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On Delay Differential Equations with Impulses

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Sufficient conditions are obtained respectively for the asymptotic stability of the trivial solution of $\dot{x}(t)+ax(t-\tau)=\sum_{j=1}^{\infty} b_j x(t_j-)\delta(t-t_j)$, $t\neq t_j$, and for the existence of a nonoscillatory solution; conditions are also obtained for all solutions to be oscillatory. The asymptotic behaviour of an impulsively perturbed delay-logistic equation is investigated as an extension to a nonlinear equation.

1. INTRODUCTION

There exists a well-developed stability theory of delay-differential and more general functional differential equations (Bellman and Cooke [2], El'sgol'ts and Norkin [6], Hale [10], Kolmanovskii and Nosov [13]. Oscillation theory of delay differential equations has also been developed extensively over the past few years. We refer to Arino, Györi, and Jawahari [1], Hunt and Yorke [12], Koplatadze and Canturiya [14], Kusano [15], Ladas [16], Lakshmikantham, Ladde, and Zhang [17], Onose [18], Fukgai and Kusano [7], Shevelo [22], and the references therein for the literature concerned with the oscillation of delay differential equations. In the opinion of these authors, delay differential equations subjected to impulsive perturbations seem to have never been considered either with respect to the stability of their steady states or oscillation of their solutions. Stability and asymptotic behaviour of certain ordinary differential equations with impulses have been considered by Pandit and Deo [19], Gurgula [9], Borisenko [3], Perestyuk and Chernikova [20].

The purpose of this article is to examine the following aspects of delay differential equations with impulses; "if the trivial solution of a delay differential system is asymptotically stable, in the absence of impulsive perturbations, under what conditions impulsive perturbations can maintain such

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an asymptotic stability?"; "if all solutions of a delay differential system are oscillatory, will they continue to do so when the system is subjected impulsive perturbations?" One of the intuitive expectations on the above questions is the following: if the impulses do not occur "too often" and if the "magnitudes" of the perturbations are not "too large" then the perturbed system will have the same qualitative behaviour as that of the unperturbed system. Our plan in the following is to consider a simple scalar delay differential system whose unperturbed behaviour is known and study its behaviour when subjected to impulsive perturbations. Our method easily generalises to vector-matrix linear systems.

2. Asymptotic Stability

We consider the following delay differential equation with impulses

$$\frac{dx(t)}{dt} + ax(t-\tau) = \sum_{j=1}^{\infty} b_j x(t_j -)\delta(t-t_j) \qquad t \neq t_j, \tag{2.1}$$

where b_j $(j=1, 2, ...)$ are real numbers, a is a positive number, τ, t_j $(j=1, 2, ...)$ are real numbers such that $\tau \geqslant 0$, $0 < t_1 < t_2 < \cdots < t_j \to \infty$ as $j \to \infty$. It has been known that when all b_j $(j=1, 2, 3, ...)$ are zero, the trivial solution of (2.1) is exponentially asymptotically stable whenever $0 < a\tau < \pi/2$. In fact the characteristic equation associated with (2.1) when $b_j = 0$, $j = 1, 2, 3, ...$ is of the form

$$\lambda + ae^{-\tau} = 0 \tag{2.2}$$

and one can show that for $0 < a\tau < \pi/2$, all the roots of (2.2) have negative real parts. Let

$$\max\{\operatorname{Re} \lambda \mid \lambda + ae^{-\lambda\tau} = 0\} = -\alpha_0, \tag{2.3}$$

where α_0 is a positive number.

By a solution of (2.1) we shall mean a real valued function x defined on $[-\tau, \infty)$ which is left continuous on $[-\tau, \infty)$ and is differentiable on $(0, t_1)$, (t_j, t_{j+1}) $(j=1, 2, 3, ...)$ satisfying

$$\frac{dx(t)}{dt} + ax(t-\tau) = 0, \qquad t \in (0, t_1) \bigcup_{j=1}^{\infty} (t_j, t_{j+1}). \tag{2.4}$$

The following result provides sufficient conditions for the asymptotic stability of the trivial solution of (2.1).

THEOREM 2.1. *Assume the following*:

(i) $0 < a\tau < \pi/2$.

(ii) $t_{j+1} - t_j \geqslant T > 0$, $j = 1, 2, 3, \ldots$; *and* $\tau < T$.

(iii) $1 + |b_j| \leqslant M$ *for* $j = 1, 2, 3, \ldots$.

(iv) $(1/T) \ln M < \alpha$ *for some* α *less then* α_0.

Then the trivial solution of (2.1) *is globally exponentially asymptotically stable.*

Proof. It is known from Corduneanu and Luca [4] that solutions of (2.1) corresponding to initial conditions of the form

$$x(t) = \varphi(t), \qquad t < 0; \quad x(0+) = x^0, \tag{2.4}$$

where $\varphi \in C([-\tau, 0), \mathbb{R})$ are given by

$$x(t) = U(t)x^0 + y(t, \varphi) + \int_0^t U(t-s)h(s)\, ds \tag{2.5}$$

in which U is defined by

$$\frac{dU(t)}{dt} + aU(t-\tau) = 0, \qquad t > 0 \tag{2.6}$$

$$U(t) = 0 \qquad \text{for} \quad t \in [-\tau, 0);\ U(0+) = 1$$

and

$$y(t, \varphi) = -a \int_{-\tau}^{0} U(t - \tau - s)\, \varphi(s)\, ds, \qquad t > 0 \tag{2.7}$$

$$h(t) = \sum_{j=1}^{\infty} b_i x(t_i -)\delta(t - t_i), \qquad t > 0. \tag{2.8}$$

In the following analysis of (2.5), we can without loss of generality assume that $\varphi(t) \equiv 0$ on $[-\tau, 0)$, since for any $\varphi \in C([-\tau, 0), \mathbb{R})$, $y(t, \varphi) \to 0$ as $t \to \infty$ by the exponential asymptotic stability of the trivial solution of (2.1) in the absence of impulses (due to the condition $0 < a\tau < \pi/2$). It follows from (2.5) and (2.8),

$$x(t) = U(t)x(0+) \qquad \text{on} \quad [0, t_1) \tag{2.9}$$

$$x(t) = U(t)x(0+) + U(t - t_1)b_1 x(t_1 -) \qquad \text{on} \quad [t_1, t_2). \tag{2.10}$$

It is not difficult to see from (2.1) that

$$x(t_j +) = (1 + b_j)x(t_j -), \qquad j = 1, 2, 3, \ldots. \tag{2.11}$$

From (2.9), (2.10), (2.11),

$$x(t) = U(t)[1 + b_1]x(0+) \qquad \text{on} \quad [t_1, t_2). \tag{2.12}$$

Similarly one derives that

$$x(t) = U(t)(1 + b_1)(1 + b_2)x(0+) \qquad \text{on} \quad [t_2, t_3). \tag{2.13}$$

Let us now suppose

$$x(t) = U(t) \prod_{j=1}^{k} (1 + b_j)x(0+) \qquad \text{on} \quad [t_k, t_{k+1}).$$

For $t \in [t_{k+1}, t_{k+2})$ we have then

$$\begin{aligned} x(t) &= U(t)x(0+) + \sum_{t_{k+1} < t < t_{k+2}} b_j x(t_j-)U(t - t_j) \\ &= U(t)x(0+) + \sum_{j=1}^{k} b_j x(t_j-)U(t - t_j) + b_{k+1} x(t_{k+1}-)U(t - t_{k+1}) \\ &= U(t) \prod_{j=1}^{k} (1 + b_j)x(0+) + b_{k+1} U(t - t_{k+1})U(t_{k+1}) \prod_{j=1}^{k} (1 + b_j)x(0+) \\ &= U(t) \prod_{j=1}^{k+1} (1 + b_j)x(0+). \end{aligned} \tag{2.14}$$

Thus by induction we have

$$x(t) = U(t) \prod_{j=1}^{n} (1 + b_j)x(0+) \qquad \text{on} \quad t \in [t_n, t_{n+1}) \tag{2.15}$$

and hence

$$\begin{aligned} |x(t)| &\leqslant Ke^{-\alpha t} \prod_{j=1}^{n} (1 + |b_j|)\, |x(0+)| \\ &\leqslant Ke^{-\alpha t} M^{n(t)}\, |x(0+)| \\ &\leqslant Ke^{-\alpha t} \exp[n(t) \ln M]\, |x(0+)| \\ &\leqslant Ke^{-\alpha t} \exp\left[\frac{(\ln M)}{T}\, t\right] |x(0+)| \\ &\leqslant K\, |x(0+)| \exp\left[-\left(\alpha - \frac{(\ln M)}{T}\right) t\right], \end{aligned}$$

where $nb(t)$ denotes the number of jumps in the interval $(0, t)$. Now if $\varphi \not\equiv 0$ on $[-\tau, 0)$ then one can easily see from (2.5) and (2.7) that $y(t, \varphi) \to 0$ exponentially as $t \to \infty$. Thus the exponential asymptotic stability of the trivial solution of (2.1) follows from (2.16) and the hypothesis of the theorem.

3. Oscillation and Nonoscillation

We consider an impulsive system of the type

$$\frac{dx(t)}{dt} + p(t)\,x(t-\tau) = 0 \qquad t \neq t_i$$

$$x(t_i+) - x(t_i-) = b_i x(t_i-) \tag{3.1}$$

$$0 < t_i < t_2 < \cdots < t_j \to \infty \qquad \text{as} \quad j \to \infty,$$

where τ is a positive real number.

As it is customary, we shall say that a nontrivial solution of (3.1) is nonoscillatory if it is eventually positive or eventually negative and otherwise it will be called oscillatory.

THEOREM 3.1. *Assume the following*:

(i) *p is continuous on $[0, \infty)$ and $p(t) \geqslant 0$ for $t \geqslant 0$.*

(ii) $t_{i+1} - t_i \geqslant T$; $i = 1, 2, 3, \ldots$.

(iii) *Either*

$$\limsup_{i \to \infty} (1 + b_i)^{-1} \int_{t_i}^{t_i + T} p(s)\, ds > 1 \qquad \textit{if} \quad \tau \geqslant T \tag{3.2}$$

or

$$\limsup_{i \to \infty} (1 + b_i)^{-1} \int_{t_i}^{t_i + \tau} p(s)\, ds > 1 \qquad \textit{if} \quad 0 < \tau < T. \tag{3.3}$$

Then all solutions of (3.1) *are oscillatory.*

Proof. Suppose the result is not true; then there exists a nonoscillatory solution of (3.1) say $x(t)$. We shall assume that $x(t) > 0$ for all $t \geqslant t*$ (if $x(t) < 0$ eventually then consider $-x(t)$). Since eventually $x(t) > 0$, $\dot{x}(t) \leqslant 0$ for all large t, x is nonincreasing on intervals of the form (t_j, t_{j+1}), $j = 1, 2, 3, \ldots$. We shall prove the result in the case of $\tau \geqslant T$.

It follows from (3.1) by an integration on (t_i, t_i+T),

$$x(t_i+T)-x(t_i+0)+\int_{t_i+0}^{t_i+T} p(s)x(s-\tau)\,ds=0. \tag{3.4}$$

By the nonincreasing nature of x, we have from (3.4),

$$x(t_i+T)-x(t_i+0)+\left[\int_{t_i+0}^{t_i+T} p(s)\,ds\right]x(t_i+T-\tau)\leqslant 0$$

and, hence,

$$x(t_i+T)-x(t_i+0)+x(t_i-0)\int_{t_i+0}^{t_i+T} p(s)\leqslant 0. \tag{3.5}$$

Now using the jump conditions of (3.1) in (3.5),

$$x(t_i+T)+x(t_i+0)\left[\frac{1}{1+b_i}\int_{t_i}^{t_i+T} p(s)\,ds-1\right]\leqslant 0. \tag{3.6}$$

But (3.6) is impossible due to the eventual positivity of x and (3.2). By a similar analysis one can derive a contradiction if (3.3) holds. The proof is complete.

The following result provides a sharper condition than that in the previous theorem.

THEOREM 3.2. *Assume the following:*

(i) $t_{i+1}-t_i\geqslant T$; $i=1, 2, 3, \ldots$ *and* $\tau<T$.

(ii) $0\leqslant b_i\leqslant M$; $i=1, 2, 3, \ldots$.

(iii) p *is continuous on* $[0, \infty)$ *and* $p(t)\geqslant 0$ *for* $t\geqslant 0$.

(iv)

$$\liminf_{t\to\infty}\int_{t-\tau}^{t} p(s)\,ds>\frac{1+M}{e}. \tag{3.7}$$

Then every solution of (3.1) *is oscillatory.*

Proof. Suppose the result is not true and there exists an eventually positive solution say $y(t)>0$ for $t\geqslant t*$. Define

$$w(t)=\frac{y(t-\tau)}{y(t)} \qquad \text{for} \quad t\geqslant t*+\tau. \tag{3.8}$$

Considering the interval $[t-\tau, t]$ and $t_i \in (t-\tau, t)$,

$$y(t-\tau) \geqslant y(t_i-) = \frac{1}{1+b_i} y(t_i+) \geqslant \frac{1}{1+b_i} y(t), \tag{3.9}$$

implying

$$w(t) = \frac{y(t-\tau)}{y(t)} \geqslant \frac{1}{1+b_i} \geqslant \frac{1}{1+M}. \tag{3.10}$$

We shall first show that $w(t)$ is bounded above. Let t_K be a jump point in $[t-2\tau, t-\tau]$. Integrating (3.1) on $[t-\tau/2, t]$,

$$y(t) - y\left(t-\frac{\tau}{2}\right) + \int_{t-\tau/2}^{t} p(s)\, y(s-\tau)\, ds = 0. \tag{3.11}$$

It follows from (3.11) that

$$\begin{aligned} y\left(t-\frac{\tau}{2}\right) &\geqslant \int_{t-\tau/2}^{t} p(s)\, y(s-\tau)\, ds \\ &\geqslant \int_{t-\tau/2}^{t_K+\tau-0} p(s)\, y(s-\tau)\, ds + \int_{t_K+\tau+0}^{t} p(s)\, y(s-\tau)\, ds \\ &\geqslant \frac{y(t-\tau)}{1+M} \int_{t-\tau/2}^{t} p(s)\, ds. \end{aligned} \tag{3.12}$$

On integrating (3.1) over $[t-\tau, t-\tau/2]$,

$$y(t-\tau) \geqslant y\left(t-\frac{3\tau}{2}\right) \int_{t-\tau}^{t-\tau/2} p(s)\, ds.$$

Thus

$$y\left(t-\frac{\tau}{2}\right) \geqslant y\left(t-\frac{3\tau}{2}\right) \left[\int_{t-\tau}^{t-\tau/2} p(s)\, ds\right] \left[\int_{t-\tau/2}^{t} p(s)\, ds\right] \frac{1}{1+M} \tag{3.13}$$

and, hence,

$$\frac{y(t-3\tau/2)}{y(t-\tau/2)} \leqslant \frac{1+M}{[\int_{t-\tau}^{t-\tau/2} p(s)\, ds][\int_{t-\tau/2}^{t} p(s)\, ds]} \leqslant N. \tag{3.14}$$

We have from (3.1) for large enough t,

$$\int_{t-\tau}^{t} \frac{y'(s)}{y(s)} + \int_{t-\tau}^{t} p(s) \frac{y(s-\tau)}{y(s)}\, ds = 0. \tag{3.15}$$

But

$$\begin{aligned}\int_{t-\tau}^{t} \frac{y'(s)}{y(s)}\,ds &= \int_{t-\tau}^{t_K-0} \frac{y'(s)}{y(s)}\,ds + \int_{t_K+0}^{t} \frac{y'(s)}{y(s)}\,ds \\ &= \ln \frac{y(t_K-0)}{y(t-\tau)}\,\frac{y(t)}{y(t_K+0)} \\ &= \ln \frac{y(t)}{y(t-\tau})\,\frac{1}{1+b_K}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16),

$$\ln \frac{y(t-\tau)}{y(t)}\,(1+b_K) = \int_{t-\tau}^{t} p(s)\,\frac{y(s-\tau)}{y(s)}\,ds. \tag{3.17}$$

if

$$l = \lim_{t\to\infty} \inf w(t) \tag{3.18}$$

then l is finite and positive and (3.17) leads to

$$\ln[(1+M)w(t)] \geqslant l \int_{t-\tau}^{t} p(s)\,ds$$

which implies that

$$\frac{1+M}{e} \geqslant \frac{\ln[(1+M)l]}{l} \geqslant \lim_{t\to\infty} \inf \int_{t-\tau}^{t} p(s)\,ds \tag{3.19}$$

and (3.19) contradicts (3.7). The result follows from such a contradiction and the proof is complete.

It is well known that the autonomous delay differential equation

$$\frac{du(t)}{dt} + au(t-\tau) = 0 \tag{3.20}$$

has a nonoscillatory solution if $0 \leqslant a\tau \leqslant 1/e$. If Eq. (3.20) is subjected to impulsive perturbations, the nonoscillatory solutions of the unperturbed system may or may not continue to persist under impulsive perturbations. The following result provides a set of sufficient conditions for the existence of nonoscillatory solutions of (3.1).

THEOREM 3.3. *Suppose the parameters of* (3.1) *satisfy the following*:

(i) *there exists a positive number c such that*

$$a\tau e \leqslant 1 - c;$$

(ii) $b_i > 0$; $i = 1, 2, 3, \ldots$ *and* $\sum_{i=1}^{\infty} b_i < \infty$.

Then (3.1) *has a nonoscillatory solution.*

Proof. Let t_0 be a real number and let $L_1[t_0 - \tau, \infty)$ denote the space of all equivalence classes of real valued functions defined on $[t_0 - \tau, \infty)$ such that

$$L_1[t_0 - \tau, \infty) = \left\{ f\colon [t_0 - \tau, \infty) \to (-\infty, \infty) \,\middle|\, \int_{t_0 - \tau}^{\infty} |f(t)|\, dt < \infty \right\}.$$

It is known that L_1 is a complete metric space with the metric ρ defined by

$$\rho(f, g) = \int_{t_0 - \tau}^{\infty} |f(t) - g(t)|\, dt.$$

Consider a set $A \subset L_1$ defined as

$$A = \{ f \in L_1[t_0 - \tau, \infty) \mid e^{-\mu_1 t} \leqslant f(t) \leqslant e^{-\mu_2 t};\ \mu_1 > \mu_2 > 0 \}, \tag{3.21}$$

where μ_2 satisfies $ae^{\mu_2 \tau} \leqslant (1 - c)\mu_2$. Define a map S,

$$S\colon A \to L_1[t_0 - \tau, \infty),$$

where

$$S(x)(t) = \begin{cases} \displaystyle\int_t^{\infty} [ax(s-\tau) - \sum_{j=1}^{\infty} b_j x(t_j -)\delta(s - t_j]\, ds, & t \geqslant t_0, \\ \displaystyle\int_{t_0}^{\infty} ax(s-\tau) - \sum_{j=1}^{\infty} b_j x(t_j -)\delta(s - t_j)]\, ds, & t_0 - \tau \leqslant t \leqslant t_0. \end{cases} \tag{3.22}$$

It is easy to see that $S(A) \subset A$; for instance,

$$S(x)(t) \leqslant \frac{ae^{\mu_2 \tau}}{\mu_2} e^{-\mu_2 t} + \sum_{n(t)}^{\infty} b_i e^{-\mu_2 t} \leqslant e^{-\mu_2 t}, \tag{3.23}$$

provided $ae^{\mu_2 \tau}/\mu_2 \leqslant 1 - c$ and $\sum_{n(t)} b_i < c$. This is possible since we can choose t_0 sufficiently large so that $n(t)$ will be a large enough positive integer. There exists a $\mu_1 > 0$ such that

$$S(x)(t) \geqslant \frac{ae^{\mu_1 \tau}}{\mu_1} e^{-\mu_1 t} \geqslant e^{-\mu_1 t}. \tag{3.24}$$

For instance, if we let $\mu_2 = 1/\tau$, we get $a\tau e \leqslant 1 - c$ from $ae^{\mu_2 \tau}/\mu_2 \leqslant 1 - c$. It is easy to see that $S(x) \in L_1$ for $x \in A$. Thus $S(A) \subset A$. Since S maps the bounded closed subset A of L_1 into itself, S is a compact map. By the Schauder's fixed point theorem, S has a fixed point x^* satisfying $Sx^* = x^*$ and this implies that x^* is a nonoscillatory solution.

4. Delay-Logistic Equation with Impulses

In this section we consider the impulsively perturbed delay-logistic nonlinear equation

$$\frac{dx(t)}{dt} = rx(t)\left\{1 - \frac{x(t-\tau)}{K}\right\} + \sum_{i=1}^{\infty} b_i[x(t_i-) - K]\,\delta(t-t_i) \tag{4.1}$$

in which r, K, τ are positive constants; b_i, t_i are real numbers such that $0 < t_1 < t_2 < \cdots < t_j \to \infty$ as $j \to \infty$. We are mainly concerned with the asymptotic behaviour of (4.1) and especially the attractivity of the steady state K with respect to solutions of (4.1) corresponding to initial conditions of the form $\varphi(s) \geqslant 0$ on $[-\tau, 0)$, $\varphi(0) > 0$, and $\varphi \in C[-\tau, 0]$. If we let $x(t) \equiv K[1 + y(t)]$ in (4.1) then y is governed by

$$\frac{dy(t)}{dt} = -r[1 + y(t)]\,y(t-\tau) + \sum_{i=1}^{\infty} b_i\, y(t_i-)\,\delta(t-t_i), \qquad t \neq t_i, \tag{4.2}$$

and it is sufficient to consider the attractivity of the trivial solution of (4.2).

Our strategy for analysing (4.2) is to consider (4.2) as perturbation of the familiar delay-logistic equation

$$\frac{dz}{dt} = -r[1 + z(t)]\,z(t-\tau). \tag{4.3}$$

If $z(t)$ is any solution of (4.3), then the variational equation corresponding to z and (4.3) is given by the linear nonautonomous equation

$$\frac{du(t)}{dt} = -rz(t-\tau)u(t) - r[1 + z(t)]\,u(t-\tau). \tag{4.4}$$

It is known from our previous works that if φ is "small" and if

$$0 < r\tau < \pi/2 \tag{4.5}$$

(for details see Zhang and Gopalsamy [23], Gopalsamy [8]) then every solution $z(t)$ of (4.3) exponentially approaches zero as $t \to \infty$. For such z, it is known that the trivial solution of (4.4) is uniformly asymptotically stable (for details see Driver [5]). That is, there exist positive constants $M \geqslant 1$ and $\alpha > 0$ such that for $(t_0, \varphi) \in [0, \infty) \times C[-\tau, 0]$ and "small" φ

$$\|u(t, t_0, \varphi)\| \leqslant M \|\varphi\| \exp[-\alpha(t - t_0)], \qquad t \geqslant t_0. \tag{4.6}$$

We are now ready to formulate our result.

THEOREM 4.1. *Suppose the positive constants r, τ satisfy $0 < r\tau < \pi/2$. Let α be the positive constant as in* (4.6). *Let $N = \sup[1 + M|b_i|, i = 1, 2, 3, ...]$. Suppose $t_{i+1} - t_i \geqslant T$, $i = 1, 2, 3, ...$ If furthermore φ is "small" and*

$$-\alpha + [\ln N)/T] < 0, \tag{4.7}$$

then every solution of (4.2) *appraoches zero exponentially as $t \to \infty$.*

Proof. By the nonlinear variation of constants formula (Shanholt [21], Hastings [11]) we have from (4.2) and (4.3),

$$\begin{aligned} y(t) &= z(t) + \int_0^t T(t, s, y_s) X_0 \sum_{i=1}^{\infty} b_i y(s_i -) \delta(s - t_i)\, ds \\ &= z(t) + \sum_{j=1}^{n(t)} T(t, t_j, y_{t_j}) X_0 b_j y(t_j -), \qquad t > n(t), \end{aligned} \tag{4.8}$$

where $T(t, t_j, y_{t_j}) X_0$ is a solution of (4.4) with initial values satisfying $u(t) = 0$ on $[-\tau, 0)$ and $u(0+) = 1$. We have from (4.8),

$$y(t) = z(t) \qquad \text{on} \quad (0, t_1) \tag{4.9}$$

$$y(t) = z(t) + T(t, t_1, y_{t_1}) X_0 b_1 y(t_1 -) \qquad \text{on} \quad (t_1, t_2) \tag{4.10}$$

and hence

$$\begin{aligned} |y(t) &\leqslant M \|z_0\| e^{-\alpha t} + M^2 |b_1| \|z_0\| e^{-\alpha t} \\ &\quad + M |b_2| e^{-\alpha(t - t_2)} \|z_0\| M(1 + M|b_1|) e^{-\alpha t_1} \\ &\leqslant M(1 + M|b_1|)(1 + M|b_2|) e^{-\alpha t} \|z_0\| \qquad \text{on} \quad (t_2, t_3). \end{aligned} \tag{4.12}$$

By induction one can prove that

$$|y(t)| \leqslant M \|z_0\| \prod_{i=1}^{n} (1 + M|b_i|) e^{-\alpha t} \qquad \text{on} \quad (t_n, t_{n+1}). \tag{4.13}$$

If we let

$$N = \sup[1 + M\,|b_i|,\ i = 1, 2, 3, ...]$$

then (4.13) becomes

$$|y(t) \leqslant MN^{n(t)}e^{-\alpha t}\,\|z_0\| \tag{4.14}$$

which, on using $t_{i+1} - t_i \geqslant T$ and $n(t) \leqslant t/T$, becomes

$$\begin{aligned} |y(t) &\leqslant M\,\|z_0\|\,\exp[-\alpha t + (t \ln N)/T] \\ &= M\,\|z_0\|\,\exp[-\{\alpha - (\ln N/T)\}\,t] \end{aligned} \tag{4.15}$$

and the result follows from (4.15) by virtue of (4.7).

We conclude with the formulation of the following result whose proof is similar to that of Theorem 3.1.

THEOREM 4.2. *In the impulsive delay-logistic system*

$$\frac{dy(t)}{dt} + p(t)[1 + y(t)]\,y(t-\tau) = 0, \qquad t \neq t_i$$

$$y(t_i + 0) - y(t_i - 0) = b_i\,y(t_i - 0), \qquad 0 < t_1 < t_2 < \cdots < t_j \to \infty \quad as \quad j \to \infty, \tag{4.16}$$

assume

(i) $t_{i+1} - t_i \geqslant T$, $\tau \geqslant T$; $i = 1, 2, 3, 4, ...$;

(ii) $p \in C(\mathbb{R}_+, \mathbb{R}_+)$;

(iii) $$\lim_{i \to \infty} \sup \frac{1}{1 + b_i} \int_{t_1}^{t_1 + T} p(s)\,ds > 1. \tag{4.17}$$

Then every solution of (4.16) *is oscillatory.*

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