity and density state variables (well known as a $P$-system). In particular, an appropriate model for stationary flow becomes the scalar equation

$$\left( \frac{mj^2}{P} + P(\rho) \right)_x = S(\rho, j).$$

Therefore, for those gases that obey the ideal gas law

$$P = k\rho T, \tag{A.4}$$

where $k$ is a constant, it follows from (A.2) that the internal energy $e$ must be a function of $T$ alone, i.e., $e = e(T)$.

In fact, for an ideal gas $e$ is a function of $P/\rho$. The form of this function could be left open, but we take a formula that arises in considerations of the specific heats which covers a wide range of phenomena in gas dynamics. That is,

$$e = c_v T \qquad \text{and} \qquad h = \left(e + \frac{P}{\rho}\right) = c_p T,$$

where $c_v$ and $c_p$ are the specific heat at constant volume and pressure, respectively. In particular the quantity $h$ is the enthalpy. The motivation of the present paper is to find steady state solutions for the stationary system associated with system (A.1) for a gas that obeys an ideal gas law with constant specific heat and for some particular force-source terms which, in addition, are time independent. These considerations lead to a system of the form

$$\begin{aligned} &(A(x)\rho v)_x = j_x = 0 \\ &\left(\frac{mj^2}{\rho} + k\rho T\right)_x = \mathscr{S}_1(\rho, T, j, x) \\ &\frac{j}{2}\left(\frac{mj^2}{\rho^2} + h\right)_x = \mathscr{S}_2(\rho, T, j, x), \end{aligned} \tag{A.5}$$

where $h = \alpha T$, with $\alpha$ constant and $m = m(x)$. This is the system we have considered in (1.1).

However, an equivalent system would arise for a different pressure law and/or different constitutive relationships among the state variables. That might be the case of different energy transport models. In such a case the momentum and energy fluxes appearing in (A.5) might be different functions of two independent state variables, in which case there is a possibility that the methodology of the present paper might work, depending on how the equation of the "sonic line" is, and, if the fluxes can be decoupled in a new set of state variables.

Before concluding this appendix, we want to point out that for an ideal gas with constant specific heat, using the thermodynamic differential relation (A.2) we can express the pressure in terms of the density and the entropy, that is,

$$P = k\rho^\gamma \exp\{S/c_v\},$$

known as the polytropic gas law. Thus, the enthalpy $h$ becomes

$$h = \gamma \rho^{\gamma - 1} \exp\{S/c_v\}.$$

Hence the stationary one-dimensional equations of motion, expressed as in cases considered under system (A.5) become

$$\begin{aligned} (A(x)\rho v)_x &= j_x = 0 \\ \left(m\frac{j^2}{\rho} + k\rho^\gamma \exp\{S/c_v\}\right)_x &= \mathscr{S}_1(\rho, \exp\{S/c_v\}, j, x) \qquad \text{(A.6)} \\ \frac{j}{2}\left(m\frac{j^2}{\rho^2} + \gamma\rho^{\gamma-1}\exp\{S/c_v\}\right)_x &= \mathscr{S}_2(\rho, \exp\{S/c_v\}, j, x). \end{aligned}$$

We use system (A.6) at the end of Section 3 to show, by a similar analysis to that for system (A.5) in Sections 2 and 3, that admissible inviscid solutions of (A.6) might have the entropy given by

$$S = c_v \log\frac{P}{k\rho^\gamma} + S_0, \qquad \text{(A.7)}$$

is discontinuous at some values of the domain, where the density $\rho$ (and pressure $P$) are discontinuous.

For references on this appendix see Courant and Friedrichs [CF], Menikoff and Phlor [MP], Zel'dovich and Raizer [ZR], and Liepmann and Roshko [LR] for a complete survey on gas dynamics.

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