

# Perfect Splines with Boundary Conditions of Least Norm\*

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Let  $A = (a_{ij})_{i=1, j=0}^{l, r-1}$  and  $B = (b_{ij})_{i=1, j=0}^{m, r-1}$  be matrices of ranks  $l$  and  $m$ , respectively. Suppose that  $\tilde{A} = ((-1)^j a_{ij}) \in SC_l$  (sign consistent of order  $l$ ) and  $B \in SC_m$ . Denote by  $\mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  the set of perfect splines with  $N$  knots which have  $n$  distinct zeros in  $(0, 1)$  with multiplicities  $v_1, \dots, v_n$ , respectively, and satisfy  $A\bar{P}(0) = 0, B\bar{P}(1) = 0$ , where  $\bar{P}(a) = (p(a), \dots, P^{(r-1)}(a))^T$ . We show that there is a unique  $P^* \in \mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  of least uniform norm and that  $P^*$  is characterized by the equioscillatory property. This is closely related to the optimal recovery of smooth functions satisfying boundary conditions by using the Hermite data. © 1994 Academic Press, Inc.

## 1. INTRODUCTION

A perfect spline  $P(t)$  of degree  $r$  with knots  $\{\xi_j\}_1^N \subset (0, 1), \xi_1 < \dots < \xi_N$ , has the representation

$$P(t) = \sum_{i=0}^{r-1} a_i t^i + \frac{1}{r!} \left[ t^r + 2 \sum_{j=1}^N (-1)^j (t - \xi_j)_+^r \right], \tag{1.1}$$

where  $\{a_i\}_{i=0}^{r-1}$  are real constants and, as usual,  $t_+ = \max\{t, 0\}$ . The set of all functions of form (1.1) is denoted by  $\mathcal{P}_{r,N}$ .

Let  $A = (a_{ij})_{i=1, j=0}^{l, r-1}$  and  $B = (b_{ij})_{i=1, j=0}^{m, r-1}$  be matrices such that

- (i)  $0 \leq l, m \leq r, \text{rank } A = l, \text{rank } B = m,$
- (ii)  $\tilde{A} = ((-1)^j a_{ij})_{i=1, j=0}^{l, r-1} \in SC_l, B \in SC_m,$

where  $A \in SC_\mu$  means that all non-zero  $\mu \times \mu$  subdeterminants of  $A$  maintain the same sign, i.e., there exists  $\sigma_A \in \{-1, 1\}$  such that

$$\sigma_A A \begin{pmatrix} i_1, \dots, i_\mu \\ j_1, \dots, j_\mu \end{pmatrix} \geq 0,$$

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for every choice of  $i_1, \dots, i_\mu$  and  $j_1, \dots, j_\mu$ , where  $A_{\substack{i_1, \dots, i_\mu \\ j_1, \dots, j_\mu}}$  is the subdeterminant of  $A$  composed of  $i_1, \dots, i_\mu$  rows and  $j_1, \dots, j_\mu$  columns, respectively.

Given  $A$  and  $B$  as above, we define functionals as follows:

$$\begin{aligned} A_i f &= \sum_{j=0}^{r-1} a_{ij} f^{(j)}(0), & i = 1, \dots, l \\ B_i f &= \sum_{j=0}^{r-1} b_{ij} f^{(j)}(1), & i = 1, \dots, m. \end{aligned} \tag{1.2}$$

For a given set  $\mathcal{F}$  of functions such that  $f^{(r-1)}(0)$  and  $f^{(r-1)}(1)$  exist for  $f \in \mathcal{F}$ , we denote by  $\mathcal{F}(A, B)$  all functions  $f$  in  $\mathcal{F}$  with  $A_i f = 0$  ( $i = 1, \dots, l$ ) and  $B_i f = 0$  ( $i = 1, \dots, m$ ).

Some problems related to boundary conditions (1.2) have been considered. The problem of existence of interpolating spline  $s$  with  $A_i s = B_j s = 0$  ( $i = 1, \dots, l, j = 1, \dots, m$ ) is discussed in Ref. [7]. The  $n$ -widths of  $W'_\infty(A, B)$  in  $C[0, 1]$  are obtained [13], where

$$W'_\infty = \{f \mid f^{(r-1)} \text{ abs. cont. on } [0, 1], \|f^{(r)}\|_\infty \leq 1\}.$$

Given  $\{v_i\}_{i=1}^n$  such that

$$1 \leq v_i \leq r, \quad r \leq \sum_{i=1}^n v_i = N + r - l - m,$$

we use  $\mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  to denote the set of functions in  $\mathcal{P}_{r,N}(A, B)$  having  $n$  distinct zeros  $\{x_i\}_{i=1}^n \subset (0, 1)$  with multiplicities  $\{v_i\}_{i=1}^n$ , respectively. This paper is devoted to study the extremal problem:

$$\inf\{\|P\| \mid P \in \mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)\}, \tag{1.3}$$

where  $\|\cdot\| = \|\cdot\|_{C[0,1]}$ .

Problem (1.3) has been discussed in [1] for  $l = m = 0$  (i.e., no boundary condition) and [9] for (1.2) being quasi-Hermite conditions, respectively. We note that [9] only gives proof for the case  $m < N$ . As for the case  $m = N$  (note that  $m \leq N$ ), we see below that the proof is somewhat more complicated.

It is well known that perfect splines are very important in the optimal recovery (see [10, 11]). In fact, as in [5, 10], the intrinsic error of the best scheme approximating  $f \in W'_\infty(A, B)$  in  $C[0, 1]$  by using information  $\{f^{(j)}(x_i) \mid i = 1, \dots, n, j = 0, \dots, v_i - 1\}$  is equal to  $\|P(\bar{X}, \cdot)\|$ , where  $P(\bar{X}, \cdot) \in \mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  vanishes at  $\bar{X} = \{(x_i, v_i)\}$  (see Lemma 1). Hence, the zeros of  $P^* \in \mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  that solves problem (1.3) determine the optimal information. Moreover, spline interpolation at the zeros of  $P^*$  is an optimal algorithm (see [5]).

The main results of this paper are as follows.

**THEOREM 1.** *Let  $\{e_i\}_{i=1}^{n+1}$  be arbitrary positive numbers. Set  $\sigma_1 = 0$ ,  $\sigma_i = \sum_{k=1}^{i-1} \nu_k$ ,  $k = 2, \dots, n + 1$ . Then there exists a unique  $P \in \mathcal{P}_{r,N}(A, B; \nu_1, \dots, \nu_n)$  with  $n$  distinct zeros  $\{x_i\}_{i=1}^n$  and a positive number  $R$  such that*

$$RP(y_i) = (-1)^{\sigma + \sigma_i} e_i, \quad i = 1, \dots, n + 1, \tag{1.4}$$

where  $\sigma \in \{-1, 1\}$  fixed and  $\{y_i\}_{i=1}^{n+1}$  satisfy

$$0 \leq y_1 < x_1 < y_2 < \dots < x_n < y_{n+1} \leq 1$$

with  $P'(y_i) = 0$  whenever  $y_i \in (0, 1)$ .

**THEOREM 2.** *There exists a unique perfect spline  $P^* \in \mathcal{P}_{r,N}(A, B; \nu_1, \dots, \nu_n)$ , which solves problem (1.3). Moreover,  $P^*$  is characterized by the equi-oscillatory property; i.e., there exist  $n + 1$  points  $\{y_i\}_{i=1}^{n+1} \subset [0, 1]$  such that*

$$P^*(y_i) = (-1)^{\sigma + \sigma_i} \|P^*\|, \quad i = 1, \dots, n + 1,$$

where  $\sigma_i$  are given as in Theorem 1.

## 2. AUXILIARY LEMMAS

Similar to [2], Li [9] proved the following result by the Hobby–Rice theorem.

**LEMMA 1.** *Given  $\{x_i\}_{i=1}^n \subset (0, 1)$ , there is a function  $P \in \mathcal{P}_{r,N}(A, B; \nu_1, \dots, \nu_n)$  such that  $P$  vanishes at  $\bar{X} = \{(x_i, \nu_i)\}_{i=1}^n$ , where  $(x_i, \nu_i)$  means that  $x_i$  is a zero with multiplicity  $\nu_i$ .*

Let  $a = (a_i)_{i=1}^n \in \mathbf{R}^n / \{0\}$ ;  $S^+(a)$  denotes the maximal number of sign changes in the sequence  $a_1, \dots, a_n$  where zero terms are arbitrarily assigned values 1 or  $-1$ . For example,  $S^+(1, 0, 1) = 2$ .

**LEMMA 2** [12, p. 163]. *For any  $P \in \mathcal{P}_{r,N}$ , it holds that*

$$Z_r(P, (0, 1)) \leq N + r - S^+((( -1)^j P^{(j)}(0))_{j=0}^r) - S^+((P^{(j)}(1))_{j=0}^r), \tag{2.1}$$

where  $Z_r(f, I)$  is the total number of zeros of  $f$  at an interval  $I$  counting multiplicities not greater than  $r$ .

We call  $P \in \mathcal{P}_{r,N}$  a perfect spline with maximal number of zeros if equality holds in (2.1) for  $P$ . For such a perfect spline, its zeros and knots satisfy the so-called interlacing conditions. To be precise, we denote by  $\{z_i\}_{i=1}^r$

and  $\{\xi_i\}_{i=1}^N$  arranged in natural increasing order the zeros of  $P$  at  $(0, 1)$  and knots, respectively, where  $s$  is the number of zeros of  $P$  at  $(0, 1)$  counting multiplicities. Then [5]

$$z_{i-\alpha_0} < \xi_i < z_{i+r-\alpha_0}, \tag{2.2}$$

whenever the subscripts are meaningful, where  $\alpha_0 = S^+((-1)^j P^{(j)}(0))_{j=0}^r$ .

It is known that (see, e.g., [13]) for any  $P \in \mathcal{P}_{r,N}(A, B)$ ,  $\alpha_0 \geq l$ ,  $\beta_0 \geq m$ . Therefore, any function  $P$  in  $\mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  is with maximal number of zeros and  $\alpha_0 = l$ ,  $\beta_0 = m$ .

LEMMA 3. *If  $P \in \mathcal{P}_{r,N}$  is with maximal number of zeros, so is  $P' \in \mathcal{P}_{r-1,N}$ .*

*Proof.* Set  $\alpha_i = S^+((-1)^j P^{(j)}(0))_{j=i}^r$ ,  $\beta_i = S^+(P^{(j)}(1))_{j=i}^r$ ,  $i = 0, 1$ . Let  $x$  and  $y$  be minimal zero and maximal zero of  $P$  in  $(0, 1)$ , respectively.

Now we should distinguish the cases according to the values of  $\alpha_i$  and  $\beta_i$ ,  $i = 0, 1$ . Suppose first that  $\alpha_0 - \alpha_1 = 1$  and  $\beta_0 - \beta_1 = 0$ . Since  $P^{(r)}(0) \neq 0$ , by [12],  $t = 0$  is a left Rolle point of  $P$  (the definition of Rolle point can be found in [12, p. 28]). By the extended Rolle theorem [12, p. 29], there exists at least a zero of  $P'$  in  $(0, x)$ . Therefore,

$$\begin{aligned} Z_{r-1}(P', (0, 1)) &\geq Z_{r-1}(P', (0, 1)) + Z_{r-1}(P', [x, y]) \\ &\geq 1 + Z_r(P, [x, y]) - 1 = Z_r(P, (0, 1)) \\ &= N + (r - 1) - \alpha_1 - \beta_1. \end{aligned}$$

This together with Lemma 2 implies that  $P' \in \mathcal{P}_{r-1,N}$  is with maximal number of zeros. The other cases may be treated similarly. So the proof is complete.

Now we introduce some notations from [7] for later use. Let

$$\begin{aligned} Z &= \{x, 0, \dots, r - 1 \mid x \in (0, 1)\}, \\ W &= \{0, \dots, r - 1, \xi \mid \xi \in (0, 1)\}, \end{aligned}$$

and  $K(z, w)$  be a function defined on  $Z \times W$  as follows:

$$\begin{aligned} K(x, i) &= u_i(x) = x^i, \\ K(x, \xi) &= (x - \xi)_+^{r-1}, \\ K(j, \xi) &= \frac{\partial^j}{\partial x^j} K(x, \xi)|_{x=1}, \\ K(j, i) &= u_i^{(j)}(x)|_{x=1}. \end{aligned}$$

For given  $z_1 < \dots < z_k$ ,  $w_1 < \dots < w_p$ , the Fredholm matrix based on  $K(z, w)$  is

$$K \begin{pmatrix} z_1, \dots, z_k \\ w_1, \dots, w_p \end{pmatrix} = (K(z_i, w_j))_{i,j=1}^{k,p}.$$

If some  $z_i$  and/or  $w_i$  coincide, the corresponding columns and/or rows are determined by successive derivatives (see [7] for the details). It is shown in [7] that  $K(z, w)$  is total positive, i.e.,

$$\det K \begin{pmatrix} z_1, \dots, z_k \\ w_1, \dots, w_k \end{pmatrix} \geq 0, \tag{2.3}$$

for every choice of  $z_1 \leq \dots \leq z_k$ ,  $w_1 \leq \dots \leq w_k$ .

LEMMA 4. Given any  $\{t_i\}_{i=1}^\tau$ ,  $\{y_i\}_{i=1}^s \subset (0, 1)$  and  $\{i_j\}_{j=1}^k \subset \{j\}_{j=0}^{\tau-1}$  ( $\tau + l + m = k + 2$ ) each of which is arranged in increasing order, we have  $\sigma_\lambda \sigma_\beta \sigma \det \Delta \geq 0$ , where  $\sigma = (-1)^{\tau l + l(l-1)/2}$  and  $\Delta$  is a  $(k + s) \times (k + s)$  matrix whose  $j$ th column is

$$(u_{i_j}(t_1), \dots, u_{i_j}(t_\tau), A_1 u_{i_j}, \dots, A_l u_{i_j}, B_1 u_{i_j}, \dots, B_m u_{i_j})^T$$

for  $j = 1, \dots, k$  and  $k + j$ th column is

$$(K(t_1, y_j), \dots, K(t_\tau, y_j), 0, \dots, 0, B_1 K(\cdot - y_j), \dots, B_m K(\cdot - y_j))^T$$

for  $j = 1, \dots, s$ , with the same interpretation as in [7] if some  $t_i$  coincide.

*Proof.* For ease of notations we assume without loss of generality that  $i_j = j - 1$ ,  $j = 1, \dots, k$ . It is obvious that  $A_l u_j = a_{ij} j!$ .

Expanding  $\Delta$  by minors based on the  $\tau + 1, \dots, \tau + l$ th rows (if  $k < \tau$ , then  $\det \Delta = 0$ ) we get

$$\det \Delta = \sigma \sum_{1 \leq j_1 < \dots < j_l \leq k} (-1)^{j_1 + \dots + j_l} A \begin{pmatrix} 1, \dots, l \\ j_1, \dots, j_l \end{pmatrix} \prod_{i=1}^l (j_i - 1)! \det \Delta_{j_1, \dots, j_l},$$

where  $\Delta_{j_1, \dots, j_l}$  is the submatrix of  $\Delta$  by eliminating its  $\tau + 1, \dots, \tau + l$ th rows and  $j_1, \dots, j_l$  columns. Let  $\{j'_i\}_{i=1}^{k-l}$  arranged in increasing order denote the complements set of  $\{j_i\}_{i=1}^l$  to  $\{i\}_{i=1}^k$ . It can be verified that  $\Delta_{j_1, \dots, j_l} = CD$ , where

$$C = \begin{pmatrix} I_\tau \\ B \end{pmatrix}$$

( $I_\tau$  stands for the  $\tau \times \tau$  identity matrix) and

$$D = K \begin{pmatrix} t_1, \dots, t_\tau, 0, \dots, r-1 \\ j'_j - 1, \dots, j'_{k-l} - 1, y_1, \dots, y_s \end{pmatrix}.$$

By the Cauchy–Binet formula (see [6, p. 1]) we have

$$\det \Delta_{j_1, \dots, j_l} = \sum_{1 \leq k_1 < \dots < k_{\tau+m} \leq \tau+r} C \begin{pmatrix} 1, \dots, \tau+m \\ k_1, \dots, k_{\tau+m} \end{pmatrix} D \begin{pmatrix} k_1, \dots, k_{\tau+m} \\ 1, \dots, \tau+m \end{pmatrix}.$$

From the definition of  $C$  it follows that  $C \begin{pmatrix} 1, \dots, \tau+m \\ k_1, \dots, k_{\tau+m} \end{pmatrix}$  is equal to zero unless  $1, \dots, \tau$  are all included among the indices  $\{k_i\}_{i=1}^{\tau+m}$  and equal to  $B \begin{pmatrix} 1, \dots, m \\ k_{\tau+1}-\tau, \dots, k_{\tau+m}-\tau \end{pmatrix}$  if  $k_i = i, i = 1, \dots, \tau$ . Therefore

$$\det \Delta_{j_1, \dots, j_l} = \sigma \sum_{1 \leq i_1 < \dots < i_m \leq \tau} \det K \begin{pmatrix} t_1, \dots, t_\tau, i_1-1, \dots, i_m-1 \\ j'_1 - 1, \dots, j'_{k-l} - 1, y_1, \dots, y_s \end{pmatrix} \times B \begin{pmatrix} 1, \dots, m \\ i_1, \dots, i_m \end{pmatrix}.$$

Since (2.3) and

$$(-1)^{j_1 + \dots + j_l} A \begin{pmatrix} 1, \dots, l \\ j_1, \dots, j_l \end{pmatrix} = \tilde{A} \begin{pmatrix} 1, \dots, l \\ j_1, \dots, j_l \end{pmatrix}$$

we have  $\sigma_{\tilde{A}} \sigma_B \sigma \det \Delta \geq 0$ . The proof is complete.

### 3. PROOF OF THEOREMS

The proof of Theorem 2 may proceed as in [2] by use of Theorem 1. So we only give the proof of Theorem 1.

*Proof of Theorem 1.* Let

$$P_0(t) = \sum_{i=0}^{r-1} a_{0i} t^i + \frac{1}{r!} \left[ t^r + 2 \sum_{j=1}^N (-1)^j (t - \xi_{0,j})^r_+ \right] \quad (3.1)$$

be an arbitrary perfect spline in  $\mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$  with  $n$  distinct zeros  $\{x_{0,i}\}_{i=1}^n \subset (0, 1)$ . Since  $P_0$  has no other zero than  $\bar{X}_0 = \{(x_{0,i}, v_i)\}_{i=1}^n$  in  $(0, 1)$ , it holds that

$$(-1)^{\sigma + \sigma_i} P_0(t) \geq 0, \quad t \in (x_{0,i-1}, x_{0,i}), \quad i = 1, \dots, n+1,$$

where  $\sigma \in \{-1, 1\}$  fixed,  $x_{0,0} = 0$  and  $x_{0,n+1} = 1$ .

In what follows we have to distinguish the cases as we do in proving Lemma 3. Let  $\alpha_i$  and  $\beta_i$  be defined as in Lemma 3 for  $P_0$  (we note that  $\alpha_0 = l$ ,  $\beta_0 = m$ ). We assume first that  $\alpha_0 - \alpha_1 = \beta_0 - \beta_1 = 1$ . By the proof of Lemma 3, there exist  $\{y_{0,i}\}_{i=1}^{n+1}$  any of which is a simple zero of  $P_0'$  such that

$$0 < y_{0,1} < x_{0,1} < y_{0,2} < \dots < x_{0,n} < y_{0,n+1} < 1.$$

Put

$$\begin{aligned} e_{0,j} &= |P_0(y_{0,j})|, & j &= 1, \dots, n+1, \\ e_j(s) &= e_{0,j} + s(e_j - e_{0,j}), & j &= 1, \dots, n+1, s \in [0, 1]. \end{aligned}$$

For  $s \in [0, 1]$ , we shall construct a function  $P(s, \cdot) \in \mathcal{P}_{r,N}(A, B; v_1, \dots, v_n)$ , with parameters  $a_i(s)$ ,  $x_i(s)$ ,  $y_i(s)$ ,  $\xi_i(s)$ , and  $R(s)$  such that

$$R(s) P(s, y_i(s)) = (-1)^{\sigma + \sigma_i} e_i(s), \quad i = 1, \dots, n+1. \quad (3.2)$$

To this end, we consider the system of equations:

$$\begin{aligned} P^{(j)}(s, t)|_{t=x_i(s)} &= 0 & i &= 1, \dots, n, j = 0, \dots, v_i - 1, \\ R(s) P(s, y_i(s)) &= (-1)^{\sigma + \sigma_i} e_i(s), & i &= 1, \dots, n+1, \\ P'(s, t)|_{t=y_i(s)} &= 0, & i &= 1, \dots, n+1, \\ A_i P(s, \cdot) &= 0, & i &= 1, \dots, l, \\ B_i P(s, \cdot) &= 0, & i &= 1, \dots, m, \end{aligned} \quad (3.3)$$

where  $P^{(j)}(s, t) = (\partial^j / \partial t^j) P(s, t)$ . It is obvious that  $(P_0, R_0 = 1)$  satisfies (3.3).

For the ease of computing Jacobian of (3.3) we reorder (3.3) as follows:

$$\begin{aligned} P^{(v_i-1)}(s, t)|_{t=x_i(s)} &= 0, & i &= 1, \dots, n, \\ P'(s, t)|_{t=y_i(s)} &= 0, & i &= 1, \dots, n+1, \\ R(s) P(s, y_1(s)) &= (-1)^{\sigma + \sigma_1} e_1(s), \\ P^{(j)}(s, t)|_{t=x_i(s)} &= 0, j = 0, \dots, v_i - 2, & i &= 1, \dots, n, \\ R(s) P(s, y_{i+1}(s)) &= (-1)^{\sigma + \sigma_{i+1}} e_{i+1}(s), \\ A_i P(s, \cdot) &= 0, & i &= 1, \dots, l, \\ B_i P(s, \cdot) &= 0, & i &= 1, \dots, m. \end{aligned} \quad (3.3')$$

Denote by  $\Delta(s)$  the Jacobi matrix of (3.3') with respect to  $\{x_i\}_{i=1}^{n+1}$ ,  $\{y_i\}_{i=1}^{n+1}$ ,  $R$ ,  $\{a_i\}_{i=0}^{r-1}$  and  $\{\xi_i\}_{i=1}^N$ . Then (cf. [2])

$$\det \Delta(s) = \det J(s) \prod_{i=1}^n P^{(v_i)}(s, x_i(s)) \prod_{j=1}^{n+1} P^j(s, y_j(s))$$

(see (3.10) below for the explanation that each  $P^{(v_i)}$  is well defined), where  $J(s)$  is a  $(N+r+1) \times (N+r+1)$  matrix with the first  $N+r+1-l-m$  rows

$R$	$a_0$	$\cdots$	$a_{r-1}$	$\xi_1$	$\cdots$	$\xi_N$
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$$\left( \begin{array}{cccccc} P(s, y_1) & Ru_0(y_1) & \cdots & Ru_{r-1}(y_1) & \frac{2R(-1)^{l+1}}{(r-1)!} (y_1 - \xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{l+N}}{(r-1)!} (y_1 - \xi_N)_+^{r-1} \\ 0 & u_0(x_1) & \cdots & u_{r-1}(x_1) & \frac{2(-1)^{l+1}}{(r-1)!} (x_1 - \xi_1)_+^{r-1} & \cdots & \frac{2(-1)^{l+N}}{(r-1)!} (x_1 - \xi_N)_+^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & u_0^{(v_1-2)}(x_1) & \cdots & u_{r-1}^{(v_1-2)}(x_1) & \frac{2(-1)^{l+1}}{(r+1-v_1)!} (x_1 - \xi_1)_+^{v_1-1} & \cdots & \frac{2(-1)^{l+N}}{(r-v_1+1)!} (x_1 - \xi_N)_+^{v_1-1} \\ P(s, y_2) & Ru_0(y_2) & \cdots & Ru_{r-1}(y_2) & \frac{2R(-1)^{l+1}}{(r-1)!} (y_2 - \xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{l+N}}{(r-1)!} (y_2 - \xi_N)_+^{r-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & u_0^{(v_2-2)}(x_1) & \cdots & u_{r-1}^{(v_2-2)}(x_1) & \frac{2(-1)^{l+1}}{(r+1-v_2)!} (x_1 - \xi_1)_+^{v_2-1} & \cdots & \frac{2(-1)^{l+N}}{(r-v_2+1)!} (x_1 - \xi_N)_+^{v_2-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P(s, y_{n+1}) & Ru_0(y_{n+1}) & \cdots & Ru_{r-1}(y_{n+1}) & \frac{2R(-1)^{l+1}}{(r-1)!} (y_{n+1} - \xi_1)_+^{r-1} & \cdots & \frac{2R(-1)^{l+N}}{(r-1)!} (y_{n+1} - \xi_N)_+^{r-1} \end{array} \right)$$

and the last  $l+m$  rows

$$\left( \begin{array}{cc} 0_{l \times 1} & (a_{ij} j!)_{l \times r} & 0_{l \times N} \\ 0_{m \times 1} & (B_i u_j)_{m \times r} & \left( \frac{2(-1)^{j+1}}{(r-1)!} B_i K(\cdot, \xi_j) \right)_{m \times N} \end{array} \right).$$

Expanding  $\det J(s)$  based on the first column, we get

$$\det J(s) = \sum_{j=1}^{n+1} (-1)^{\sigma_j} P(s, y_j) \det J_j(s), \tag{3.4}$$

where  $J_j(s)$  is the submatrix of  $J(s)$  eliminated 1 column and  $\sigma_j$  row.



We use  $\{z_{0,1} \leq z_{0,2} \leq \dots \leq z_{0,p}\}$  ( $p = N + r - l - m + 1$ ) to denote the sequence

$$\{y_{0,1}, (x_{0,1}, v_1 - 1), y_{0,2}, \dots, (x_{0,n}, v_n - 1), y_{0,n+1}\},$$

which is the set of zeros of  $P'_0 \in \mathcal{P}_{r-1,N}$  counting multiplicities. It follows from Lemma 3 and (2.2) that

$$z_{0,j-(l-1)} < \xi_{0,j} < z_{0,j+(r-1)-(l-1)} \tag{3.5}$$

whenever the subscripts are meaningful.

Given any  $i$ , put  $\{z_{0,j}^{(i)}\}_{j=1}^{p-1} = \{z_{0,j}\}_{j=1}^p / \{y_i\}$  arranged in increasing order. Then (3.5) yields

$$z_{0,j-l}^{(i)} < \xi_{0,j} < z_{0,j+r-l}^{(i)}. \tag{3.6}$$

On the other hand we see that  $J_i(s)$  has the form of matrix given as in Lemma 4 only different in some rows and columns to some non-zero constants, respectively, independent of  $i$ . It follows from Lemma 4, (3.1), and (3.4) that

$$|\det J(0)| = \sum_{j=1}^{n+1} |P_0(y_{0,j})| |\det J_i(0)|. \tag{3.7}$$

By the proof of Lemma 4 we have for some positive number  $\gamma$  that

$$|\det J_i(0)| = \gamma \sum_{\substack{1 \leq i_1 < \dots < i_l \leq r \\ 1 \leq j_1 < \dots < j_m \leq r}} \left| \tilde{A} \begin{pmatrix} 1, \dots, l \\ i_1, \dots, i_l \end{pmatrix} \prod_{j=1}^l (i_j - 1)! \right. \\ \left. \times B \begin{pmatrix} 1, \dots, m \\ j_1, \dots, j_m \end{pmatrix} \det K \begin{pmatrix} z_{0,1}^{(i)}, \dots, z_{0,p-1}^{(i)}, j_1 - 1, \dots, j_m - 1 \\ i'_1 - 1, i'_{r-l} - 1, \xi_{0,1}, \dots, \xi_{0,N} \end{pmatrix} \right|.$$

If  $m < N$ , then it follows from (3.6) and [7, Theorem 1'] that for any  $\{i_k\}_{k=1}^l$  and  $\{j_k\}_{k=1}^m$ ,

$$\det K \begin{pmatrix} z_{0,1}^{(i)}, \dots, z_{0,p-1}^{(i)}, j_1 - 1, \dots, j_m - 1 \\ i'_1 - 1, \dots, i'_{r-l} - 1, \xi_{0,1}, \dots, \xi_{0,N} \end{pmatrix} > 0. \tag{3.9}$$

Hence, (3.7), rank  $A = l$  and rank  $B = m$  give  $\det J_i(0) \neq 0$ .

If  $m = N$ , the interlacing conditions (3.6) cannot guarantee that (3.9) hold for all  $\{i_k\}_{k=1}^l$  and  $\{j_k\}_{k=1}^m$ . However, we also have  $\det J_i(0) \neq 0$  in this case. In fact, in view of Lemma 4 and [13, Lemma 10], there exists a unique function (i.e., zero element)  $s \in \Pi_r(A, B)$  which satisfies

$$s^{(j)}(x_{0,i}) = 0, \quad i = 1, \dots, n, \quad j = 0, \dots, v_i - 1, \tag{3.10}$$

where

$$\Pi_r = \left\{ s \mid s = \sum_{i=0}^{r-1} a_i t^i + \sum_{i=r}^{r+N-1} a_i (t - \xi_{0,i})_+^{r-1} \right\}.$$

(We have to say a few words about (3.10). By the method of proof of Theorem C of [3] we can conclude that any  $\xi_{0,i}$  cannot be a zero of  $P_0$  with multiplicity  $r$ ,  $i = 1, \dots, N$ . Since  $\Pi_r \subset C^{r-2}[0, 1] \cap C^r\{[0, 1]/\{\xi_{0,i}\}_{i=1}^N\}$ , all  $s^{(j)}(x_{0,i})$  in (3.10) are well defined).

Appealing to [7 Theorem 1'] once again we get two sets of indices  $\{i_k^*\}_{k=1}^l$  and  $\{j_k^*\}_{k=1}^m$  such that

$$A \begin{pmatrix} 1, \dots, l \\ i_1^*, \dots, i_l^* \end{pmatrix} B \begin{pmatrix} 1, \dots, m \\ j_1^*, \dots, j_m^* \end{pmatrix} \neq 0, \quad (3.11)$$

and

$$j_k^* \leq i_{k+r+N-l-m}^*. \quad (3.12)$$

Therefore, it follows from (3.6), (3.12), and [7, Theorem 1'] that (3.9) holds for  $\{i_k\}_{k=1}^l = \{i_k^*\}_{k=1}^l$  and  $\{j_k\}_{k=1}^m = \{j_k^*\}_{k=1}^m$ . Noting (3.11) we get  $\det J_i(0) \neq 0$ .

Since (3.7) and other factors in  $\det \Delta(0)$  are not zero, we have  $\det \Delta(0) \neq 0$ . Now we can continue the proof as in Theorem 3.1 in [2] and get the unique  $P(\cdot) := P(1, \cdot) \in \mathcal{P}_{r,N}(A, B; \nu_1, \dots, \nu_n)$  satisfying the theorem. The detail is omitted and referred to [2].

If  $\alpha_0 - \alpha_1 = 1$  and  $\beta_0 - \beta_1 = 0$ , then  $P'_0$  has a zero  $y_{0,1} \in (0, x_{0,1})$  and not zero in  $(x_{0,n}, 1)$ . Instead of (3.3) we consider the following system of equations:

$$\begin{aligned} P^{(j)}(s, t)|_{t=x_i(s)} &= 0, & i = 1, \dots, n, j = 0, \dots, \nu_i - 1, \\ R(s)P(s, y_i(s)) &= (-1)^{\sigma + \sigma_i} e_i(s), & i = 1, \dots, n+1, \\ P'(s, t)|_{t=y_i(s)} &= 0, & i = 1, \dots, n, \\ A_i P(s, \cdot) &= 0, & i = 1, \dots, l, \\ B_i P(s, \cdot) &= 0, & i = 1, \dots, m, \end{aligned}$$

where  $y_{n+1}(s) = 1$ .

As before, we can get a unique  $P \in \mathcal{P}_{r,N}(A, B; \nu_1, \dots, \nu_n)$  satisfying the theorem. The other cases of  $\alpha_i$  and  $\beta_i$  may be treated similarly. So Theorem 1 is proved.

*Remark 1.* Since perfect splines and monosplines may be treated in a unified way (see [4]), Theorems 1 and 2 hold for monosplines. This is related to optimal quadrature formula.

*Remark 2.* It is not difficult to generalize the results of this paper to the case where separated boundary conditions are replaced by mixed boundary conditions, which were studied, e.g., in [8].

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